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The Singular Adams Inequality in Lorentz Sobolev Space for Bounded Domains in $\mathbb{R}^n$

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Abstract: In this paper, we obtain a singular Adams inequalities in Lorentz Sobolev space for bounded domains in $\mathbb{R}^n$.We generalize the Adams inequality in lorentz space for bounded domains to singular results.Furthermore, we construct an experimental function to verify the optimality of the conclusion. Our approach is based on the rearrangement argument developed by A.Alberico and the lemma used by G.Lu, H.Tang.

Keywords: Adams inequality;Lorentz space;lorentz inequality;singular; rearrangement

1 Introduction

Let Ω be a domain with finite measure in $\mathbb{R}^n (n \geq 2)$,$p \geq 1$,k be a nonnegative integer.We use $W_0^{k,p}(\Omega)$ to denote the usual Sobolev space consisting of functions vanishing on boundary $\partial\Omega$ together with their derivatives of order less than $k-1$, that is, the completion of $C_0^{+\infty}(\Omega)$ under the norm

$$\left[\|f\|_{L^p(\Omega)}^p + \sum_{j=1}^{k} \left\|\nabla^j f\right\|_{L^p(\Omega)}^p \right]^{1/p}.$$

The classical sobolev embedding theorem tells us $\forall p < \frac{n}{k}, W_0^{k,p}(\Omega) \subseteq L^q(\Omega), q = \frac{np}{n-p}$,but $W_0^{k,\frac{n}{k}}(\Omega) \subseteq L^\infty(\Omega)$ is no true.Later, Moser (see [6]) and Trudinger (see [9]) further studied the case $k = 1$, namely the $W_0^{1,n}(\Omega)$ embedding property, and obtained the following well-known Trudinger-Moser inequality:

$$\sup_{||\nabla f||_n \leq 1} \int_\Omega \exp(\alpha_n |f|^{n/(n-1)}) dx < C\,|\Omega| \tag{1}$$

where $\alpha_n = \left(n v_n^{\frac{1}{n}}\right)^{\frac{n}{n-1}}$,$v_n = \pi^{n/2}/\Gamma(1+n/2)$,the measure of the unit ball in $\mathbb{R}^n$,Γ is the Gamma function,$|\Omega|$ stands for the Lebesgue measure of a subset Ω of $\mathbb{R}^n$. Moreover, the result is optimal, in the sense that, if α_n is replaced by any larger constant in (1), the integral is still finite, but not uniformly bounded.

If we divide the integral by monomial of $|x|$ so that the value of the function goes to infinity at the origin, is there a similar result?The question has been studied by K. Sandeep in [10]:

Theorem 1.1 Let Ω be an bounded domain with finite measure in Euclidean space $\mathbb{R}^n$.Then there exists a constant $C > 0$ such that

$$\sup_{f \in W_0^{1,n}(\Omega), \int_\Omega |\nabla f|^n dx \leq 1} \int_\Omega \frac{\exp\left(\alpha |f|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx < C$$

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if and only if $\frac{\alpha}{\alpha_n}+\frac{\beta}{n}\leq 1$ where $\alpha>0\,,\beta\in[0\,,n)$.This constant α_n is sharp in the sense that if α_n is replaced by any $\alpha>\alpha_n$, then the supremum is infinite.

By observing the above theorem, we can see that when β=0, it's the same as the Trudinger-Moser inequality.If $\beta\neq 0$,the optimal constant is going to be less than α_n, and it depends on β.

In this paper, we will focus on Trudinger-Moser inequality for higher order derivatives.The analogue of (1) for higher order Sobolev space $W_0^{m,\frac{n}{m}}(\Omega)$ is obtained by Adams ([1]), which can be stated as following:

Theorem 1.2 (Adams inequality). Let Ω be an open and bounded set in $\mathbb{R}^n$. If m is a positive integer less than n,then there exists a constant $C_0=C(n,m)>0$ such that for any $f\in W_0^{m,\frac{n}{m}}(\Omega)$ and $\|\nabla^m f\|_{L^{\frac{n}{m}}(\Omega)}\leq 1$, where

$$\nabla^m f=\begin{cases}\Delta^{\frac{m}{2}}f & for\ m\ even\\ \nabla\Delta^{\frac{m-1}{2}}f & for\ m\ odd\end{cases}$$

then

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\alpha|f\left(x\right)|^{\frac{n}{n-m}}\right)dx\leq C_0 \tag{2}$$

for all $\alpha\leqslant\beta_{n,m}$ where

$$\beta_{n,m}=\begin{cases}\left(\frac{\pi^{n/2}2^m\Gamma\left(\frac{m}{2}\right)}{v_n^{\frac{n-m}{n}}\Gamma\left(\frac{n-m}{2}\right)}\right)^{\frac{n}{n-m}} & m\ is\ odd\\ \left(\frac{\pi^{n/2}2^m\Gamma\left(\frac{m+1}{2}\right)}{v_n^{\frac{n-m}{n}}\Gamma\left(\frac{n-m+1}{2}\right)}\right)^{\frac{n}{n-m}} & m\ is\ even\end{cases}$$

Furthermore, for any $\alpha>\beta_{n,m}$, the integral can be made as large as possible.

In this paper, we are interested in the Trudinger-moser and Adams inequalities under the restriction of lorentz norm. Let us first recall some of the them. The first improvement of (1) by exploiting a Lorentz space has been addressed in [3]:

Theorem 1.3 Let Ω be an bounded domain with finite measure in Euclidean space $\mathbb{R}^n$,$n\geq 2$,$1<q<+\infty$.Then there exists a constant $c_0>0$ and a sharp constant $\alpha_{n,q}=\left(nv_n^{1/n}\right)^{\frac{q}{q-1}}$ such that

$$\int_\Omega \exp\left(\alpha_{n,q}|f\left(x\right)|^{\frac{q}{q-1}}\right)dx\leq C$$

for any $f\in W_0^1L^{n,q}(\Omega)$ with $\|\nabla f\|_{n,q}\leq 1$.This constant $\alpha_{n,q}$ is sharp in the sense that if $\alpha_{n,q}$ is replaced by any $\alpha>\alpha_{n,q}$, then the supremum is infinite.

Just as K. Sandeep studied singular results in Trudinger-moser inequality,Lu and Tang studied singular results in Trudinger-moser inequality under the restriction of lorentz norm[4]:

Theorem 1.4 Let Ω be an open domain with finite measure in Euclidean space $\mathbb{R}^n$,$n\geq 2$, $q\in(1,+\infty)\,,0\leq\beta<n$.Then there exists a constant $C=C\left(n,\beta,|\Omega|\,,q\right)$ such that

$$\frac{1}{|\Omega|^{1-\beta/n}}\int_\Omega\frac{\exp\left[\left(1-\frac{\beta}{n}\right)\alpha_{n,q}|f|^{q/(q-1)}\right]}{|x|^\beta}dx\leq C \tag{3}$$

for any $f\in W_0^1L^{n,q}(\Omega)$ with $\|\nabla f\|_{n,q}\leq 1$.This constant $\alpha_{n,q}$ is sharp in the sense that if $\alpha_{n,q}$ is replaced by any $\alpha>\alpha_{n,q}$, then the supremum is infinite.

Later,Alberico [2] proved the Moser type inequalities for higher-order derivatives in Lorentz spaces:

Theorem 1.5 Let Ω be an open domain with finite measure in Euclidean space $\mathbb{R}^n$,$n\geq 2$,$1<q<+\infty$.Then there exists a constant $C=C\left(n,m,|\Omega|\,,q\right)$ such that

$$\int_\Omega \exp\left(\beta_{n,m}|f\left(x\right)|\right)^{\frac{q}{q-1}}dx\leq C \tag{4}$$

for any $f\in W_0^mL^{\frac{n}{m},q}(\Omega)$ with $\|\nabla^m f\|_{\frac{n}{m},q}\leq 1$.This constant $\beta_{n,m}$ is sharp in the sense that $\beta_{n,m}$ is replaced by any $\beta>\beta_{n,m}$, then the supremum is infinite.

A nature question is whether the Moser type inequalities for higher-order derivatives in Lorentz spaces still hold for singular case?In this paper,we will answer this question primarily and our result can be stated as following:

Theorem 1.6 let Ω be an open domain with finite measure in Euclidean space $\mathbb{R}^n$,$0 \leq \beta < n$,$1 < q < +\infty$. Then there exist a constant $C = C(n, m, |\Omega|, q)$ such that

$$\sup_{f\in W_0^m L^{\frac{n}{m},q}(\Omega),\|\nabla^m f\|_{L^{\frac{n}{m},q}(\Omega)}\leq 1} \int_\Omega \frac{\exp\left[\left(1-\frac{\beta}{n}\right)|\beta_{n,m} f|^{q/q-1}\right]}{|x|^\beta} dx \leq C \tag{5}$$

for any $f \in W_0^m L^{\frac{n}{m},q}(\Omega)$ with $\|\nabla^m f\|_{\frac{n}{m},q} \leq 1$.This constant $\beta_{n,m}$ is sharp in the sense that $\beta_{n,m}$ is replaced by any $\alpha > \beta_{n,m}$, then the supremum is infinite.

When $q = +\infty$,our result can be stated as following:

Theorem 1.7 let Ω be an open domain with finite measure in Euclidean space $\mathbb{R}^n$,$0 \leq \beta < n$, Then there exist a constant $C = C(n, m, |\Omega|, q)$ such that for any $f \in W_0^{m,\frac{n}{m}}(\Omega)$ and $\|\nabla^m f\|_{L^{\frac{n}{m},+\infty}(\Omega)} \leq 1$, then

$$\int_\Omega \frac{\exp\left(\left(1-\frac{\beta}{n}\right)\alpha |f(x)|\right)}{|x|^\beta} dx \leq C \tag{6}$$

for all $\alpha < \beta_{n,m}$. Furthermore, for any $\alpha > \beta_{n,m}$, the integral can be made as large as possible.

2 Preliminaries

Let $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f^*(s) = \sup\{t > 0, d_f(t) > s\}$$

where $d_f(t) := \{x \in \mathbb{R}^n | f(x) > t\}$.Now, define $f^\# : \mathbb{R}^n \to \mathbb{R}$ by

$$f^\#(x) = f^*(v_n|x|^n)$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$.Then for every continuous increasing function $\Psi : [0, +\infty) \to [0, +\infty)$, we have from (see [5]) that

$$\int_{\mathbb{R}^n} \psi(f)\, dx = \int_{\mathbb{R}^n} \psi(f^\#)\, dx.$$

Since f^* is nonincreasing, the maximal function f^{**} of the rearrangement of f^*,defined by $f^{**} := \frac{1}{s}\int_0^s f^*\, dt$ for $s \geqslant 0$ is also nonincreasing and $f^* \leqslant f^{**}$. For more properties of the rearrangement (see [5],[7]and [8]). The Lorentz spaces $L^{p,q}$, with $1 < p < \infty$, and $1 \leqslant q < \infty$, is defined to consist of all functions Ψ such that

$$\|\Psi\|_{p,q} = (\int_0^\infty [\Psi^*(t)t^{\frac{1}{p}}]^q \frac{1}{t}\, dt)^{\frac{1}{q}} < \infty.$$

It is well known that $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and when $q > p$ the quantity $\|\Psi\|_{p,q}$ is not a norm. But the quantity

$$\|\Psi\|^*_{p,q} = (\int_0^\infty [\Psi^{**}(t)t^{\frac{1}{p}}]^q \frac{1}{t}\, dt)^{\frac{1}{q}}$$

is a norm for any p and q. It is easy to prove by using the Hardy¡¯s inequalities (see [7]) that these two quantities are equivalent in the sense that

$$\|\Psi\|_{p,q} \leqslant \|\Psi\|^*_{p,q} \leqslant C(p, q)\|\Psi\|_{p,q}.$$

As the classic Sobolev spaces are built up from the Lebesgue spaces, one similarly can introduce Lorentz-Sobolev spaces which consist of functions having weak derivatives belonging to the Lorentz space $L^{p,q}(\mathbb{R}^n)$. For more details of Lorentz spaces, we refer the reader to the book by Stein and Weiss (see [7]). In particular, let $1 < q < \infty$ and Ω be an open domain with finite measure in Euclidean space $\mathbb{R}^n$, define

$$W_0^m L^{p,q}(\Omega) = the\ closure\ of \left\{ f \in C_0^m(\Omega) : \|f\|_{m,(p,q)} < +\infty \right\}$$

where the norm

$$\|f\|_{m,(p,q)}^q = \sum_{i=0}^{m} \left\|\nabla^i u\right\|_{p,q}^q.$$

Next, we recall the rearrangement inequality calculated by A. Alberico in [2]:If $f \in W_0^m L^{\frac{n}{m},q}(\Omega)$,then

$$f^*(s) \le \frac{1}{\beta_{n,m}} \left(\frac{n}{m} s^{-\frac{n-m}{n}} \int_0^s |\nabla^m f|^*(x)\,dx + \int_s^{|\Omega|} |\nabla^m f|^*(x)\, x^{-\frac{n-m}{n}} dx \right). \tag{7}$$

we will convert it to one-dimensional problems to simplify the original integral.The estimates which have to be faced after this rearrangement process require the use of an extension of Moser¡¯s one-dimensional lemma used by G. Lu and H. Tang in [4]:

Lemma 2.1 Let $0 < \alpha \le 1, 1 < q < \infty$,and $a(s,t)$ be a non-negative measurable function on $(-\infty, +\infty) \times [0, +\infty]$ such that

$$a(s,t) \le 1 \quad when\ 0 < s < t \tag{8}$$

and

$$\sup_{t>0} \left(\int_{-\infty}^{0} + \int_t^{\infty} a(s,t)^{q'} ds \right)^{\frac{1}{q'}} = b < +\infty. \tag{9}$$

If

$$\int_{-\infty}^{+\infty} \phi(s)^q ds \le 1 \quad for\ \phi \ge 0$$

Then, there exists a constant $C = C(q,b)$ such that

$$\int_0^\infty \mathrm{e}^{-F_\alpha(t)} dt \le C$$

where $F_\alpha(t) = \alpha t - \alpha \left(\int_{-\infty}^{+\infty} \phi(s) a(s,t)\, ds \right)^{q'}$.

3 Proof of theorem 1.6

Proof. By inequality (7),we have

$$\begin{aligned} f^*(|\Omega| e^{-t}) &\le \tfrac{1}{\beta_{n,m}} \left(\tfrac{n}{m} (|\Omega| e^{-t})^{-\frac{n-m}{n}} \int_0^{|\Omega| e^{-t}} |\nabla^m f|^*(x)\,dx + \int_{|\Omega| e^{-t}}^{|\Omega|} |\nabla^m f|^*(x)\, x^{-\frac{n-m}{n}} dx \right) \\ &\le \tfrac{|\Omega|^{\frac{m}{n}}}{\beta_{n,m}} \left(\tfrac{n}{m} e^{\frac{n-m}{n} t} \int_t^{+\infty} |\nabla^m f|^*(|\Omega| e^{-s}) e^{-s} ds + \int_0^t |\nabla^m f|^*(|\Omega| e^{-s}) e^{-\frac{m}{n} s} ds \right) = \tfrac{1}{\beta_{n,m}} \int_{-\infty}^{+\infty} \phi(s) a(s,t)\, ds. \end{aligned}$$

set

$$\phi(s) = |\Omega|^{\frac{m}{n}} |\nabla^m f|^* \left(|\Omega| e^{-s} \right) e^{-\frac{m}{n} s} \qquad \text{if } s > 0$$

and

$$\mathrm{a}(s,\mathrm{t}) = \begin{cases} 0 & \text{if } s \le 0 \\ 1 & \text{if } 0 \le s < t < +\infty \\ \frac{n}{m} e^{\frac{n-m}{n}(t-s)} & \text{if } 0 \le t < s < +\infty. \end{cases} \tag{10}$$

Assumption (8) is obviously satisfied if a is given by (10).As far as (9) is concerned we have

$$\left(\int_t^\infty a(s,t)^{q'} ds \right)^{\frac{1}{q'}} = \left(\int_t^\infty \left(\tfrac{n}{m}\right)^{q'} e^{\frac{n-m}{n}(t-s)q'} ds \right)^{\frac{1}{q'}} < +\infty \qquad for\ t > 0,$$

whence

$$\sup_{t>0}(\int_t^{\infty} a\,(s,t)^{q'}\,ds)^{\frac{1}{q'}} < +\infty.$$

By calculation,

$$\begin{aligned}&\int_0^{+\infty} \phi\,(s)^q ds = \int_0^{+\infty} \left(|\Omega|^{\frac{m}{n}} |\nabla^m f|^* \left(|\Omega|\, e^{-s}\right) e^{-\frac{m}{n}s}\right)^q ds\\ &= \int_0^{|\Omega|} \left(|\nabla^m f|^* \left(|x|\right) |x|^{\frac{m}{n}}\right)^q \frac{d|x|}{|x|} = \||\nabla^m f|\|^q_{L^{\frac{n}{m},q}(\Omega)} \le 1 \qquad for\ q \in (1,+\infty),\end{aligned}$$

and

$$\left(|x|^{-\beta}\right)^* (z) = \left(\frac{z}{v_n}\right)^{-\frac{\beta}{n}}, \left(\exp\left[\left(1-\frac{\beta}{n}\right)\alpha |f|^{\frac{q}{q-1}}\right]\right)^* (z) = \exp\left[\left(1-\frac{\beta}{n}\right)\alpha (f^*\,(z))^{\frac{q}{q-1}}\right].$$

Thus,by Lemma 2.1,

$$\begin{aligned}&\int_\Omega \frac{\exp\left[\left(1-\frac{\beta}{n}\right)|\beta_{n,m} f|^{q/q-1}\right]}{|x|^\beta} dx \le \int_0^{|\Omega|} \exp\left[\left(1-\frac{\beta}{n}\right)(\beta_{n,m})^{q/q-1} f^*(z)^{q/q-1}\right]\left(\frac{z}{v_n}\right)^{-\frac{\beta}{n}} dz\\ &= \int_0^{+\infty} \exp\left[\left(1-\frac{\beta}{n}\right)(\beta_{n,m})^{q/q-1} f^*(e^{-t}\,|\Omega|)^{q/q-1}\right]\left(\frac{e^{-t}|\Omega|}{v_n}\right)^{-\frac{\beta}{n}} e^{-t}\,|\Omega|\, dt\\ &\le \int_0^{+\infty} \exp\left[\left(1-\frac{\beta}{n}\right)\left(\int_{-\infty}^{+\infty} \phi\,(s) a\,(s,t)\, ds\right)^{q/q-1}\right]\left(\frac{e^{-t}|\Omega|}{v_n}\right)^{-\frac{\beta}{n}} e^{-t}\,|\Omega|\, dt\\ &\le (v_n)^{\frac{\beta}{n}} |\Omega|^{1-\frac{\beta}{n}} \int_0^{+\infty} \exp\left[-F_{1-\frac{\beta}{n}}\,(t)\right] dt \le C\end{aligned}$$

for some constant $C = C(n,m,|\Omega|,q)$. In order to prove the optimality of (10), we construct a sequence $\{f_k\}_{k\in N} \subset W_0^m L^{\frac{n}{m},q}\,(\Omega)$ such that

$$\lim_{k\to\infty} \int_\Omega \frac{\exp\left[\left(1-\frac{\beta}{n}\right)\left(\alpha \frac{|f_k|}{\||\nabla^m f_k|\|_{L^{\frac{n}{m},q}(\Omega)}}\right)^{q/q-1}\right]}{|x|^\beta} dx = +\infty \qquad \forall \alpha > \beta_{n,m}.$$

Let B be the unit ball of $\mathbb{R}^n$ centered at the origin. Up to rescaling and translating, we may assume, without loss of generality, that $B \subset\subset \Omega$. Let φ be an increasing smooth function (of class C^m, say) defined in $\mathbb{R}$ as equals to zero, if $t \le 0$, and equals to t , if $t \ge 1$. Given $\varepsilon \in \left(0,\frac{1}{2}\right)$,let $H_\varepsilon : R \to [0,1]$ be defined as

$$\mathrm{H}_\varepsilon\,(t) = \begin{cases} 0 & \text{if} \quad t<0\\ \varepsilon\varphi\left(\frac{t}{\varepsilon}\right) & \text{if} \quad 0\le t\le\varepsilon\\ t & \text{if} \quad \varepsilon<t<1-\varepsilon\\ 1-\varepsilon\varphi\left(\frac{1-t}{\varepsilon}\right) & \text{if} \quad 1-\varepsilon\le t\le 1\\ 1 & \text{if} \quad t>1.\end{cases}$$

Observe that H_ε is of class C^m. For $k \in \mathbb{N}$, define $f_{\varepsilon,k} : \Omega \to \mathbb{R}$ as

$$f_{\varepsilon,k}\,(x) = \begin{cases} H_\varepsilon\left(\frac{\log\frac{1}{|x|}}{\log k}\right) & if \quad x\in B\\ 0 & otherwise.\end{cases}$$

Notice that $f_{\varepsilon,k} \in C_0^m(\Omega)$.Computations show that

$$\nabla^m f_{\varepsilon,k}\,(x) = \begin{cases} \sum\limits_{i=1}^{2h} c_i H_\varepsilon^{(i)}\left(\frac{\log\frac{1}{|x|}}{\log k}\right)\frac{1}{|x|^{2h}(\log k)^i}, & m=2h\\ \sum\limits_{i=1}^{2h+1} c_i H_\varepsilon^{(i)}\left(\frac{\log\frac{1}{|x|}}{\log k}\right)\frac{1}{|x|^{2h+1}(\log k)^i}, & m=2h+1\end{cases} \qquad (h\in N, x\in B)$$

with $h \in \mathbb{N}$,if $x \in B$,and $\nabla^m f_{\varepsilon,k}\,(x) = 0$ otherwise.Since φ is smooth (of class C^m,then a constant $C = C(\varphi,m)$),which be assumed to be larger than $n-m$,exist such that on setting

$$G(\varepsilon,m)=C\left(1+\frac{1}{\varepsilon\log k}+\frac{1}{\varepsilon^2(\log k)^2}+...+\frac{1}{\varepsilon^{m-1}(\log k)^{m-1}}\right)$$

we have

$$|\nabla^m f_{\varepsilon,k}(x)|\begin{cases}=0 & \text{if}\quad 0\le|x|<\frac{1}{k}\\ \le\frac{G(\varepsilon,m)}{|x|^m\log k} & \text{if}\quad \frac{1}{k}\le|x|\le\frac{1}{k^{1-\varepsilon}}\\ =\frac{\beta_{n,m}}{nv_n^{m/n}}\frac{1}{|x|^m\log k} & \text{if}\quad \frac{1}{k^{1-\varepsilon}}<|x|<\frac{1}{k^{\varepsilon}}\\ \le\frac{G(\varepsilon,m)}{|x|^m\log k} & \text{if}\quad \frac{1}{k^{\varepsilon}}\le|x|<1\end{cases}\tag{11}$$

for $x\in B$.Let $g_{\varepsilon,k}:[0,+\infty)\to[0,+\infty)$ be the function defined as

$$g_{\varepsilon,k}(s)=\begin{cases}\frac{G(\varepsilon,m)v_n^{m/n}}{s^{m/n}\log k} & if\quad \frac{1}{k^n}\le s<\frac{1}{k^{(1-\varepsilon)n}}\\ \frac{\beta_{n,m}}{ns^{m/n}}\frac{1}{\log k} & if\quad \frac{1}{k^{(1-\varepsilon)n}}\le s<\frac{1}{k^{\varepsilon n}}\\ \frac{G(\varepsilon,m)v_n^{m/n}}{s^{m/n}\log k} & if\quad \frac{1}{k^{\varepsilon n}}\le s<1\\ 0 & otherwise.\end{cases}$$

Then,by (11),$|\nabla^m f_{\varepsilon,k}(x)|\le g_{\varepsilon,k}(v_n|x|^n)$ for $x\in B$,and hence $(\nabla^m f_{\varepsilon,k})^*(s)\le g^*_{\varepsilon,k}(s)\,(s>0)$,for $s>0$. In order to get the rearrangement function of $g_{\varepsilon,k}$ (s),we will compare the value of each piecewise function separately. We set $\frac{1}{k^n}\le s<\frac{1}{k^{(1-\varepsilon)n}}$ as the first interval,$\frac{1}{k^{(1-\varepsilon)n}}\le s<\frac{1}{k^{\varepsilon n}}$ as the second interval, and $\frac{1}{k^{\varepsilon n}}\le s<1$ as the third interval.In the first interval,the minimum value is $\frac{G(\varepsilon,m)k^{(1-\varepsilon)m}}{\log k}$.Since $f_{\varepsilon,k}\in C_0^m(\Omega)$,$[|\nabla^m f_{\varepsilon,k}|$ is continuous and $\left|\nabla^m f_{\varepsilon,k}\left(\frac{1}{k^{1-\varepsilon}}\right)\right|=\frac{\beta_{n,m}}{nv_n^{m/n}}\frac{k^{(1-\varepsilon)m}}{\log k}$.By (11),$\left|\nabla^m f_{\varepsilon,k}\left(\frac{1}{k^{1-\varepsilon}}\right)\right|\le\frac{G(\varepsilon,m)k^{(1-\varepsilon)m}}{\log k}$,So we get the relation between the minimum value of the first piecewise function and the maximum value of the second piecewise function:

$$\frac{\beta_{n,m}}{n}\frac{k^{(1-\varepsilon)m}}{\log k}\le\frac{G(\varepsilon,m)k^{(1-\varepsilon)m}v_n^{m/n}}{\log k}.\tag{12}$$

In other words, the first segment is at the top of the whole function.By (12),we see The maximum value of the third segment function is greater than or equal to the minimum value of the second segment function:

$$\frac{\beta_{n,m}}{n}\frac{k^{\varepsilon m}}{\log k}\le\frac{G(\varepsilon,m)k^{\varepsilon m}v_n^{m/n}}{\log k}\left(\frac{\beta_{n,m}}{nv_n^{m/n}}\le G(\varepsilon,m)\right).\tag{13}$$

If k is sufficiently large,the maximum value of the second segment is greater than the maximum value of the third segment:

$$\frac{\beta_{n,m}}{n}\frac{k^{(1-\varepsilon)m}}{\log k}>\frac{G(\varepsilon,m)k^{\varepsilon m}v_n^{m/n}}{\log k}.$$

and the second segment minimum is larger than the third segment minimum:

$$\frac{\beta_{n,m}}{n}\frac{k^{\varepsilon m}}{\log k}>\frac{G(\varepsilon,m)v_n^{m/n}}{\log k}.$$

Thus,we can get $g^*_{\varepsilon,k}(s)\,(s>0)$:

$$g^*_{\varepsilon,k}(s)=\begin{cases}\frac{G(\varepsilon,m)v_n^{m/n}}{(s+k^{-n})^{m/n}(\log k)} & 0\le s<\frac{1}{k^{(1-\varepsilon)n}}-\frac{1}{k^n}\\ \frac{\beta_{n,m}}{n(s+k^{-n})^{m/n}}\frac{1}{\log k} & \frac{1}{k^{(1-\varepsilon)n}}-\frac{1}{k^n}\le s<\frac{1}{k^{\varepsilon n}}r_{\varepsilon,n,m}-\frac{1}{k^n}\\ \left(\frac{\left(v_n^{-m/n}n^{-1}\beta_{n,m}\right)^{n/m}+G(\varepsilon,m)^{n/m}}{s+k^{-n}+k^{-\varepsilon n}}\right)^{\frac{m}{n}}\frac{v_n^{m/n}}{\log k} & \frac{1}{k^{\varepsilon n}}r_{\varepsilon,n,m}-\frac{1}{k^n}\le s<\frac{1}{k^{\varepsilon n}}r^{-1}_{\varepsilon,n,m}-\frac{1}{k^n}\\ \frac{G(\varepsilon,m)v_n^{m/n}}{(s+k^{-n})^{m/n}\log k} & \frac{1}{k^{\varepsilon n}}r^{-1}_{\varepsilon,n,m}-\frac{1}{k^n}\le s<1-\frac{1}{k^n}\\ 0 & s\ge 1-\frac{1}{k^n}\end{cases}$$

where $r_{\varepsilon,n,m} = \left(\frac{w_n^{-m/n} n^{-1} \beta_{n,m}}{G(\varepsilon,m)}\right)^{n/m}$. Fix $k > 1$,we have

$$\int_\Omega \frac{\exp\left[\left(1-\frac{\beta}{n}\right)\left(\alpha \frac{\left|f_{\varepsilon,k}(x)\right|}{\left\|\left|\nabla^m f_{\varepsilon,k}(x)_k\right|\right\|_{L^{\frac{n}{m},q}(\Omega)}}\right)^{q/q-1}\right]}{|x|^\beta} dx \geq \int_{B\cap\left\{|x|<\frac{1}{k}\right\}} \frac{\exp\left[\left(1-\frac{\beta}{n}\right)\left(\alpha \frac{\left|f_{\varepsilon,k}(x)\right|}{\left\|g^*_{\varepsilon,k}\right\|_{L^{\frac{n}{m},q}(\Omega)}}\right)^{q/q-1}\right]}{|x|^\beta} dx$$
$$= \frac{v_{n-1}}{n-\beta} \exp\left[\left(1-\frac{\beta}{n}\right)\left(\left(\frac{\alpha^{q/q-1}}{\left\|g^*_{\varepsilon,k}\right\|^{q/q-1}_{L^{\frac{n}{m},q}(\Omega)} \log k} - n\right)\log k\right)\right].$$

Hence, $\lim\limits_{k\to\infty} \int_\Omega \frac{\exp\left[\left(1-\frac{\beta}{n}\right)\left(\alpha \frac{|f_k|}{\||\nabla^m f_k|\|_{L^{\frac{n}{m},q}(\Omega)}}\right)^{q/q-1}\right]}{|x|^\beta} dx < +\infty$ only if

$$\lim_{k\to\infty}\left(\frac{\alpha^{q/q-1}}{\left\|g^*_{\varepsilon,k}\right\|^{q/q-1}_{L^{\frac{n}{m},q}(\Omega)} \log k} - n\right)\log k < +\infty. \tag{14}$$

Since

$$\lim_{k\to\infty}\left(\frac{\alpha^{q/q-1}}{\left\|g^*_{\varepsilon,k}\right\|^{q/q-1}_{L^{\frac{n}{m},q}(\Omega)} \log k} - n\right)\log k$$
$$= \lim_{k\to\infty}\left(-n + \frac{\alpha^{q/q-1}}{\log k}\left(\begin{array}{l} \int_0^{\frac{1}{k^{(1-\varepsilon)n}}-\frac{1}{k^n}} \left(\frac{G(\varepsilon,m) v_n^{m/n}}{(s+k^{-n})^{m/n}(\log k)} s^{\frac{m}{n}}\right)^q \frac{ds}{s} \\ + \int_{\frac{1}{k^{(1-\varepsilon)n}}-\frac{1}{k^n}}^{\frac{1}{k^{\varepsilon n}} r_{\varepsilon,n,m}-\frac{1}{k^n}} \left(\frac{\beta_{n,m}}{n(s+k^{-n})^{m/n}} \frac{1}{\log k} s^{\frac{m}{n}}\right)^q \frac{ds}{s} \\ + \int_{\frac{1}{k^{\varepsilon n}} r_{\varepsilon,n,m}-\frac{1}{k^n}}^{\frac{1}{k^{\varepsilon n}} r^{-1}_{\varepsilon,n,m}-\frac{1}{k^n}} \left(\left(\frac{\left(v_n^{-m/n} n^{-1}\beta_{n,m}\right)^{n/m} + G(\varepsilon,m)^{n/m}}{s+k^{-n}+k^{-\varepsilon n}}\right)^{\frac{m}{n}} \frac{v_n^{m/n}}{\log k} s^{\frac{m}{n}}\right)^q \frac{ds}{s} \\ + \int_{\frac{1}{k^{\varepsilon n}} r^{-1}_{\varepsilon,n,m}-\frac{1}{k^n}}^{1-\frac{1}{k^n}} \left(\frac{\beta_{n,m}}{n(s+k^{-n})^{m/n}} \frac{1}{\log k} s^{\frac{m}{n}}\right)^q \frac{ds}{s} \end{array}\right)^{-\frac{1}{q-1}}\right)\log k$$
$$= \lim_{k\to\infty}\left(-n + \frac{\alpha^{q/q-1}}{v_n^{\frac{mq}{n(q-1)}}}\left(\begin{array}{l} \frac{G(\varepsilon,m)^q}{\log k} \int_0^{k^{\varepsilon n}-1} \frac{t^{mq/n-1}}{(t+1)^{mq/n}} dt \\ + \left(\frac{\beta_{n,m}}{n v_n^{\frac{m}{n}}}\right)^q \frac{1}{\log k} \int_{k^{\varepsilon n}-1}^{k^{(1-\varepsilon)} r_{\varepsilon,n,m}-1} \frac{t^{mq/n-1}}{(t+1)^{mq/n}} dt \\ + \left(\frac{\left(v_n^{-m/n} n^{-1}\beta_{n,m}\right)^{n/m} + G(\varepsilon,m)^{n/m}}{\log k}\right)^q \int_{\frac{\frac{1}{k^{\varepsilon n}} r_{\varepsilon,n,m}-\frac{1}{k^n}}{\left(\frac{1}{k^n}+\frac{1}{k^{\varepsilon n}}\right)}}^{\frac{\frac{1}{k^{\varepsilon n}} r^{-1}_{\varepsilon,n,m}-\frac{1}{k^n}}{\left(\frac{1}{k^n}+\frac{1}{k^{\varepsilon n}}\right)}} \frac{t^{mq/n-1}}{(t+1)^{mq/n}} dt \\ + \frac{G(\varepsilon,m)^q}{\log k} \int_{k^{(1-\varepsilon)n} r_{\varepsilon,n,m}-1}^{k^n-1} \frac{t^{mq/n-1}}{(t+1)^{mq/n}} dt \end{array}\right)\right)^{-\frac{1}{q-1}} \log k$$
$$= \lim_{k\to\infty}\left(-n + \frac{\alpha^{q/q-1}}{v_n^{\frac{mq}{n(q-1)}}}\left(\varepsilon n G(\varepsilon,m)^q + \left(\frac{\beta_{n,m}}{n v_n^{m/n}}\right)^q n(1-2\varepsilon) + \varepsilon n G(\varepsilon,m)^q\right)^{-\frac{1}{q-1}}\right)\log k,$$

inequality (14) holds if and only if

$$-n + \frac{\alpha^{q/q-1}}{v_n^{\frac{mq}{n(q-1)}}}\left(2\varepsilon n G(\varepsilon,m)^q + \left(\frac{\beta_{n,m}}{n v_n^{m/n}}\right)^q n(1-2\varepsilon)\right)^{-\frac{1}{q-1}} \leq 0.$$

This condition is in turn equivalent to

$$\alpha \leq n v_n^{\frac{m}{n}}\left[2n\varepsilon G(\varepsilon,m)^q + \left(\frac{\beta_{n,m}}{n v_n^{m/n}}\right)^q n(1-2\varepsilon)\right]^{\frac{1}{q}}.$$

Thanks to the arbitrarinees of ε and (13), the conclusion follows. ■

Proof of theorem 1.7

Proof. Since we are assuming that $\||\nabla^m f|\|_{L^{\frac{n}{m},\infty}(\Omega)} \leq 1$,according to the lorentz norm (1.4) in [3],

$$(\nabla^m f)^*(t) \leq t^{-\frac{m}{n}} \quad for\ t \in (0, |\Omega|). \tag{15}$$

From (7) and (15),we infer that

$$\begin{aligned} f^*(t) &\leq \frac{1}{\beta_{n,m}} \left(\frac{n}{m} t^{-\frac{n-m}{n}} \int_0^t |\nabla^m f|^*(x)\,dx + \int_t^{|\Omega|} |\nabla^m f|^*(x)\, x^{-\frac{n-m}{n}} dx \right) \\ &\leq \frac{1}{\beta_{n,m}} \left(\frac{n}{m} t^{-\frac{n-m}{n}} \int_0^t x^{-\frac{m}{n}} ds + \int_t^{|\Omega|} x^{-1} ds \right) = \frac{1}{\beta_{n,m}} \left(\frac{n^2}{m(n-m)} + \log \frac{|\Omega|}{t} \right) \quad for\ \ t \in (0, |\Omega|). \end{aligned}$$

Thus a constant $C = C(n,m)$ exists such that

$$\begin{aligned} &\int_\Omega \frac{\exp\left(\left(1-\frac{\beta}{n}\right)\alpha|f(x)|\right)}{|x|^\beta} dx = \int_0^{|\Omega|} \exp\left(\left(1-\frac{\beta}{n}\right)\alpha\,|f^*(t)|\right)\left(\frac{t}{v_n}\right)^{-\frac{\beta}{n}} dt \\ &\leq \int_0^{|\Omega|} \exp\left(\left(1-\frac{\beta}{n}\right)\alpha \frac{1}{\beta_{n,m}} \left(\frac{n^2}{m(n-m)} + \log\frac{|\Omega|}{t}\right)\right)\left(\frac{t}{v_n}\right)^{-\frac{\beta}{n}} dt < +\infty \end{aligned}$$

for every $\alpha < \beta_{n,m}$. We conclude by exhibiting a sequence of functions $\{f_a\}_{a\in N} \subset W_0^m L^{\frac{n}{m},\infty}(\Omega)$such that

$$\lim_{a\to+\infty} \int_\Omega \frac{\exp\left(\left(1-\frac{\beta}{n}\right)\frac{\alpha|f_a|}{\||\nabla^m f|\|_{L^{\frac{n}{m},\infty}(\Omega)}}\right)}{|x|^\beta} dx = +\infty. \tag{16}$$

As in the proof of theorem (1.6), we assume that $B \subset\subset \Omega$. Let us consider an increasing smooth function (of class C^m, say) $\varphi : \mathbb{R} \to [0, +\infty)$ such that $\varphi(t) = 0$, for $t \leq 0$ and $\varphi(t) = t - 0.5$,for $t \geq 1$. Let $a > 1$ and define the function $f_a : \Omega \to \mathbb{R}$ as

$$f_a(x) = \begin{cases} \varphi\left(\frac{\log\frac{1}{|x|}}{\log a}\right) & if \quad x \in B \\ 0 & otherwise. \end{cases} \tag{17}$$

Observe that $|\varphi'(t)| \leq 1$ for every $t \in \mathbb{R}$ and that the h-th order derivative $\varphi^h(t)$ is bounded for every $t \in \mathbb{R}$ and for every $h \in \mathbb{N}$. Furthermore, $f_a \in C_0^m(\Omega)$. Calculations show that, for every $h \in \mathbb{N}$,

$$\nabla^m f_a(x) = \begin{cases} \sum\limits_{i=1}^{2h} c_i \varphi^{(i)}\left(\frac{\log\frac{1}{|x|}}{\log a}\right) \frac{1}{|x|^{2h}(\log a)^i}, & if \quad m = 2h \\ \sum\limits_{i=1}^{2h+1} c_i \varphi^{(i)}\left(\frac{\log\frac{1}{|x|}}{\log a}\right) \frac{1}{|x|^{2h+1}(\log a)^i}, & if \quad m = 2h+1 \end{cases} \tag{18}$$

for $x \in B$,and $\nabla^m f_a(x) = 0$ otherwise.Here $c_1 = s(m)\frac{\beta_{n,m}}{nw_n^{m/n}}$,where $s(m) = \begin{cases} (-1)^{m/2} & m = 2h \\ (-1)^{(m+2)/2} & m = 2h+1 \end{cases}$ $(h \in N)$ and $c_i, i = 2, ..., m$, are constants depending only on n and m.Owing to (18),one has

$$|\nabla^m f_a|(x) \leq \frac{\beta_{n,m}}{nv_n^{m/n}} \frac{1}{|x|^m \log a}\left(1 + \frac{K}{\log a} + ... + \frac{K}{(\log a)^{m-1}}\right)$$

for some constant K depending on φ. Fix any $\alpha > \beta_{n,m}$, choose $\varepsilon > 0$ such that

$$\alpha > \beta_{n,m}(1+\varepsilon). \tag{19}$$

If a is sufficiently large, we have

$$|\nabla^m f_a|(x) \leq \frac{\beta_{n,m}}{nv_n^{m/n}} \frac{1}{|x|^m \log a}(1+\varepsilon), (|\nabla^m f_a|)^*(t) \leq \frac{\beta_{n,m}}{nv_n^{m/n}} \frac{1}{\log a}\left(\frac{t}{v_n}\right)^{-\frac{m}{n}}(1+\varepsilon) = \frac{\beta_{n,m}}{nt^{\frac{m}{n}}} \frac{1}{\log a}(1+\varepsilon).$$

Hence,

$$|||\nabla^m f_a|||_{L^{\frac{n}{m},\infty}(\Omega)} \leq \frac{\beta_{n,m}(1+\varepsilon)}{n\log a}.$$

Equation (16) follows since

$$\begin{aligned}
&\int_\Omega \frac{\exp\left(\left(1-\frac{\beta}{n}\right)\frac{\alpha|f|}{|||\nabla^m f|||_{L^{\frac{n}{m},\infty}(\Omega)}}\right)}{|x|^\beta}dx \geq \int_B \frac{\exp\left(\left(1-\frac{\beta}{n}\right)\frac{\alpha\varphi\left(\frac{\log\left(|x|^{-1}\right)}{\log a}\right)n\log a}{\beta_{n,m}(1+\varepsilon)}\right)}{|x|^\beta}dx \\
&\geq \int_{B\cap\{x:|x|\leq 1/a\}} \frac{\exp\left(\left(1-\frac{\beta}{n}\right)\frac{\alpha\left(\frac{\log\left(|x|^{-1}\right)}{\log a}-\frac{1}{2}\right)n\log a}{\beta_{n,m}(1+\varepsilon)}\right)}{|x|^\beta}dx \\
&= w_{n-1}\int_0^{1/a}\exp\left(\left(1-\frac{\beta}{n}\right)\frac{\alpha n\log a}{2\beta_{n,m}(1+\varepsilon)}\right)\exp\left(\left(1-\frac{\beta}{n}\right)\frac{\alpha\log\left(r^{-1}\right)n}{\beta_{n,m}(1+\varepsilon)}\right)r^{n-\beta-1}dr \\
&= C(n,m)\int_0^{1/a} r^{\left(1-\frac{\beta}{n}\right)\left(n-\frac{\alpha n}{\beta_{n,m}(1+\varepsilon)}\right)-1}dr = +\infty
\end{aligned}$$

by (18) and (19). ■

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