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Traveling Wave Solutions of the Fornberg-Whitham Equation with Dispersion Perturbation Term

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Abstract: In this paper, we study the traveling wave solutions of the Fornberg-Whitham equation with dispersion perturbation term by the bifurcation method of dynamical systems. The solitons, peakons and periodic wave solutions for the Fornberg-Whitham equation with dispersion perturbation term are obtained.

Keywords: Fornberg-Whitham equation; Dispersion term; Traveling wave solution.

1 Introduction

The Fornberg-Whitham (FW) equation is

$$u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}uu_{xxx} - \frac{3}{2}uu_x + u_x = 0, \tag{1.1}$$

where $(x, t) \in R \times R^+$ and $u = u(x, t)$ represents the fluids free surface above a flat bottom or equivalently, the fluid velocity at time $t > 0$ in the spatial x direction. In 1978, Fornberg and Whitham [1] introduced the FW equation to study qualitative behaviors of wave-breaking, and obtained a peaked solution of the form

$$u(x, t) = \frac{8}{9}e^{-\frac{1}{2}|x-\frac{4}{3}t|}.$$

Recently, Zhou and Tian [2,3] investigated the exact traveling wave solutions for the FW equation by the bifurcation method of dynamical systems. Chen et al. obtained the smooth traveling solutions of the FW equation, for details refer to [4] and references therein.

The goal of this paper is to study the traveling wave solutions of the Fornberg-Whitham equation with dispersion perturbation term

$$u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}uu_{xxx} - \frac{3}{2}uu_x + u_x + \sigma u_{xxx} = 0, \tag{1.2}$$

where u_{xxx} is the dispersion term and $\sigma > 0$ is the perturbation factor.

The remainder of the paper is organized as follows. In Section 2, we discuss the bifurcation curves and equilibrium point of traveling wave system. In Section 3, we obtain the implicit expression for solitons and the explicit expressions for peakons and periodic cusp wave solutions.

2 Bifurcation and equilibrium point of traveling wave system

In this section, we discuss the traveling wave solutions of Eq.(1.2) by using the theory of dynamical systems. Let $u(x, t) = -\frac{2}{3}\varphi(\xi)$ with $\xi = x + ct$ be the solution of Eq.(1.2), then we have

$$c\varphi''' - c\varphi' - 3\varphi'\varphi'' - \varphi\varphi''' + \varphi\varphi' + \varphi' + \sigma\varphi''' = 0. \tag{2.1}$$

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Integrating Eq. (2.1) once we have

$$\varphi''(\varphi - c - \sigma) = g - c\varphi + \varphi + \frac{1}{2}\varphi^2 - (\varphi')^2, \tag{2.2}$$

where g is the integral constant.

Let $y = \varphi'$, then we get the following planar dynamical system:

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{g - c\varphi + \varphi + \frac{1}{2}\varphi^2 - y^2}{\varphi - c - \sigma}, \end{cases} \tag{2.3}$$

with a first integral

$$H(\varphi, y) = (\varphi - c - \sigma)^2 (y^2 - \frac{1}{4}\varphi^2 + (\frac{1}{2}c - \frac{1}{6}\sigma - \frac{2}{3})\varphi + \frac{1}{2}(c + \sigma)(\frac{1}{2}c - \frac{1}{6}\sigma - \frac{2}{3}) - g) = h, \tag{2.4}$$

where h is a constant. Such a systems is called a singular traveling wave system, because there is a singular line $\varphi = c + \sigma$. To avoid the line temporarily, we make transformation $d\xi = (\varphi - c - \sigma)d\eta$. Under this transformation, system (2.3) becomes

$$\begin{cases} \frac{d\varphi}{d\eta} = (\varphi - c - \sigma)y, \\ \frac{dy}{d\eta} = g - c\varphi + \varphi + \frac{1}{2}\varphi^2 - y^2. \end{cases} \tag{2.5}$$

Systems (2.3) and (2.5) have the same first integral as (2.4). Consequently, system (2.5) has the same topological phase portraits as system (2.3) except for the straight line $\varphi = c + \sigma$. Obviously, $\varphi = c + \sigma$ is an invariant straight-line solution for system (2.5).

From the first integral and system (2.5), we obtain three bifurcation curves as follows:

$$g_1(c) = \frac{1}{2}(c - 1)^2, \tag{2.6}$$

$$g_2(c) = \frac{1}{2}(c - 1)^2 - \frac{1}{18}(\sigma + 1)^2, \tag{2.7}$$

$$g_3(c) = \frac{1}{2}(c - 1)^2 - \frac{1}{2}(\sigma + 1)^2. \tag{2.8}$$

Obviously, the three curves have no intersection point and $g_3(c) < g_2(c) < g_1(c)$ for arbitrary constant c.

Let $M(\varphi_e, y_e)$ be the coefficient matrix of the linearized version of (2.5) at the equilibrium point (φ_e, y_e), and define $J = detM(\varphi_e, y_e)$, then

$$M(\varphi_e, y_e) = \begin{pmatrix} y_e & \varphi_e - c - \sigma \\ \varphi_e - c + 1 & -2y_e \end{pmatrix}, \tag{2.9}$$

$$J = detM(\varphi_e, y_e) = -2y_e^2 - (\varphi_e - c - \sigma)(\varphi_e - c + 1). \tag{2.10}$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar dynamical system: (1) if $J < 0$, then this equilibrium point is a saddle point; (2) if $J > 0$, then this equilibrium point is a center point; (3) if $J = 0$ and the Poincare index of the equilibrium point is 0, then the equilibrium point is a cusp.

Using the above qualitative analysis, we have the following results:

Theorem 1 *For given any constant wave speed $c \neq 0$, let*

$$\varphi_{1\pm} = c - 1 \pm \Delta, \quad for \quad g \leq g_1(c), \tag{2.11}$$

$$y_{1\pm} = \pm\sqrt{g - \frac{1}{2}(c - 1)^2 + \frac{1}{2}(\sigma - 1)^2}, \quad for \quad g \geq g_3(c). \tag{2.12}$$

where $\Delta = \sqrt{(c - 1)^2 - 2g}$, then for system (2.5), we have

(1) for $g < g_3(c)$, there are two equilibrium points $(\varphi_{1-}, 0)$ and $(\varphi_{1+}, 0)$, which are saddle points.

(2) for $g = g_3(c)$, there are two equilibrium points $(c + \sigma, 0)$ and $(c - \sigma - 2, 0)$. $(c - \sigma - 2, 0)$ is a saddle point and $(c + \sigma, 0)$ is a cusp.

(3) for $g_3(c) < g < g_2(c)$, there are four equilibrium points $(\varphi_{1-}, 0)$, $(\varphi_{1+}, 0)$, $(c+\sigma, y_{1-})$ and $(c+\sigma, y_{1+})$. $(\varphi_{1-}, 0)$ is a saddle point and $(\varphi_{1+}, 0)$ is a center point enclosing the orbit which connects the saddle points $(c + \sigma, y_{1-})$ and $(c + \sigma, y_{1+})$.

(4) for $g = g_2(c)$, there are four equilibrium points $(c-\frac{4}{3}-\frac{1}{3}\sigma, 0)$, $(c-\frac{2}{3}+\frac{1}{3}\sigma, 0)$, $(c+\sigma, -\frac{2}{3}-\frac{2}{3}\sigma)$ and $(c+\sigma, \frac{2}{3}+\frac{2}{3}\sigma)$, which satisfy $H(c - \frac{4}{3} - \frac{1}{3}\sigma, 0) = H(c + \sigma, -\frac{2}{3} - \frac{2}{3}\sigma) = H(c + \sigma, \frac{2}{3} + \frac{2}{3}\sigma) = 0$ and form a triangular orbit which encloses the center point $(c - \frac{2}{3} + \frac{1}{3}\sigma, 0)$.

(5) for $g_2(c) < g < g_1(c)$, there are four equilibrium points $(\varphi_{1-}, 0)$, $(\varphi_{1+}, 0)$, $(c+\sigma, y_{1-})$ and $(c+\sigma, y_{1+})$. $(\varphi_{1+}, 0)$ is a center point enclosing the orbit which is homoclinic for the saddle point $(\varphi_{1-}, 0)$.

(6) for $g = g_1(c)$, there are three equilibrium points $(c-1, 0)$, $(c+\sigma, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\sigma)$ and $(c+\sigma, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sigma)$. $(c-1, 0)$ is a cusp. $(c + \sigma, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\sigma)$ and $(c + \sigma, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\sigma)$ are two saddle points.

The phase portraits of system (2.5) are similar to the Fig.1 in the [2] or [4].

3 Several types of bounded traveling wave solutions

In this section ,we obtain the following three types traveling wave solutions:

3.1 Smooth solitary wave solution (soliton)

When $g_2(c) < g < g_1(c)$, there exist a smooth solitary wave solution of Eq.(2.1), which corresponds to the homoclinic orbit connecting the saddle point $(\phi_{1-}, 0)$. The homoclinic orbit of system (2.5) can be expressed as

$$y = \frac{(\varphi - \varphi_{1-})\sqrt{\varphi^2 + m\varphi + n}}{2(\varphi - c - \sigma)} \quad for \quad \varphi_{1-} < \varphi <_{2+}, \tag{3.1}$$

where

$$m = \frac{2}{3}(1 - 3c - 2\sigma - 3\Delta), \tag{3.2}$$

$$n = \frac{2}{3}(1 - 4c + 3c^2 - 3g + 2c\sigma - 2\sigma + (1 + 3c + 4\sigma)\Delta), \tag{3.3}$$

$$\varphi_{2+} = -\frac{1}{3}(1 - 3c - 2\sigma - 3\Delta + 2\sqrt{(1 + \sigma)^2 - 3(1 + \sigma)\Delta}). \tag{3.4}$$

Substituting Eq.(3.1) into the first equation of system (2.3) and integrating along the homoclinic orbits, we have the implicit expression for smooth solitary wave solutions of Eq.(2.1)

$$\int_{\varphi}^{\varphi_{2+}} \frac{s - c - \sigma}{(s - \varphi_{1-})\sqrt{s^2 + ms + n}} ds = -\frac{1}{2}|\xi|. \tag{3.5}$$

When $g_2(c) < g < g_1(c)$ and g tends to $g_2(c)$, we know that $m \to -2(c+\sigma), n \to (c+\sigma)^2$, and $\varphi_{2+} \to c+\sigma$. This means that (3.5) tends to

$$\int_{\varphi}^{c+\sigma} \frac{s - c - \sigma}{(s - c + \frac{4}{3} + \frac{1}{3}\sigma)\sqrt{(s - c - \sigma)^2}} ds = -\frac{1}{2}|\xi|, \tag{3.6}$$

from which we obtain

$$\varphi = (\frac{4}{3} + \frac{4}{3}\sigma)e^{-\frac{1}{2}|\xi|} + c - \frac{1}{3}\sigma - \frac{4}{3}. \tag{3.7}$$

The solitons lose their smoothness and tend to peakons. Then the solution of Eq.(1.2) tends to

$$u(x, t) = -(\frac{8}{9}\sigma + \frac{8}{9})e^{-\frac{1}{2}|x+ct|} - \frac{2}{3}c + \frac{2}{9}\sigma + \frac{8}{9}. \tag{3.8}$$

3.2 Solitary cusp wave solution (peakon)

When $g = g_2(c)$, there exist a solitary cusp wave solution of Eq.(1.2), which corresponds to the heteroclinic orbits defined by $H(\varphi, y) = 0$. And the heteroclinic orbits is a triangle with the following three line segments:

$$y = \pm\frac{1}{2}(\varphi - (c - \frac{1}{3}\sigma - \frac{4}{3})) \quad for \quad \varphi_{1-} \leq \varphi \leq \varphi_{1+}, \tag{3.9}$$

and

$$\varphi = c + \sigma \quad for \quad y_{1-} \leq y \leq y_{1+}. \tag{3.10}$$

Thus, we have the following exact parametric representations of solitary cusp wave solution (the peakon) of Eq.(2.1)

$$\varphi = Ae^{-\frac{1}{2}|\xi|} + c - \frac{1}{3}\sigma - \frac{4}{3}. \tag{3.11}$$

We get the solution of the Eq.(1.2)

$$u(x,t) = Ae^{-\frac{1}{2}|\xi|} - \frac{2}{3}c + \frac{2}{9}\sigma + \frac{8}{9}, \tag{3.12}$$

where A is a constant.

3.3 Periodic cusp wave solution

When $g_3(c) < g < g_2(c)$, there exist a periodic cusp wave solution of Eq.(2.1), which correspond to the heteroclinic orbits. The periodic orbit can be expressed as

$$y = \pm\frac{1}{2}\sqrt{\varphi^2 - (2c - \frac{2}{3}\sigma - \frac{8}{3})\varphi - 2(c+\sigma)(\frac{1}{2}c - \frac{1}{6}\sigma - \frac{2}{3}) + 4g} \quad for \quad \varphi_{2-} \leq \varphi \leq c + \sigma, \tag{3.13}$$

and

$$\varphi = c + \sigma \quad for \quad y_{1-} \leq y \leq y_{1+}, \tag{3.14}$$

where

$$\varphi_{2-} = c - \frac{1}{3}\sigma - \frac{4}{3} + \frac{1}{3}\sqrt{18c^2 - 2\sigma^2 - 36c - 4\sigma - 36g + 16}. \tag{3.15}$$

Substituting (3.13) into the first equation of system (2.3) and integrating along the periodic orbit, we have

$$\int_{\varphi}^{c+\sigma} \frac{1}{\sqrt{s^2 - (2c - \frac{2}{3}\sigma - \frac{8}{3})s - 2(c+\sigma)(\frac{1}{2}c - \frac{1}{6}\sigma - \frac{2}{3}) + 4g}} = \frac{1}{2}\xi, \quad \xi > 0, \tag{3.16}$$

and

$$\int_{\varphi}^{c+\sigma} \frac{1}{\sqrt{s^2 - (2c - \frac{2}{3}\sigma - \frac{8}{3})s - 2(c+\sigma)(\frac{1}{2}c - \frac{1}{6}\sigma - \frac{2}{3}) + 4g}} = -\frac{1}{2}\xi, \quad \xi < 0. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\varphi = l_+e^{-\frac{1}{2}|\xi|} + l_-e^{\frac{1}{2}|\xi|} + c - \frac{1}{3}\sigma - \frac{4}{3}, \tag{3.18}$$

where

$$l_\pm = \frac{1}{6}(4\sigma + 4 \pm 3\sqrt{-2c^2 + 2\sigma^2 + 4c + 4\sigma + 4g}). \tag{3.19}$$

Let

$$T = 2|\ln(\varphi_{2-} - c + \frac{1}{3}\sigma + \frac{4}{3}) - \ln(2l_-)|, \tag{3.20}$$

then

$$u(x,t) = -\frac{2}{3}\varphi(x + ct - 2nT) \quad for \quad (2n-1)T < x + ct < (2n+1)T, \tag{3.21}$$

are periodic cusp wave solutions for Eq.(1.2) with $2T$ period. Clearly, when $g_3(c) < g < g_2(c)$ and $g \to g_2(c)$, $T \to \infty$, $l_+ \to \frac{4}{3}\sigma + \frac{4}{3}$, $l_- \to 0$, and $u(x,t)$ in (3.21) tends to

$$u(x,t) = -(\frac{8}{9}\sigma + \frac{8}{9})e^{-\frac{1}{2}|x+ct|} - \frac{2}{3}c + \frac{2}{9}\sigma + \frac{8}{9}. \tag{3.22}$$

This shows that $g_3(c) < g < g_2(c)$ and g tends to $g_2(c)$, the periodic cusp wave solutions also tend to the peakons.

4 Conclusion

In this paper, we obtained three types traveling wave solutions of the Fornberg-Whitham equation with dispersion perturbation term by the bifurcation method: the implicit expression for solitons, the explicit expressions for peakons and periodic cusp wave solutions. Furthermore, we show that the limits of soliton solutions and periodic cusp wave solutions are peakons. And when $c = \frac{4}{3}$, $\sigma \to 0$, the solution (3.8) or (3.22) of Eq.(1.2) agrees with the solution of Eq.(1.1). From (3.8), we can see that the dispersion disturbance term has no effect on the existence of traveling wave solution of Fornberg-Whitham equation, but changes the amplitude and position of traveling wave solution.

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