ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.29(2020) No.2/3,pp.94-102

Blow-up of Solution Near the Peakon for Cauchy Problem of the Degasperis-Procesi Equation

Qiaoli Zhou*, Danping Ding
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China
(Received 15 April 2020, accepted 21 May 2020)

Abstract: This paper studies the blow-up of solution near the peakon for Cauchy problem of the Degasperis-Procesi equation. A sufficient condition for the blow-up of solution is obtained. Applying the extended pseudo-conformal transformation, a equivalent proposition of the solution breaking in finite time near the peakon is constructed.

Keywords: Degasperis-Procesi equation; Blow-up; Pseudo-conformal transformation

1 Introduction

In 1999, Degasperis and Procesi [1] derived a class of nonlinear dispersive shallow water wave equations as follows:

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, x \in R, \tag{1.1}$$

which is called the Degasperis-Procesi (DP) equation.

Degasperis, Holm and Hone [2] constructed a Lax pair proving that DP equation is the completely integrable system with bi-Hamiltonian structure. Yin [3] obtained the local well-posedness of Cauchy problem for the DP equation. Liu and Yin [4] established the global existence and uniqueness results for entropy weak solutions which belong to the class of $L^1(\mathrm{R})\bigcap \mathrm{BV}(\mathrm{R})$ and the class of $L^2(\mathrm{R})\bigcap \mathrm{L}^4(\mathrm{R})$. Liu and Yin [7] studied the blow-up solution of Cauchy problem for DP equation. Liu and Yin [8] constructed a Lyapunov function proving the orbital stability of peakons for the DP equation in the L^2 norm. Kabakouala [9] proved the orbital stability of multi-peakons for the DP equation.

The DP equation has conservation laws:

$$E_1(u) = \int_R y dx, \quad E_2(u) = \int_R ym dx, \quad E_3(u) = \int_R u^3 dx,$$

where $y = (1-\partial_x^2)u$ and $m = (4-\partial_x^2)^{-1}u$. By the Fourier transform, it gives

$$E_2(u) = \int_R ym dx = \int_R \frac{1+\xi^2}{4+\xi^2}|\hat{u}(\xi)|^2 d\xi \backsim \|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2. \tag{1.2}$$

For simplicity, equation (1.1) can be written as:

$$u_t + \partial_x\left(\frac{1}{2}u^2 + P * \left(\frac{3}{2}u^2\right)\right) = 0, \quad t > 0, x \in \mathrm{R}, \tag{1.3}$$

where $P(x) = \frac{1}{2}\mathrm{e}^{-|x|}$ for $x \in \mathrm{R}$.

The DP equation possesses solitary waves which are called peakons and defined by

$$u(t,x) = Q_c(x-ct) = cQ(x-ct) = ce^{-|x-ct|}, \quad c \in R^+, (t,x) \in R^+ \times R,$$

*Corresponding author. *E-mail address*: 1467730951@qq.com

IJNS.2020.06.15

but they are not smooth since $Q_c \notin C^1(R)$. Without loss of generality, we will take $c = 1$ and let $Q(x-t) = Q_1(x-t)$ in the remainder of this paper. The equation (1.3) is rewritten as follows:

$$u_t + uu_x - \frac{3}{2}\int_R P_z(x-z)u^2(z)dz = 0. \tag{1.4}$$

We consider the following Cauchy problem for the DP equation in this paper

$$\begin{cases} u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \ (t,x) \in \mathrm{R}^+ \times \mathrm{R}, \\ u(0,x) = u_0(x), \ x \in \mathrm{R}, \end{cases} \tag{1.5}$$

for $u_0 \in H^2(\mathrm{R})$.

Definition 1 *Let $u(t,x)$ be the solution of Cauchy problem (1.5). If there exists $0 < T_1 < +\infty$, such that*

$$\lim_{t\to T_1} \inf_{x\in R}\{u_x(t,x)\} = -\infty,$$

then it is said to be the $u(t,x)$ blows up in finite time.

On the other hand, Martel and Merle [10] proved the orbital instability of solitons for the critical generalized Korteweg-de Vries equation via the extended pseudo-conformal transformation method. Martel [11] further studied the blow-up of critical generalized Korteweg-de Vries equation through pseudo-generic transformation.

Inspired by Martel and Merle, in this paper, we apply the extended pseudo-conformal transformation to the solution of (1.5) and decompose

$$\lambda^{1/2}(t)u(t, y + x(t)) = \varepsilon(t,y) + Q(y), \tag{1.6}$$

where $u(t,x)$ is a solution of the Cauchy problem(1.5), ε is small, Q is the peakon of the DP equation, $\lambda(t) > 0$, $x(t) \in \mathrm{R}$ are two parameters which will be determined later.

Changing the time variable yields:

$$s = \int_0^t \frac{dt'}{\lambda^{1/2}(t')}, \quad \text{or equivalently,} \quad \frac{ds}{dt} = \frac{1}{\lambda^{1/2}(t)}.$$

Setting $\beta > 0$ and defining the neighborhood of radius β around Q, we have

$$U_\beta = \left\{ u \in H^1(\mathrm{R}); \inf_{r\in\mathrm{R}} \|u(\cdot) - Q(\cdot - r)\|_{L^2} \le \beta \right\}.$$

Now, we state our main results.

Theorem 1 *Let $u_0 \in H^3(R)$ and $u_0 = Q + \varepsilon_0$. Assume there exists at least one point $x_0 \in R$, such that $\frac{\sqrt{6}}{3}u_0'(x_0) < -2 \parallel Q + \varepsilon_0 \parallel_{L^2}$, then the corresponding solution to equation (1.5) blows up in finite time.*

Theorem 2 *Blow-up of the solution $u(t,\cdot)$ in finite time occurs if and only if blow-up of the residual term $\varepsilon(t,\cdot)$ in finite time.*

The paper is organized as follows. Section 2 gives a sufficient condition of blow-up of solution $u(t)$. Section 3 is devoted to discussing the relationship of blow-up of solution u and blow-up of residual term ε.

2 The blow-up of solution u

Let

$$v(t,y) = \lambda^{1/2}(t)u(t, y + x(t)), \tag{2.1}$$

where u is a solution of (1.5), and $\lambda(t) > 0, x(t) \in \mathrm{R}$. From (1.6), we have

$$\varepsilon(t,y) = v(t,y) - Q(y) = \lambda^{1/2}(t)u(t, y + x(t)) - Q(y). \tag{2.2}$$

Lemma 3 *([12],Theorem 2.1) Let $T > 0$, and $u(t,\cdot) \in C^1([0,T); H^2(R))$. Then for every $t \in [0,T)$ there exists at least one point $\theta(t) \in R$ with*

$$J(t) = \inf_{x\in R}\{u_x(t,x)\} = u_x(t,\theta(t)),$$

and the function J is almost everywhere differentiable on $(0,T)$ with

$$\frac{dJ}{dt}(t) = u_{tx}(t,\theta(t)) \quad a.e.on \quad (0.T).$$

Lemma 4 *Suppose $J(0) < 0, 0 < \mu_1, \mu_2 < 1$, and $J(t)$ satisfies*

$$\frac{dJ}{dt}(t) \le -J^2(t) + \mu_1(1-\mu_2)J^2(0), \quad t \in [0,T), T > 0,$$

then

$$J^2(t) > (1-\mu_2)J^2(0), \quad t \in [0,T).$$

Proof: By contradiction, we assume there exists one point $t_0 \in [0,T)$, such that $J^2(t) > (1-\mu_2)J^2(0)$ for all $t \in [0,t_0)$, but $J^2(t_0) \le (1-\mu_2)J^2(0)$. Then, we obtain

$$\frac{dJ}{dt}(t) \le -J^2(t) + \mu_1(1-\mu_2)J^2(0) \le -(1-\mu_1)J^2(t), \quad t \in [0,t_0).$$

(i) If $dJ/dt \equiv 0$, performing integral over $(0,t_0)$, we have $J(t_0) \equiv J(0)$, which contradicts with the $J^2(t_0) \le (1-\mu_2)J^2(0)$.

(ii) If $dJ/dt < 0$, then $J(t)$ is monotone decreasing on $[0,t_0)$, and $J(t_0) \le J(0) < 0$. Hence we derive $J^2(t_0) \ge J^2(0)$, which contradicts with $J^2(t_0) \le (1-\mu_2)J^2(0)$.

Combining (i) and (ii), we have $J^2(t) > (1-\mu_2)J^2(0), t \in [0,T)$. The proof of this Lemma is finished.

Lemma 5 *([3], Theorem 2.3) Suppose $u_0 \in H^s, s > \frac{3}{2}$, there exists a maximal $T = T(u_0) > 0$ and a unique solution u to the equation (1.1) (or equation(1.4)) such that*

$$u = u(\cdot,u_0) \in C([0,T); H^s(\mathrm{R})) \cap C^1([0,T); H^{s-1}(\mathrm{R})).$$

Proof of Theorem 1: From Lemma 5, we get $u(t,x) \in C^1([0,T); H^2(R))$. Differentiating (1.4) with respect to the spatial variable x, we obtain

$$u_{tx} + u_x^2 + uu_{xx} - \frac{3}{2}\int_R P_{zx}(x-z)u^2(z)dz = 0, \tag{2.3}$$

by the Sobolev embedding theorem, we conclude that $u(t) \in H^3(R) \in H^2(R)$. From the definition of $\theta(t)$ (Lemma3), we derive $\partial_x^2 u(t,x)\mid_{x=\theta(t)}= 0$. Substituting $x = \theta(t)$ into equation (2.3), we obtain

$$\frac{dJ}{dt} + J^2 = \frac{3}{2}\int_R P_{zx}(x-z)u^2(z)dz.$$

Utilizing the Hölder inequality and $\|u\|_{L^2}^2 \le \|u_0\|_{L^2}^2$ (Lemma 3.1 in [4]), we have

$$\frac{3}{2}\int_R P_{zx}(x-z)u^2(z)dz \le \frac{3}{4}\|u\|_{L^2}^2 \le 3\|u_0\|_{L^2}^2.$$

Therefore,

$$\frac{dJ}{dt} \le -J^2 + \frac{3}{4}\|u\|_{L^2}^2 \le -J^2 + 3\|u_0\|_{L^2}^2. \tag{2.4}$$

As $\frac{\sqrt{6}}{3}u_0'(x_0) < -2\|Q+\varepsilon_0\|_{L^2}$, there exists $\eta \in (0,1)$ such that $(1-\eta)(u_0'(x_0))^2 \ge 6\|u_0\|_{L^2}^2$. Noticing that $J(0) = \inf_{x\in R} u_x(0,x) \le u_0'(x_0) < 0$, we derive

$$(1-\eta)J^2(0) \ge (1-\eta)(u_0'(x_0))^2 \ge 6\|u_0\|_{L^2}^2. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\frac{dJ}{dt} \leq -J^2 + \frac{1}{2}(1-\eta)(u_0'(x_0))^2 \leq -J^2 + \frac{1}{2}(1-\eta)J^2(0).$$

By Lemma 4, we deduce $J^2(t) > (1-\eta)J^2(0), t \in (0,T)$, then

$$\frac{dJ}{dt} \leq -J^2 + \frac{1}{2}(1-\eta)J^2(0) \leq -\frac{1}{2}J^2(t), \quad a.e. \quad t \in (0,T_1).$$

Direct calculations show that

$$\frac{1}{J(t)} \geq \frac{t}{2} + \frac{1}{J(0)}, \quad 0 \leq t \leq T_1.$$

Hence, $J(t) \to -\infty$, as $t \to T_1 = 2/|J(0)| = 2/|\inf_{x\in R} u_x(0,x)|$. It shows that the solution of (1.5) blows up in finite time. The proof is completed.

3 An equivalent proposition of blow-up of solution u

This section studies the relationship of blow-up of solution u and blow-up of residual term ε, and completes the proof of Theorem 2.

For our purpose, we first present the following Lemma.

Lemma 6 *For all $s \geq 0$, there is*

$$\varepsilon_s = \frac{1}{2}\frac{\lambda_s}{\lambda}Q + (x_s - 1)Q_y + \frac{1}{2}\frac{\lambda_s}{\lambda}\varepsilon + x_s\varepsilon_y - \varepsilon\varepsilon_y - (Q\varepsilon)_y + R(\varepsilon), \tag{3.1}$$

where

$$R(\varepsilon) = \frac{3}{2}\int_R P_z\,(y-z)\,\varepsilon^2(z)dz + 3\int_R P_z\,(y-z)\,\varepsilon(z)Q(z)dz. \tag{3.2}$$

Proof: A direct computation shows that

$$v_t = \frac{1}{2}\lambda^{-1/2}\lambda_t u + \lambda^{1/2}u_t + \lambda^{1/2}x_t u_y, \quad v_y = \lambda^{1/2}u_y, \quad v_{yy} = \lambda^{1/2}u_{yy}.$$

Then, we obtain

$$\lambda^{1/2}v_t + vv_y - \frac{3}{2}\int_R P_z(y-z)v^2(z)dz = \frac{1}{2}\lambda^{-1/2}\lambda_t v + \lambda^{1/2}x_t v_y.$$

Since $ds/dt = 1/\lambda^{1/2}(t)$, then

$$v_s + vv_y - \frac{1}{2}\frac{\lambda_s}{\lambda}v - x_s v_y = \frac{3}{2}\int_R P_z(y-z)v^2(z)dz.$$

Noticing that $v(s,y) + Q(y) + \varepsilon(s,y)$ and $\frac{3}{2}\int_R P_z(y-z)Q^2(z)dz = Q_y(Q-1)$, we derive

$$\varepsilon_s = \frac{1}{2}\frac{\lambda_s}{\lambda}Q + (x_s - 1)Q_y + \frac{1}{2}\frac{\lambda_s}{\lambda}\varepsilon + x_s\varepsilon_y - \varepsilon\varepsilon_y - (Q\varepsilon)_y + R(\varepsilon),$$

where

$$R(\varepsilon) = \frac{3}{2}\int_R P_z\,(y-z)\,\varepsilon^2(z)dz + 3\int_R P_z\,(y-z)\,\varepsilon(z)Q(z)dz.$$

The claim of this Lemma is shown.

Select $\lambda(s) > 0$ and $x(s) \in \mathrm{R}$ such that $\varepsilon(s,y) \perp Q(y)$ and $\varepsilon(s,y) \perp Q_y(y)$, for all $s \geq 0$. That is,

$$\int_R \varepsilon Q dy = 0, \quad \int_R \varepsilon Q_y dy = 0.$$

The Q and Q_y are linearly independent. Assume that there exists a constant $\mu^* \neq 0$ such that $Q = \mu^* Q_y$, multiplying the equation by Q and performing integral over R, we get $0 \neq \int_R Q^2 dy = \mu^* \int_R QQ_y dy = 0$.

Supposing that $u(t) \in H^1(\mathrm{R})$, $\lambda_1 > 0$ and $x_1 \in \mathrm{R}$, we denote

$$\varepsilon_{\lambda_1,x_1}(y) = {\lambda_1}^{1/2} u(y + x_1) - Q(y). \tag{3.3}$$

Lemma 7 *There exists $\hat{\alpha} > 0$, $\bar{\lambda} > 0$, and a unique C^1-map $(\lambda_1, x_1) : U_{\hat{\alpha}} \to (1 - \bar{\lambda}; 1 + \bar{\lambda})$, such that if $u(t) \in U_{\hat{\alpha}}$, and $\varepsilon_{\lambda_1,x_1}$ is denoted by (3.3), then*

$$\varepsilon_{\lambda_1,x_1} \perp Q \quad and \quad \varepsilon_{\lambda_1,x_1} \perp Q_y. \tag{3.4}$$

Moreover, there exists a constant $C_1 > 0$, if $u(t) \in U_\alpha$, with $0 < \alpha < \hat{\alpha}$, then

$$\|\varepsilon_{\lambda_1,x_1}\|_{L^2} \leq C_1 \alpha \quad and \quad |\lambda_1 - 1| \leq C_1 \alpha, \tag{3.5}$$

where $U_\alpha = \left\{ u \in H^1(\mathrm{R}); \inf_{r \in \mathrm{R}} \|u(\cdot) - Q(\cdot - r)\|_{L^2} \leq \alpha \right\}$.

Proof: Define the functionals:

$$\rho^1_{\lambda_1,x_1}(u) = \int_R \varepsilon_{\lambda_1,x_1} Q dy\ , \quad \rho^2_{\lambda_1,x_1}(u) = \int_R \varepsilon_{\lambda_1,x_1} Q_y dy.$$

Since

$$\left.\frac{\partial \varepsilon_{\lambda_1,x_1}}{\partial \lambda_1}\right|_{\lambda_1=1, x_1=0} = u_y \quad \text{and} \quad \left.\frac{\partial \varepsilon_{\lambda_1,x_1}}{\partial x_1}\right|_{\lambda_1=1,x_1=0} = \frac{1}{2}u,$$

the Jacobian determinant

$$\left.\frac{\partial(\rho^1_{\lambda_1,x_1}, \rho^2_{\lambda_1,x_1})}{\partial(\lambda_1, x_1)}\right|_{\lambda_1=1,x_1=0,u=Q} = \left.\left|\begin{array}{cc} \frac{1}{2}\int_R Q^2 dy & \int_R QQ_y dy \\ \frac{1}{2}\int_R QQ_y dy & \int_R Q_y^2 dy \end{array}\right|\right|_{\lambda_1=1,x_1=0,u=Q} = \frac{1}{2}\int_R Q^2 dy \int_R Q_y^2 \neq 0.$$

According to the implicit function theorem, there exists $\overline{\alpha} > 0$, a neighborhood $\Omega_{1,0}$ of (1,0) in $\mathrm{R}^+ \times \mathrm{R}$, and a unique C^1-map $(\lambda_1, x_1) : \left\{ u \in H^1(\mathrm{R}); \|u - Q\|_{L^2} < \overline{\alpha} \right\} \to \Omega_{1,0}$, such that

$$\int_R \varepsilon_{\lambda_1,x_1} Q dy = \int_R \varepsilon_{\lambda_1,x_1} Q_y dy = 0,$$

then (3.4) holds. Furthermore, if $\|u - Q\|_{L^2} < \alpha \leq \overline{\alpha}$, then U_α determines a closed region, from the properties of continuous mappings, there exists a constant $C_1 > 0$ such that $|\lambda_1 - 1| \leq C_1\alpha$. Since

$$\|\varepsilon_{\lambda_1,x_1}\|_{L^2} = \|\lambda_1^{1/2} u(x) - Q(x - x_1)\|_{L^2} \leq |\lambda_1^{1/2} - 1| \|u\|_{L^2} + \|u(x) - Q(x - x_1)\|_{L^2},$$

we have $\|\varepsilon_{\lambda_1,x_1}\|_{L^2} \leq C_1\alpha$, for some $C_1 > 0$. Restrict the map (λ_1, x_1) to the tube U_α. Applying the implicit function theorem again, there exists $\hat{\alpha} < \bar{\alpha}$, and a unique C^1-map $r : U_{\hat{\alpha}} \to \mathrm{R}$, such that if $u \in U_{\hat{\alpha}}$, then

$$\|u(\cdot) - Q(\cdot - r)\|_{L^2} = \inf_h \|u(\cdot) - Q(\cdot - h)\|_{L^2} < \hat{\alpha} < \bar{\alpha}.$$

Denote $\lambda_1(u) = \lambda_1(u(\cdot - r(u)))$ and $x_1(u) = x_1(u(\cdot - r(u))) + r(u)$ which satisfy (3.4) and (3.5). The proof is finished.

Lemma 8 *For all $\hat{\alpha} > 0$, there exists $\alpha_0 > 0$, such that if $u_0 \in U_{\alpha_0} \cap H^2(\mathrm{R})$, then $u(t) \in U_{\hat{\alpha}}$, where $t \geq 0$, and u_0 is the initial value of $u(t)$.*

Proof: By contradiction, we assume $u_0 \notin U_{\alpha_0} \cap H^2(\mathrm{R})$ for any $\alpha_0 > 0$, but the $u(t) \in U_{\hat{\alpha}}$, $u(t)$ is a solution of (1.5) corresponding to initial value $u_0(x)$. It follow from Lemma 5 that $u(t) \in C([0, T), H^2(\mathrm{R}))$. We obtain

$$\|u(\cdot) - Q(\cdot - r)\|^2_{L^2} = \int_R u^2(x)dx + \int_R Q^2(x - r)dx - 2\int_R u(x)Q(x - r)dx.$$

From the conservation law (1.2), we have $\|u\|_{L^2}^2 \sim \|u_0\|_{L^2}^2$, then there exist constants $\vartheta_1 > 0$ and $\vartheta_2 > 0$ such that $\vartheta_1\|u_0\|_{L^2}^2 \le \|u\|_{L^2}^2 \le \vartheta_2\|u_0\|_{L^2}^2$. By the Hölder inequality, we have

$$\begin{aligned}\hat{\alpha}^2 \ge \inf_{r\in \mathrm{R}} \|u(\cdot) - Q(\cdot - r)\|_{L^2}^2 >&\vartheta_1 \inf_{r\in \mathrm{R}} \|u_0 - Q(\cdot - r)\|_{L^2}^2 + (1-\vartheta_1)\inf_{r\in \mathrm{R}} \|Q(\cdot - r)\|_{L^2}^2 \\ &- 2\sqrt{\vartheta_2}\|u_0\|_{L^2} \inf_{r\in \mathrm{R}} \|Q(\cdot - r)\|_{L^2} \\ >&\vartheta_1\alpha_0^2 + (1-\vartheta_1)\inf_{r\in \mathrm{R}} \|Q(\cdot - r)\|_{L^2}^2 - 2\sqrt{\vartheta_2}\|u_0\|_{L^2} \inf_{r\in \mathrm{R}} \|Q(\cdot - r)\|_{L^2} \\ =&\vartheta_1\alpha_0^2 + 1 - \vartheta_1 - 2\sqrt{\vartheta_2}\|u_0\|_{L^2}.\end{aligned} \tag{3.5}$$

In (3.5), taking $\alpha_0^2 = \frac{\hat{\alpha}^2 + \vartheta_1 + 2\sqrt{\vartheta_2}\|u_0\|_{L^2}}{\vartheta_1}$, we have

$$\hat{\alpha}^2 > \vartheta_1\alpha_0^2 + 1 - \vartheta_1 - 2\sqrt{\vartheta_2}\|u_0\|_{L^2} = \hat{\alpha}^2 + 1.$$

A contradiction appears. Then there exist some $\alpha_0 > 0$, such that if $u_0 \in U_{\alpha_0} \cap H^2(\mathrm{R})$, then $u(t) \in U_{\hat{\alpha}}$. The proof is completed.

Support $u(t,x) \in U_{\hat{\alpha}}$, for all $t \ge 0$. We define λ and x as follows: define $\varepsilon(s,y) = \varepsilon_{\lambda(s),x(s)}$, which satisfies $(\varepsilon(s,y), Q(y)) = (\varepsilon(s,y), Q_y(y)) = 0$, for all $s \ge 0$ and some $\lambda(s) > 0$, $x(s) \in \mathrm{R}$.

Lemma 9 *Assume $u_0 \in U_{\alpha_0} \cap H^2(\mathrm{R})$, then $\lambda(s)$, $x(s) \in C^1(\mathrm{R}^+)$. Furthermore,*

$$\frac{1}{2}\frac{\lambda_s}{\lambda}\int_R Q^2 dy = -\frac{1}{2}\int_R \varepsilon^2 Q_y dy - \int_R \varepsilon Q Q_y dy - \int_R R(\varepsilon) Q dy,$$

where $R(\varepsilon)$ is given by (3.2).

Proof: Let $\chi(x)$ be a $C(\mathrm{R})-$function such that

$$|\chi(x)| \le C\mathrm{e}^{-\frac{|x|}{2}}, \forall x \in \mathrm{R}.$$

From Lemma 5, Lemma 7 and Lemma 8, we obtain $u(t,x) \in C([0,T]; H^2(\mathrm{R}))$ and $\lambda(t), x(t) \in C^1(\mathrm{R}^+)$. By (1.3), we have

$$\begin{aligned}\frac{d}{ds}\int_R \chi v(s) dy =&\frac{d}{ds}\left(\lambda^{1/2}\int_R \chi u(s, y + x(s)x) dy\right) \\ =&\frac{1}{2}\frac{\lambda_s}{\lambda}\int_R \chi v dy + x_s\int_R \chi v_y dy - \int_R \chi v v_y dy - \int_R \chi\left(P * \left(\frac{3}{2}v^2\right)_y\right) dy.\end{aligned}$$

Noticing $v(s,y) = Q(y) + \varepsilon(s,y)$, then

$$\begin{aligned}\frac{d}{ds}\int_R \chi\varepsilon(s) dy =&\frac{1}{2}\frac{\lambda_s}{\lambda}\int_R \chi(Q+\varepsilon) dy + x_s\int_R \chi(Q_y + \varepsilon_y) dy - \int_R \chi(Q\varepsilon)_y dy \\ &- \int_R \chi\varepsilon\varepsilon_y dy - \int_R \chi Q Q_y dy + \int_R \chi R(\varepsilon) dy.\end{aligned}$$

Let $\chi = Q$. Since $(\varepsilon(s), Q) = \int_R Q\varepsilon dy = 0$, we drive

$$\frac{1}{2}\frac{\lambda_s}{\lambda}\int_R Q^2 dy = -\frac{1}{2}\int_R \varepsilon^2 Q_y dy - \int_R \varepsilon Q Q_y dy - \int_R R(\varepsilon) Q dy,$$

where $R(\varepsilon)$ is defined in (3.2). The proof is finished.

Lemma 10 *Suppose $u_0 \in H^2(\mathrm{R})$, with $u_0 = Q + \varepsilon_0$, then there exist constants $C_3 > 0, C_4 > 0, C_5 > 0$, such that*

$$\|\varepsilon\|_{L^2}^2 \le C_3\lambda\|\varepsilon_0\|_{L^2}^2 + C_4(\lambda - 1) + C_5\lambda\|\varepsilon_0\|_{L^2}, \tag{3.6}$$

for all $s \ge 0$.

Proof: By the conservation law (1.2) follows

$$E(u(t)) = \int_R ymdx = \int_R \frac{1+\xi^2}{4+\xi^2}|\hat{u}(\xi)|^2 d\xi = E(u_0).$$

Since $E(Q+\varepsilon(s)) = E(v(s)) = \lambda(s)E(u_0)$ and $\int_R \frac{1+\xi^2}{4+\xi^2}|\hat{Q}|^2 d\xi = \frac{1}{3}$, we obtain

$$\begin{aligned}\int_R \frac{1+\xi^2}{4+\xi^2}|\hat{\varepsilon}|^2 d\xi =& \lambda \int_R \frac{1+\xi^2}{4+\xi^2}|\hat{\varepsilon_0}|^2 d\xi + \frac{1}{3}(\lambda-1) \\ &+ 2\left(\lambda \int_R \frac{1+\xi^2}{4+\xi^2}|\hat{Q}\hat{\varepsilon_0}| d\xi - \int_R \frac{1+\xi^2}{4+\xi^2}|\hat{Q}\hat{\varepsilon}| d\xi\right).\end{aligned} \tag{3.7}$$

We derive

$$\int_R \frac{1+\xi^2}{4+\xi^2}|\hat{Q}\hat{\varepsilon}| d\xi \le \frac{1}{4}\left(\int_R |\hat{Q}|^2 d\xi\right)^{\frac{1}{2}}\left(\int_R |\hat{\varepsilon}|^2 d\xi\right)^{\frac{1}{2}} = \frac{1}{4}\|\hat{\varepsilon}\|_{L^2}. \tag{3.8}$$

Recalling $\int_R \frac{1+\xi^2}{4+\xi^2}|\hat{\varepsilon}(\xi)|^2 d\xi \backsim \|\varepsilon\|_{L^2}^2$ and $\int_R \frac{1+\xi^2}{4+\xi^2}|\hat{\varepsilon_0}(\xi)|^2 d\xi \backsim \|\varepsilon_0\|_{L^2}^2$, we deduce by adding (3.7) and (3.8) that there exist constants $C_3 > 0, C_4 > 0, C_5 > 0$, such that

$$\|\varepsilon\|_{L^2}^2 \le C_3\lambda\|\varepsilon_0\|_{L^2}^2 + C_4(\lambda - 1) + C_5\lambda\|\varepsilon_0\|_{L^2}, \quad s \ge 0.$$

The proof is finished.

Lemma 11 *Suppose $u_0 \in U_{\alpha_0} \cap H^2(\mathrm{R})$, with $u_0 = Q + \varepsilon_0$, then there exists a constant $C_6 > 0$, such that*

$$\|\varepsilon(s)\|_{L^2} \le C_6\|\varepsilon_0\|_{L^2}^{1/2},$$

for all $s \ge 0$.

Proof: Since $u_0 \in U_{\alpha_0} \cap H^2(\mathrm{R})$, it follows from Lemma 8 and Lemma 7 that there exists $0 < \alpha < \hat{\alpha}$, such that $\|\varepsilon(s)\|_{L^2} + |\lambda(s) - 1| \le C_1\alpha$. Thus there exists a constant $C_7 > 0$, such that $|\lambda(s) - 1| \le C_7 \|\varepsilon(s)\|_{L^2}$, for all $s \ge 0$. According to (3.6), we obtain

$$\|\varepsilon(s)\|_{L^2}^2 \le b_1 \|\varepsilon_0\|_{L^2} + b_2 \|\varepsilon(s)\|_{L^2}, \quad s \ge 0, \tag{3.9}$$

where $b_1 = C_3\|\varepsilon_0\|_{L^2} + C_5, \quad b_2 = C_3C_7\|\varepsilon_0\|_{L^2}^2 + C_4C_7 + C_5C_7\|\varepsilon_0\|_{L^2}$.

By contradiction. we assume there is a $s_0 > 0$, such that $\|\varepsilon(s_0)\|_{L^2} \ne 0$, and for any $0 < C_0 < +\infty$, we have

$$\|\varepsilon(s_0)\|_{L^2} > C_0\|\varepsilon_0\|_{L^2}^{1/2}.$$

By (3.9), we obtain

$$\|\varepsilon(s)\|_{L^2} \le \frac{b_2 + \sqrt{b_2^2 + 4b_1\|\varepsilon_0\|_{L^2}}}{2}, \quad s \ge 0.$$

Due to the arbitrariness of the constant C_0, taking

$$C_0 = \frac{b_2 + \sqrt{b_2^2 + 4b_1\|\varepsilon_0\|_{L^2}}}{2\|\varepsilon_0\|_{L^2}^{1/2}} + 1.$$

Therefore,

$$\frac{b_2 + \sqrt{b_2^2 + 4b_1\|\varepsilon_0\|_{L^2}}}{2} + \|\varepsilon_0\|_{L^2}^{1/2} < \|\varepsilon(s_0)\|_{L^2} \le \frac{b_2 + \sqrt{b_2^2 + 4b_1\|\varepsilon_0\|_{L^2}}}{2}.$$

A contradiction appears. Then there exists a constant $C_6 > 0$, such that

$$\|\varepsilon(s)\|_{L^2} \le C_6\|\varepsilon_0\|_{L^2}^{1/2}, \quad s \ge 0.$$

The proof is completed.

Proof of Theorem 2: Assume the solution of Cauchy problem (1.5) blows up in finite time, that is there exists $0 < T_1 < +\infty$ such that $\lim\limits_{t\to T_1}\inf\limits_{x\in R}\{u_x(t,x)\} = -\infty$.

By contradiction. we assume the solution of Cauchy problem for equation (3.1) with initial value $\varepsilon(0,y) = \varepsilon_0(y)$ not blows up, hence $|\varepsilon_y(t,\cdot)| < +\infty$ for all $t > 0$. Particularly, by (2.2), we have

$$\lim_{t\to T_1}\inf_{x\in R}\{\varepsilon_y(t,y)\} > -\infty.$$

On the other hand,

$$\begin{aligned}\inf_{x\in R}\{u_x(t,x)\} &= \inf_{y\in R}\{\lambda^{-1/2}(t)\varepsilon_x(t,y) + \lambda^{-1/2}(t)Q_y(y)\}\\ &\geq \inf_{t>0}\{\lambda^{-1/2}(t)\}\inf_{y\in R}\varepsilon_x(t,y) + \inf_{t>0}\{\lambda^{-1/2}(t)\}\inf_{y\in R}\{Q_y(y)\},\end{aligned}$$

where $y = x - x(t)$.

From (3.5), we have $|\lambda - 1| \leq C_1\alpha$. Taking $0 < C_1\alpha < 1$, then $\frac{1}{\sqrt{1+C_1\alpha}} \leq \lambda_{\inf}^{-1/2} \leq \frac{1}{\sqrt{1-C_1\alpha}}$, here $\lambda_{\inf}^{-1/2} = \inf\limits_{t>0}\{\lambda^{-1/2}(t)\}$. Let $t \to T_1$, it follows from the decay properties of Q_y that

$$-\infty < \lim_{t\to T_1}\inf_{y\in R}\varepsilon_x(t,y) \leq \lim_{t\to T_1}\inf_{y\in R}\{u_x(t,x)\} + \lambda_{\inf}^{-1/2} = -\infty.$$

That yields a contradiction, hence $\varepsilon(t,\cdot)$ blows up in finite time.

On the contrary. Assume the solution of Cauchy problem for equation (3.1) with initial value $\varepsilon_0(y)$ blows up in finite time, it shows that there exists $0 < T_2 < +\infty$, such that $\lim\limits_{t\to T_2}\inf\limits_{y\in R}\{\varepsilon_y(t,y)\} = -\infty$.

By contradiction. we assume the solution of Cauchy problem (1.5) not blows up, that is the $|u_x(t,\cdot)| < +\infty$ for all $t > 0$. According to (2.2), we have

$$\lim_{t\to T_2}\inf_{x=y+x(t)\in R}\{u_y(t,x)\} > -\infty.$$

On the other hand,

$$\begin{aligned}\inf_{x\in R}\{\varepsilon_y(t,y)\} &= \inf_{y,x\in R}\{\lambda^{1/2}(t)u_y(t,x) - Q_y(y)\}\\ &\geq \inf_{t>0}\{\lambda^{1/2}(t)\}\inf_{x\in R}u_y(t,x) + \inf_{y\in R}\{-Q_y(y)\}.\end{aligned}$$

Let $t \to T_2$, recalling the decay properties of Q_y, we deduce by adding $\sqrt{1-C_1\alpha} \leq \lambda_{\inf}^{1/2} \leq \sqrt{1+C_1\alpha}$ that

$$-\infty < \lim_{t\to T_2}\inf_{x\in R}u_y(t,x) \leq \lim_{t\to T_2}\inf_{y\in R}\{\varepsilon_y(t,y)\} + 1 = -\infty.$$

A contradiction appears, hence $u(t,\cdot)$ blows up in finite time. The proof is completed.

4 Acknowledgements

This work was supported by National Natural Science Foundation of China (No. 11371175).

References

[1] A. Degasperis and M. Procesi. Asymptotic Integrability. *Symmetry and Perturbation Theory*, 1999, 23-27.

[2] A. Degasperis, D. D. Holm and A. N. W. Hone. A New Integrable Equation with Peakon Solutions. *Theoretical and Mathematical Physics*, 2002, 133(2): 1463-1474.

[3] Z. Yin. On the cauchy problem for an integrable equation with peakon solutions. *Illinois Journal of Mathematics*, 2003, 47: 649-666.

[4] Y. Liu, Z. Yin. Global Existence and Blow-Up Phenomena for the Degasperis-Procesi Equation. *Communications in Mathematical Physics*, 2006, 267(3): 801-820.

[5] Y. Matsuno. Multisolition solutions of the Degasperis-Procesi equation and their peakon limit. *Invere Problems*, 2005, 21: 1553-1570.

[6] G. M. Coclite, K. H. Karlsen. On the well-posedness of the DegasperisC Procesi equation. *Journal of Functional Analysis*, 2006, 233(1): 60-91.

[7] Y. Liu, Z. Yin. On the Blow-up Phenomena for the Degasperis-Procesi Equation. *International Mathematics Research Notices*,2015, 2007(23).

[8] Z. Lin, Y. Liu. Stability of Peakons for the Degasperis-Procesi Equation. *Communications on Pure and Applied Mathematics*,2010, 62(1): 125-146.

[9] A. Kabakouala. Stability in the energy space of the sum of N peakons for the Degasperis-Procesi equation. *Journal of Differential Equations*, 2015, 259(5): 1841-1897.

[10] Y. Martel, F. Merle. Instability of solitons for the critical generalized KortewegCde Vries equation. *Geometric Functional Analysis Gafa*, 2001, 11(1): 74-123.

[11] Y. Martel, F. Merle, P. Raphaël. Blow up for the critical gKdV equation I:dynamics near the soliton. *Acta Mathematica*, 2014, 212: 59-140.

[12] A. Constantin, J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Mathematica*, 1998, 181(2): 229-243.