Generalized Exact Travelling Wave Solutions of mch and mdp Equations

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Abstract: A procedure for finding exact travelling wave solutions with its graphical explanation to the modified Camassa Holm and Degasperis Procesi equations is proposed. Many solutions for sin, cos, sinh and cosh functions is obtained. Furthermore, these solutions are in general forms.

Keywords: Camassa-Holm equation; Degasperis-Procesi equation; travelling wave solution; symbolic computation; partial differential equations

1 Introduction

The family of nonlinear evolution equations that extended its status because of its presence in several scientific uses and physical phenomena. The representation of Camassa-Holm (CH) \cite{1} family of equations is

\begin{equation}
    u_t - u_{xxt} + au_x + buu_x = k u_x u_{xx} + uu_{xxx}
\end{equation}

Where a,b and k are constants, and u(x,t) is unknown function depending on time variable t and three-dimensional variable x. For b = 3 and k = 2, (1) reduces to Camassa-Holm equation (CH)

\begin{equation}
    u_t - u_{xxt} + au_x + 3uu_x = 2 u_x u_{xx} + uu_{xxx}
\end{equation}

Eq. (2) models unidirectional propagation of nonlinear shallow waves of water over flat bottom. In eq. (2),u(x,t) is fluid velocity or waters free surface. One phase of the Camassa-Holm equation (CH) \cite{1} is the existence of many conservation laws and its bi-Hamiltonian structure. This property shows that the Camassa-Holm equation (CH) is integrable completely. Another phase is that for a = 0, it shapes a new kind of solitary waves contains discontinuous slope at its crest.

The peaked solitary wave is called peakon which has non analytic nature dissimilar to smooth soliton. For a = 0, Camassa-Holm equation (CH) provides multi-soliton solutions of peaked solitary waves. Peakon has discontinuities in the three dimensional derivative. In this paper we will discuss two well-known equations of this family for the following particular values of the constants b and k \cite{1}. For a = 0, the set of constants

\begin{align*}
    b &= 3, k = 2, \\
    b &= 4, k = 3,
\end{align*}

are used to attain the following equations.

\begin{align}
    u_t - u_{xxt} + 3uu_x &= 2 u_x u_{xx} + uu_{xxx} \\
    u_t - u_{xxt} + 4uu_x &= 3 u_x u_{xx} + uu_{xxx}
\end{align}
Modified form of CH and DP equations

For the function \( u(x, t) \), the Camassa-Holm (CH) equation and the Degasperis-Procesi (DP) equation have been investigated by many researchers. The CH equation (3) is a shallow water equation and was originally derived as an approximation to the incompressible Euler equation, while the DP equation (4) can be considered as a model for shallow water dynamics [5]. Since the CH and DP equations have rich structures, Wazwaz [2] suggested a modified form of the Camassa-Holm equation (called mCH)

\[
  u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}
\]

and a modified form of the Degasperis-Procesi equation (called mDP)

\[
  u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + uu_{xxx}
\]

They are obtained by changing the nonlinear convection term \( uu_x \) in equations (3) and (4) to \( u^2u_x \). Wazwaz obtained two bell-shaped travelling wave solutions of wave speed \( c = 2 \) for the mCH equation (5), and two bell-shaped travelling wave solutions of wave speed \( c = 5/2 \) for the mDP equation (6). Later Wazwaz [3] found more solutions of wave speeds \( c = 1 \) and \( c = 2 \) for the mCH equation, and more solutions of wave speed \( c = 5/2 \) for the mDP equation. Recently, author [4] proposed an algorithm for solving travelling wave solution to non-linear partial differential equation and further author [5] proposed the procedure for finding the travelling wave solution to mCH and mDP equation. In this paper, based on work in [5], a procedure is proposed of travelling wave solution for those functions which cannot solve by author [5] procedure. It turns out many travelling wave solutions obtained. Most importantly, these solutions in general form. Consequently, these new solutions would be useful for better understanding the physical phenomena associated with these equations.

2 The procedure

The proposed procedure basically based on tanh method [7][8][9] and mainly it is extracted from the procedure proposed in [5]. The procedure explain in [5] only solve following list of functions [rational, exp, csch, sech, tanh, sec, tan, cn, sn] but still 4 functions [sin, cos, sinh, cosh] cannot be solved by the procedure [5]. In this paper the proposed procedure will solve the 4 functions [sin, cos, sinh, cosh] and it is also capable to solve [rational, exp, csch, sech, tanh, sec, tan, cn, sn] but here we only concentrate on 4 functions i.e [sin, cos, sinh, cosh]. The main steps of the procedure are as follows, where pde is the mCH equation (5) or the mDP equation (6).

**Step 1** Substitute \( u(x, t) = U(\zeta) \) where \( \zeta = \omega_1 x + \omega_2 t \)

\[
  \zeta = \omega_1 x + \omega_2 t
\]

**Step 2** To find the balancing number \( b \), we start putting the balancing number from \( b = 1 \) so on and where \( f(\zeta) = \sin(\zeta) \) in equation (8). For every balancing number we apply the further steps of the procedure and get a solution if the solution satisfies the given partial differential equation then at that balancing number we obtain the solution and if solution does not satisfy the balancing number then we neglect the balancing number and take the next number. The reason why we do not take \( b = 0 \) because at this we get the trivial solution.

**Step 3** Substitute

\[
  U(\zeta) = \sum_{i=-b}^{b} a_i T^i
\]

where \( T = f(\zeta) = \sin(\zeta) \) into ordinary differential equation and eliminate the common denominator to obtain an equation. Any order derivative of \( f(\zeta) \) w.r.t \( \zeta \) is a polynomial in \( f(\zeta) \), therefore, after putting (8) into ordinary differential equation and removing the common denominator which is a power of \( f(\zeta) \), we get the desired equation.

**Step 4** Setting all the coefficients of the different powers of \( T \) equal to zero gives a system of polynomial equations.

**Step 5** Solving system of polynomial equations involving variables \( a_i, a_{-i}, \omega_1 \) and \( \omega_2 \), where \( i = 1 \ldots b \).

**Step 6** Substitute the solutions obtained into \( U(\zeta) = \sum_{i=-b}^{b} a_i T^i \) one after another to obtain the travelling wave solutions of \( f \) type.

3 Application

As an explanation of above procedure, we solve mCH equation

\[
  u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}
\]

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we will get the travelling wave solutions for the function $\tilde{f}(\zeta) = \sin(\zeta)$.

In step 1, substituting the formula $u(x, t) = \tilde{U}(\zeta)$ into equation (3.1) gives the following ODE:

$$
\omega_1 U' - \omega_1^2 \omega_2 u''' + \omega_1 (U^3)' - \frac{1}{2} \omega_1^3 ((U')^2)' - \omega_1^3 (UU''')' = 0
$$

In step 2 we find out that the balancing number $b = 2$, because at $b = 1$ the solution we get does not satisfy the mCH equation.

In step 3, substituting

$$U(\zeta) = \sum_{i=-b}^{b} a_i T^i$$

where $T = \sin(\zeta)$, into eq. (10) and eliminating the common denominator give an equation of the form which have different power of $\sin(\zeta)$

In step 4, set all the coefficient of different powers of $\sin(\zeta)$ from equation which is get through step 3 to zero, we have the system of polynomial equations as follows

$$
8\omega_1^3 a_2^2 - 8\omega_1^2 \omega_2 a_2 - 6\omega_1 a_0^2 a_2 - 6\omega_1 a_0^2 a_{-2} - 6\omega_1 a_2^2 a_0 - 8\omega_1^3 a_2 a_0 - 2\omega_2 a_2 - 3\omega_1^2 a_1^2 - 12\omega_1 a_1 a_{-1} a_2 = 0 \tag{12}
$$

$$-6\omega_1 a_{-2} a_1 a_2 - 6\omega_1 a_0 a_{-1} a_2 - \omega_2 a_1 - \omega_1^2 a_0 a_1 - 3\omega_1^2 a_2 a_{-1} - 3\omega_1^2 a_0^2 a_1 - 3\omega_1 a_1 a_{-1} - 3\omega_1^2 a_2 a_{-1} + 4\omega_1^3 a_1 a_2 = 0 \tag{13}
$$

$$-2\omega_1^3 a_{-1} a_2 + \omega_1^2 \omega_2 a_{-1} + 3\omega_1 a_2^2 a_{-1} + 3\omega_1^2 a_{-1} a_1 + a_1^3 a_0 a_{-1} + \omega_2 a_{-1} + 6\omega_1 a_{-2} a_{-1} a_2 + 6\omega_1 a_{-2} a_1 a_0 = 0 \tag{14}
$$

$$3\omega_1^3 a_{-1}^2 + 2\omega_2 a_{-2} + 8\omega_1 a_{-2} a_0 + 6\omega_1 a_2 a_{-2} + 6\omega_1 a_2 a_{-2} + 8\omega_1^2 a_{-2} a_0 - 8\omega_1^3 a_{-2} a_2 + 12\omega_1 a_{-2} a_{-1} a_1 - 2\omega_1^3 a_{-1} a_1 = 0 \tag{15}
$$

$$-6\omega_1^3 a_{0} a_{-1} - 6\omega_1^2 \omega_2 a_{-1} - 12\omega_1^3 a_1 a_{-2} + 9\omega_1 a_2 a_{-1} - 21\omega_1^3 a_{-2} a_{-1} - 3\omega_1 a_1 a_{-1} a_{-1} - 18\omega_1 a_{-2} a_{-1} a_0 = 0 \tag{16}
$$

$$24\omega_1^3 a_{-2}^2 + 12\omega_1 a_{-2}^2 a_{-2} + 12\omega_1 a_{-2}^2 a_{-2} - 10\omega_1^3 a_{-1}^2 - 24\omega_1^2 a_0 a_{-2} - 24\omega_1^2 a_{-2} a_0 = 0 \tag{17}
$$

$$-18\omega_1 a_0 a_{-1} a_2 - 3\omega_1 a_{-1} - 21\omega_1^2 a_2 a_{1} - 9\omega_1 a_2 a_{-1} a_{-1} = 0 \tag{18}
$$

$$-12\omega_1 a_{-1} a_2 - 12\omega_1 a_2 a_{-1} a_2 - 6\omega_1 a_2 a_{-1} a_2 = 0 \tag{19}
$$

$$15\omega_1 a_{-1} a_{-1} - 50\omega_1 a_{-2} a_{-1} = 0 \tag{20}
$$

$$-6\omega_1 a_{-2} a_{-2} - 48\omega_1^2 a_{-2} = 0 \tag{21}
$$

$$-15\omega_1 a_2 a_{1} = 0 \tag{22}
$$

$$-6\omega_1 a_{2} = 0 \tag{23}
$$

Solving the system above leads to the following roots

1. $a_{-2} = 0, a_{-1} = 0, a_0 = a_0, a_1 = 0, a_2 = 0, \omega_1 = \omega_1, \omega_2 = \omega_2$

2. $a_{-2} = a_{-2}, a_{-1} = a_{-1}, a_0 = a_0, a_1 = a_1, a_2 = a_2, \omega_1 = 0, \omega_2 = 0$

3. $a_{-2} = 8\omega_1^2, a_{-1} = 0, a_0 = -\frac{2}{3} \omega_1^2 a_{-1} - \frac{1}{3} \omega_1^2 a_{-1} a_{-1} = 0$

4. $a_{-2} = 8\omega_1^2, a_{-1} = 0, a_0 = -\frac{3}{2} \omega_1^2 a_{-1} - \frac{1}{2} \omega_1^2 a_{-1} a_{-1} = 0$

Finally in step 6, substituting the roots into eq. (10) one by one gives a solution for sin function to mCH equation which is the solutions $u_1(x, t)$ and $u_2(x, t)$.
4 Solutions to the mCH equation

We get the following travelling wave solutions to the mCH equation. All the solutions have been verified. **Two solutions for sin function**

\[
\begin{align*}
\text{for } & \sin \text{ function} \\
\text{u}_1(x, t) & = -\frac{8}{3} \omega_1^2 - \frac{1}{2} + \frac{1}{6} \sqrt{9 - 128 \omega_1^4} + \frac{8 \omega_1^2}{\sin^2(\omega_1 x + (-\frac{3}{2} + \frac{1}{2} \sqrt{9 - 128 \omega_1^4}) \omega_1 t))} \tag{24} \\
\text{u}_2(x, t) & = -\frac{8}{3} \omega_1^2 - \frac{1}{2} - \frac{1}{6} \sqrt{9 - 128 \omega_1^4} + \frac{8 \omega_1^2}{\sin^2(\omega_1 x + (-\frac{3}{2} - \frac{1}{2} \sqrt{9 - 128 \omega_1^4}) \omega_1 t))} \tag{25}
\end{align*}
\]

**Two solutions for cos function**

\[
\begin{align*}
\text{u}_3(x, t) & = -\frac{8}{3} \omega_1^2 - \frac{1}{2} + \frac{1}{6} \sqrt{9 - 128 \omega_1^4} + \frac{8 \omega_1^2}{\cos^2(\omega_1 x + (-\frac{3}{2} + \frac{1}{2} \sqrt{9 - 128 \omega_1^4}) \omega_1 t))} \tag{26} \\
\text{u}_4(x, t) & = -\frac{8}{3} \omega_1^2 - \frac{1}{2} - \frac{1}{6} \sqrt{9 - 128 \omega_1^4} + \frac{8 \omega_1^2}{\cos^2(\omega_1 x + (-\frac{3}{2} - \frac{1}{2} \sqrt{9 - 128 \omega_1^4}) \omega_1 t))} \tag{27}
\end{align*}
\]

**Two solutions for sinh function**

\[
\begin{align*}
\text{u}_5(x, t) & = \frac{8}{3} \omega_1^2 - \frac{1}{2} + \frac{1}{6} \sqrt{9 - 128 \omega_1^4} + \frac{8 \omega_1^2}{\sinh^2(\omega_1 x + (-\frac{3}{2} + \frac{1}{2} \sqrt{9 - 128 \omega_1^4}) \omega_1 t))} \tag{28}
\end{align*}
\]
Two solutions for cosh function

\[ u_7(x, t) = \frac{8}{3} \omega_1^2 - \frac{1}{2} - \frac{1}{6} \sqrt{9 - 128\omega_1^4} + \frac{8\omega_1^2}{cosh^2(\omega_1 x + (-\frac{\omega_1}{2} + \frac{1}{2} \sqrt{9 - 128\omega_1^4})\omega_1 t))} \]  
\[ u_8(x, t) = \frac{8}{3} \omega_1^2 - \frac{1}{2} - \frac{1}{6} \sqrt{9 - 128\omega_1^4} + \frac{8\omega_1^2}{cosh^2(\omega_1 x + (-\frac{\omega_1}{2} - \frac{1}{2} \sqrt{9 - 128\omega_1^4})\omega_1 t))} \]  

Real solutions of mCH equation can be obtained by putting different values for \( \lambda \). Complex solutions can also be obtained by putting suitable values for \( \lambda \), in other words, selecting those values for \( \lambda \) such that the values inside the radicals are negative.

5 Solutions to the mDP equation

We obtain the following travelling wave solutions to the modified DP equation. All the solutions have been verified.

Two solutions for sin function

\[ u_9(x, t) = -\frac{5}{2} \omega_1^2 - \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 15\omega_1^4)} + \frac{15\omega_1^2}{2sin^2(\omega_1 x + 2(-1 + \sqrt{(1 - 15\omega_1^4)})\omega_1 t))} \]
u_{10}(x, t) = -\frac{5}{2}\omega^2_1 - \frac{1}{2} - \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\sin^2(\omega_1 x + 2(-1 - \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{33}

Two solutions for cos function

u_{11}(x, t) = \frac{5}{2}\omega_1^2 - \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\cos^2(\omega_1 x + 2(1 + \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{34}

u_{12}(x, t) = -\frac{5}{2}\omega^2_1 - \frac{1}{2} - \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\cos^2(\omega_1 x + 2(-1 - \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{35}

Two solutions for sinh function

u_{13}(x, t) = \frac{5}{2}\omega^2_1 - \frac{1}{2} - \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\sinh^2(\omega_1 x + 2(-1 - \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{36}

u_{14}(x, t) = \frac{5}{2}\omega^2_1 - \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\sinh^2(\omega_1 x + 2(1 + \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{37}

Two solutions for cosh function

u_{15}(x, t) = \frac{5}{2}\omega^2_1 - \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\cosh^2(\omega_1 x + 2(-1 - \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{38}

u_{16}(x, t) = \frac{5}{2}\omega^2_1 - \frac{1}{2} + \frac{1}{2} \sqrt{(1 - 15\omega^4_1)} + \frac{15\omega^2_1}{2\cosh^2(\omega_1 x + 2(1 + \sqrt{(1 - 15\omega^4_1)})\omega_1 t))} \tag{39}
Figure 13: Singular Solution for $\alpha_1 = \frac{1}{2}$ of (36)

Figure 14: Singular Solution for $\alpha_1 = \frac{1}{2}$ of (37)

Figure 15: Peakon Solution for $\alpha_1 = \frac{1}{2}$ of (38)

Figure 16: Peakon Solution for $\alpha_1 = \frac{1}{2}$ of (39)
6 Conclusion

The procedure is suitable for finding exact travelling wave solutions of modified Camassa-Holm equation and Degasperis-Procesi equations and most importantly, these solutions are in generalized form. The solutions are also explained by graphically on particular values of $\omega_1$.

The procedure that is proposed in [5] is very sophisticated method for finding the balancing number but the balancing number is obtained only for 9 functions i.e. rational, csc, sec, tan, cot, csch, sech, tanh, coth and still 4 functions i.e. sin, sinh, cos, cosh were missing. The procedure we extract is obtained the solution of remaining 4 functions, although the method we proposed is applicable for all above functions.

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