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Sharp Liouville Type Results for a Class of Parabolic Inequality

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Abstract: In this paper, we study the existence and nonexistence of nonnegative or positive solutions of the system of parabolic inequality

$$\begin{cases} u_t - \Delta u \geq \mu_1 u^p + \beta u^r v^{s+1}, & \text{in } \mathbb{R}^N \times \Omega_T, \\ v_t - \Delta v \geq \mu_2 v^p + \beta u^{r+1} v^s, & \text{in } \mathbb{R}^N \times \Omega_T, \end{cases}$$

where $\Omega_T = \mathbb{R}$ or $\Omega_T = (0, \infty)$. Under some assumptions on the coefficients μ_1, μ_2, β and parameters p, r, s, we obtain the sharp Liouville type results for this kind of system.

Keywords: Positive solutions; Parabolic inequality; Liouville type results.

1 Introduction

In this paper we prove the existence and nonexistence of positive or nonnegative solutions to the following the system of parabolic inequality

$$\begin{cases} u_t - \Delta u \geq \mu_1 u^p + \beta u^r v^{s+1}, & \text{in } \mathbb{R}^N \times \Omega_T, \\ v_t - \Delta v \geq \mu_2 v^p + \beta u^{r+1} v^s, & \text{in } \mathbb{R}^N \times \Omega_T, \end{cases} \tag{1}$$

where $\Omega_T = \mathbb{R}$ or $\Omega_T = (0, \infty)$, and μ_1, $\mu_2 > 0$, $r, s, \beta \in \mathbb{R}$.

In recent years, there are many results about elliptic and parabolic systems. In order to state the results, we define the Sobolev exponent and Bidaut-Veron exponent (see [5])

$$p_S = \begin{cases} \frac{N+2}{N-2}, & N \geq 3, \\ \infty, & N = 1,\ 2, \end{cases} \quad \text{and} \quad p_B = \begin{cases} \frac{N(N+2)}{(N-1)^2}, & N \geq 2, \\ \infty, & N = 1. \end{cases}$$

Mitidieri [15] proved that the inequalities

$$\begin{cases} -\Delta u \geq v^p, \\ -\Delta v \geq u^q \end{cases}$$

has no positive solutions of class of $C^2(\mathbb{R}^N)$, which p, q satisfy $\frac{N-2}{N}(pq-1) \leq \max\{p+1, q+1\}$. Taliaferro [28] studied the nonnegative solutions of nonlinear parabolic inequalities

$$au^\lambda \leq u_t - \Delta u \leq u^\lambda,\ \lambda > \frac{N+2}{N},\ a \in (0,1).$$

Moreover, the paper [28] also shown that changing the value of a in the open interval (0,1) can dramatically affect the blow-up of these solutions. Armstrong [3] gives the nonexistence of positive solutions of the semilinear inequality

$$-\Delta u \geq f(u)$$

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in any exterior domain of $\mathbb{R}^N$ with $N \geq 3$. The nonlinearity satisfies $f = f(s) > 0$ and continuous on $(0, \infty)$, and $\liminf_{s \searrow 0} s^{-\frac{N}{N-2}} f(s) > 0$.

Kurta [12, 13] studied the Liouville comparison principle for solutions of semilinear parabolic inequalities

$$u_t - \mathfrak{L}u - |u|^{q-1}u \geq v_t - \mathfrak{L}v - |v|^{q-1}v$$

in the space $\mathbb{R}^N \times \mathbb{R}$, $q > 0$, where $\mathfrak{L}$ is a differential operator. Considering the following parabolic inequalities systems in $\mathbb{R}^N \times I$, I is an interval of $\mathbb{R}$,

$$\begin{cases} u_t - \Delta u \geq \ v^p, \quad p \in \mathbb{R}, \\ v_t - \Delta v \geq \ u^q, \quad q \in \mathbb{R}. \end{cases} \tag{2}$$

Duong and Souplet [8] proved the Liouville results about systems (2). Precisely, the system (2) has nonexistence of nontrivial nonnegative solutions when

$$pq > 1, \ \max\left\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right\} \geq N.$$

On the other hand, the system (2) has nonexistence of positive solutions if

$$p \leq 0, \text{ or } q \leq 0, \text{ or } pq < 1, p, q > 0, pq > 1 \text{ and } \max\left\{\tfrac{2(p+1)}{pq-1}, \tfrac{2(q+1)}{pq-1}\right\} \geq N.$$

In the paper [18], the authors proved the Liouville type results for the following system

$$\begin{cases} u_t - \Delta u = a_{11}u^p + a_{12}u^r v^{s+1}, \quad (x,t) \in \Omega \times I, \\ v_t - \Delta v = a_{22}v^p + a_{21}u^{r+1}v^s, \quad (x,t) \in \Omega \times I. \end{cases} \tag{3}$$

They proved that if $1 < p < p_B$, the system (3) does not possess any nontrivial nonnegative classical solution in the whole space $\mathbb{R}^N \times \mathbb{R}$. Later, Duong and Phan [7] generalized the results of [18]. Note that since $p_B < p_S$, we can make a conjuncture that the system (3) has nonexistence of positive solution if $1 < p < p_S$. Let $r = s$, $a_{11} = a_{22} = 1$, $a_{12} = a_{21} < \frac{r}{3r+2}$ in (3), Phan [17] proved that if $1 < p < p_S$, (3) doesn't possess any radial, nontrivial nonnegative, classical solution in $\mathbb{R}^N \times \mathbb{R}$. Recently, the paper [24] prove that the system (3) has nonexistence of positive solution if $1 < p < p_S$. For more results on this direction we refer the readers to [2, 10, 14, 17, 18, 23, 25–27] and the references therein.

Motivated by the previous works [8, 24, 26], in this paper we establish the existence and nonexistence of nonnegative or positive solutions of (1) for $\beta \in \mathbb{R}$. This Liouville type results are sharp for this kind of system. To accomplish this, for the cases $\beta > 0$, the nonexistence of positive solutions and nonnegative nontrivial solutions can be done by using the method as in [8]. If $\beta = 0$, the system (1) is equivalent to the single equation. If $\beta < 0$, the main method of nonexistence of positive solutions is used the contradiction arguments, which suppose (1) has a positive solution. Then we prove some dedicates upper and lower estimates to obtain the contradiction and show that the solution must be zero.

1.1 Nontrivial nonnegative solutions of (1)

In this part we study the existence and nonexistence of nonnegative solutions of (1). In order to state the nonexistence of nonnegative solutions of (1), we need the following conditions. If $\beta < 0$, we suppose that $p(2r+1) > 0$, $p(2s+1) > 0$ and β satisfies

$$\frac{-p[\mu_1(2s+1) + \mu_2(2r+1)] + p|\mu_1(2s+1) - \mu_2(2r+1)|}{2(2r+1)(2s+1)} < \beta < 0. \tag{4}$$

Then the following main results hold.

Theorem 1 *Assume that μ_1, $\mu_2 > 0$.*

(1) If $1 < p \leq 1 + \frac{2}{N}$, $\beta > 0$, the system (1) has no nontrivial nonnegative solutions in Ω_T.

(2) If $1 < p \leq 1 + \frac{2}{N}$, β satisfies (4) and $p = r + s + 1$, then the system (1) has no nontrivial nonnegative solutions in Ω_T.

(3) If $0 < p \leq 1$*, the system* (1) *has a nonnegative solution* (7) *in* Ω_T.

(4) If $p \geq 1 + \frac{2}{N}$*, the system* (1) *has a nonnegative solution* (8) *in* Ω_T.

Remark 2 *It is well known that the exponent* $\alpha_F := 1 + 2/N$*, which is called the Fujita exponent, plays an important role for the existence of the nontrivial nonnegative global in time solutions of*

$$\begin{cases} \phi_t - \Delta\phi = |\phi|^{\alpha-1}\phi, & (t,x) \in \mathbb{R}^+ \times \Omega \\ \phi(0,x) = \phi_0(x), & x \in \Omega. \end{cases} \tag{5}$$

Precisely, if $\phi_0 \geq 0$*,* $\phi_0 \not\equiv 0$ *and* $1 < \alpha \leq \alpha_F$*, the solution of* (5) *must blow up in finite time. On the other hand, if* $\alpha > \alpha_F$*, there exist global in time solutions of* (5) *for suitable small initial data(see [11]). From the results of the above table, we know that this Fujita exponent play the same important role for the parabolic inequality* (1).

We borrow the idea of [8, 16] to give the proof of Theorem 1. The Fujita result ensures the nonexistence of nontrivial nonnegative solutions of the scalar inequality

$$u_t - \Delta u \geq u^p, \text{ in } \mathbb{R}^N \times \Omega_T, \tag{6}$$

where $\Omega_T = \mathbb{R}$ or $\Omega_T = (0,\infty)$. Hence if $\beta > 0$, the nonexistence of nontrivial nonnegative solutions of (1) is the same as the single parabolic inequality (6) in the same space. On the other hand, if $0 < p \leq 1$, the explicit nonnegative solution of systems (1) in $\mathbb{R}^N \times \Omega_T$ is given by

$$(u,v) = \begin{cases} (e^{t\alpha_1}, e^{t\alpha_2}), & t > 0, \\ (0,0), & t \leq 0. \end{cases} \tag{7}$$

If $\beta > 0$, then α_1, α_2 should satisfy

$$0 < \mu_1 + \beta \leq \alpha_1, \ 0 < \mu_2 + \beta \leq \alpha_2$$

and

$$\alpha_1(r-1) + \alpha_2(s+1) \leq 1, \ \alpha_1(r+1) + \alpha_2(s-1) \leq 1.$$

If $\beta < 0$, then α_1, α_2 should satisfy $\alpha_1 \geq \mu_1$, $\alpha_2 \geq \mu_2$.

As in [12], we know that if $p > 1 + \frac{2}{N}$, the nontrivial nonnegative solution of systems (1) is given by

$$(u,v) = \begin{cases} (\alpha_1 t^{\frac{1}{1-p}} e^{-\lambda\frac{1+|x|^2}{t}}, 0), & t > 0, \\ (0, \alpha_2 e^t), & t \leq 0, \end{cases} \tag{8}$$

where the constants α_1, α_2, λ are chosen such that

$$\frac{1}{2N(p-1)} < \lambda \leq \frac{1}{4}, \ 0 < \alpha_1 \leq \left[\frac{1}{\mu_1}\left(\frac{1}{1-p} + 2\lambda N\right)\right]^{\frac{1}{p-1}}, \ 0 < \alpha_2 \leq \mu_2^{\frac{1}{1-p}}.$$

1.2 Existence and nonexistence of positive solutions of (1)

In this subsection we shall give the existence and nonexistence of positive solution of (1) in Theorem 3-Theorem 5.

Theorem 3 *Assume that* μ_1, $\mu_2 > 0$.

(1) If $\beta > 0$*,* $1 < p \leq 1 + \frac{2}{N}$ *or* β *satisfies* (4) *and* $1 < p = r + s + 1 \leq 1 + \frac{2}{N}$*, there is no positive solutions* (1) *in* $\mathbb{R}^N \times (0,\infty)$.

(2) If $\beta > 0$*,* $p < 1$*, or* β *satisfies* (4) *and* $1 < p = r + s + 1 \leq 1 + \frac{2}{N}$*, there is no positive solutions of* (1) *in* $\mathbb{R}^N \times \mathbb{R}$.

Theorem 4 *Assume that* μ_1, $\mu_2 > 0$.

(1) If β *satisfy* (4)*,* $p > 1 + \frac{2}{N}$*, there is a positive solution of* (1) *in* $\mathbb{R}^N \times \mathbb{R}$.

(2) *If* $\beta > 0$ *and* $p = r + s + 1 > 1 + \frac{2}{N}$, *there is a positive solution of* (1) *in* $\mathbb{R}^N \times \mathbb{R}$.

Theorem 5 *Assume that* $\mu_1, \mu_2 > 0$.

(1) *If* $p < 1$, $\beta > 0$, *then* (1) *has a positive solution* (7) *in* $\mathbb{R}^N \times (0, \infty)$, *and if* $p = 1$, $\beta < 0$, (1) *has a positive solution* (7) *in* $\mathbb{R}^N \times \Omega_T$.

(2) *If* $p < 1$, $r + s + 1 = \frac{p}{1-p}$, $\beta < 0$, *then* (1) *has a positive solution*

$$(u, v) = (|t|^{\frac{1}{1-p}}, |t|^{\frac{1}{1-p}}), \quad \text{where } \mu_1 + \beta,\ \mu_2 + \beta \leq \frac{1}{p-1} < 0 \tag{9}$$

in $\mathbb{R}^N \times \mathbb{R}$ *and* $\mathbb{R}^N \times (0, \infty)$.

(3) *If* $p = 1$, $r = -s$, $\beta > 0$, *then* (1) *has a positive solution*

$$(u, v) = (e^{t\alpha}, e^{t\alpha}),\ 0 < \mu_1 + \beta,\ \mu_2 + \beta \leq \alpha \tag{10}$$

in $\mathbb{R}^N \times \mathbb{R}$ *and* $\mathbb{R}^N \times (0, \infty)$.

(4) *If* $p > \frac{N+2}{N}$, $\beta < 0$, *then the positive solution of* (1) *in* $\mathbb{R}^N \times (0, \infty)$ *is given by*

$$(u, v) = (\alpha_1 t^{\frac{1}{1-p}} e^{-\lambda_1 \frac{1+|x|^2}{t}}, \alpha_2 t^{\frac{1}{1-p}} e^{-\lambda_2 \frac{1+|x|^2}{t}}), \tag{11}$$

where the constants α_1, α_2, λ_1, λ_2 *satisfy*

$$\begin{cases} 0 < \alpha_1 \leq \left[\frac{1}{\mu_1}\left(2\lambda_1 N + \frac{1}{1-p}\right)\right]^{\frac{1}{p-1}},\ 0 < \alpha_2 \leq \left[\frac{1}{\mu_2}\left(2\lambda_2 N + \frac{1}{1-p}\right)\right]^{\frac{1}{p-1}}, \\ \frac{1}{2N(p-1)} < \lambda_1,\ \lambda_2 \leq \frac{1}{4}. \end{cases}$$

For the case $\beta > 0$, $p = r + s + 1$, *we let* $\alpha_1 = \alpha_2 = \alpha$ *in* (6). *Then* λ_1, λ_2 *satisfy*

$$\frac{1}{2N(p-1)} < \lambda_1,\ \lambda_2 \leq \frac{1}{4}$$

and

$$(\alpha(\mu_1 + \beta))^{p-1} \leq 2N\lambda_1 + \frac{1}{1-p},\ (\alpha(\mu_2 + \beta))^{p-1} \leq 2N\lambda_2 + \frac{1}{1-p}.$$

In particular, if $\mu_1 = \mu_2 = 0$, from the above results, we know that the following results hold.

Corollary 6 *When* $\mu_1 = \mu_2 = 0$, *the systems* (1) *becomes*

$$\begin{cases} u_t - \Delta u \geq \beta u^r v^{s+1}, \\ v_t - \Delta v \geq \beta u^{r+1} v^s, \end{cases} \tag{12}$$

where β, r, $s \in \mathbb{R}$.

(1) *If* $-1 < r$, $s < 0$, $0 < \beta \leq \min\{r + 1, s + 1\}$, (12) *has a positive solution* $(u, v) = (e^{(s+1)t}, e^{(r+1)t})$ *in* $\mathbb{R}^N \times (0, \infty)$.

(2) *If* $r + s + 1 > \frac{N+2}{N}$, β, r, $s > 0$, (12) *has a positive solution in* $\mathbb{R}^N \times \mathbb{R}$.

(3) *If* $r = s$, $r \leq \frac{1}{N+2}$ *or* $r \neq s$, $r + s \leq \frac{2}{N+2}$ *and* $\beta > 0$, (12) *does not possess any positive solution in* $\mathbb{R}^N \times \mathbb{R}$.

(4) *If* $\beta < 0$, *all positive constants are solutions of* (12).

Remark 7 *If* $\beta < 0$, $p \neq r + s + 1$, $1 < p \leq 1 + \frac{2}{N}$ *or* $\beta > 0$, $p \neq r + s + 1 > 1 + \frac{2}{N}$, *the existence of solutions of* (1) *is not obviously. We leave these as open questions and pursue it in the near future.*

The structure of this paper is as follows. In sections 2-3 we give the prove of nonexistence result of (1). In section 4 we prove the existence result of (1).

Finally, we give the outline of the proofs of the main results. If $1 < p \le \frac{N+2}{N}$, from the Fujita results, we get the nonexistence of nontrivial nonnegative solutions of (1) in $\mathbb{R}^N \times \mathbb{R}$. From the above tables, we find that the range of the nonexistence of nontrivial nonnegative solutions coincides with the positive solutions. In order to prove the nonexistence of nontrivial nonnegative solutions, we use the method of variable transformation, including a cut-off function, the properties of maximum arguments, the heat semigroup and Gaussian kernel. According to [20, 21], suppose in contrary that (u, v) is a nontrivial nonnegative solution of (1) in the space $\mathbb{R}^N \times (0, \infty)$. By using the similar arguments in [1, 28], for t large enough and all $x \in \mathbb{R}^N$, then there holds

$$(u+v)(x,t) \ge Ct^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \log(t+1).$$

On the other hand, by the arguments of integral estimates and inequalities, we have

$$\int_{U_R} (u+v)^{\frac{1}{p}} \,\mathrm{d}x\mathrm{d}t \le CR^{N+2+\frac{2}{p}-\frac{2}{p-1}}.$$

Combining the two estimates, we arrive at the contradiction.

Remark 8 *If $p > 1$, let $f(t) = t^{\frac{1}{p}}$, then $f''(t) = \frac{1-p}{p^2} t^{\frac{1-2p}{p}} < 0$ is strictly concave function, so for arbitrary $\lambda \in (0,1)$, $f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda) f(x_2)$. If $\lambda = \frac{1}{2}$, we have*

$$f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = \left(\frac{1}{2}\right)^{\frac{1}{p}} (x_1+x_2)^{\frac{1}{p}} > \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) = \frac{1}{2}x_1^{\frac{1}{p}} + \frac{1}{2}x_2^{\frac{1}{p}}.$$

Hence for any x_1, $x_2 > 0$, $p > 1$,

$$x_1^{\frac{1}{p}} + x_2^{\frac{1}{p}} < \left(\frac{1}{2}\right)^{\frac{1}{p}-1} (x_1+x_2)^{\frac{1}{p}}.$$

2 Proof of Theorems 1-3

2.1 The case $\beta > 0$ and $p < 1$

In this subsection we focus on the proof of the case $\beta > 0$ and $p < 1$ of Theorem 3 (2). To accomplish this we use contradiction arguments. Assume that (u, v) is a positive solution of (1). By the Young's inequality, we can get an inequality of the solution of (1). Then from the Liouville results about the inequality $w_t - \Delta w \ge w^p$, one can get a contradiction.

Proof of Theorem 3 (2) when $\beta > 0$ and $p < 1$. Suppose in contrary that (1) has a positive solution (u, v) in $\mathbb{R}^N \times \mathbb{R}$. If $\beta > 0$, $p < 1$, let $w = \sqrt{uv}$. A simple calculation shows that

$$w_t = \frac{1}{2} u_t u^{-\frac{1}{2}} v^{\frac{1}{2}} + \frac{1}{2} v_t v^{-\frac{1}{2}} u^{\frac{1}{2}} \tag{13}$$

and

$$-\Delta w = \frac{1}{4} u^{-\frac{3}{2}} v^{\frac{1}{2}} |\nabla u|^2 + \frac{1}{4} v^{-\frac{3}{2}} u^{\frac{1}{2}} |\nabla v|^2 - \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} \Delta u - \frac{1}{2} v^{-\frac{1}{2}} u^{\frac{1}{2}} \Delta v - \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} \nabla u \cdot \nabla v. \tag{14}$$

From elementary inequality, we have

$$\frac{1}{4} u^{-\frac{3}{2}} v^{\frac{1}{2}} |\nabla u|^2 + \frac{1}{4} v^{-\frac{3}{2}} u^{\frac{1}{2}} |\nabla v|^2 \ge \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} |\nabla u||\nabla v|. \tag{15}$$

Combining (14) and (15), we have

$$\begin{aligned} -\Delta w &\ge \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} (-\Delta u) + \frac{1}{2} v^{-\frac{1}{2}} u^{\frac{1}{2}} (-\Delta v) \\ &\ge \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} (\mu_1 u^p + \beta u^r v^{s+1} - u_t) + \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} (\mu_2 v^p + \beta u^{r+1} v^s - v_t) \\ &\ge \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} (\mu_1 u^p - u_t) + \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} (\mu_2 v^p - v_t). \end{aligned} \tag{16}$$

One infers from (13) and (16) that

$$\begin{aligned}w_t-\Delta w=&\frac{1}{2}u_tu^{-\frac{1}{2}}v^{\frac{1}{2}}+\frac{1}{2}v_tv^{-\frac{1}{2}}u^{\frac{1}{2}}-\frac{1}{4}u^{-\frac{3}{2}}v^{\frac{1}{2}}|\nabla u|^2+\frac{1}{4}v^{-\frac{3}{2}}u^{\frac{1}{2}}|\nabla v|^2\\&-\frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}}\Delta u-\frac{1}{2}v^{-\frac{1}{2}}u^{\frac{1}{2}}\Delta v-\frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}}\nabla u\cdot\nabla v\\&\geq\frac{1}{2}\sqrt{uv}(\mu_1u^{p-1}+\mu_2v^{p-1})\end{aligned}\tag{17}$$

On the other hand, we deduce from the Young's inequality that

$$\frac{\mu_1}{2}u^{p-1}+\frac{\mu_2}{2}v^{p-1}\geq\sqrt{\mu_1\mu_2}(uv)^{\frac{p-1}{2}}.\tag{18}$$

Substituting (18) into (17), one gets

$$w_t-\Delta w\geq\sqrt{\mu_1\mu_2}\sqrt{uv}(uv)^{\frac{p-1}{2}}=\sqrt{\mu_1\mu_2}w^p.$$

This contradicts [8, Lemma 2.1] for $p<1$. Hence if $\beta>0$, $p<1$, there is no positive solutions of (1) in $\mathbb{R}^N\times\mathbb{R}$. ■

2.2 The case $1<p\leq 1+\frac{2}{N}$ in Theorems 1-3

In this subsection we prove Theorems 1-3. To accomplish this we divide into the following two steps. We first prove the lower estimate of positive solution (u,v) in Lemma 9. On the other hand, we get the upper estimate of positive solution (u,v) in Lemma 10. Then combining these two estimates, we can obtain the contradiction and prove the results.

Lemma 9 *Assume that* $1<p\leq 1+\frac{2}{N}$ *and* β *satisfies* (4)*. Let* (u,v) *be a positive solution of* (1) *in* $\mathbb{R}^N\times[0,\infty)$*. Then there is some* $\bar{t}>0$ *s.t. for all* $x\in\mathbb{R}^N$*,* $t\geq\bar{t}$*, there holds*

$$(u+v)(x,t)\geq Ct^{-\frac{N}{2}}e^{-\frac{|x|^2}{2t}}\log(t+1).$$

Proof. We define the heat kernel in $\mathbb{R}^N\times\mathbb{R}$

$$G(x,t)=\begin{cases}(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}, & t>0,\ x\in\mathbb{R}^N,\\0, & t\leq 0,\ x\in\mathbb{R}^N.\end{cases}$$

First, by using the fundamental arguments of partial differential equation, we can use the integral inequality to represent u and v. For $\forall\, t>0$, $G_n(x-y,t)$ is the Green's function of the operator $-\frac{\partial}{\partial t}+\Delta$ on the set $B_n=\{x\in\mathbb{R}^N:|x|<n\}$ and $G_n(x-y,t)=0$ on ∂B_n. Since (u,v) is a positive solution of (1), it follows that

$$\begin{aligned}u(x,t)\geq&\int_{B_n}G_n(x-y,t)u(y,0)\,\mathrm{d}y+\int_0^t\int_{B_n}G_n(x-y,t-s)(\mu_1u^p+\beta u^rv^{s+1})(y,s)\,\mathrm{d}y\mathrm{d}s\\&-\int_0^t\int_{\partial B_n}\frac{\partial G_n}{\partial\nu}(x-y,s)(\mu_1u^p+\beta u^rv^{s+1})(y,s)\,\mathrm{d}y\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}v(x,t)\geq&\int_{B_n}G_n(x-y,t)v(y,0)\,\mathrm{d}y+\int_0^t\int_{B_n}G_n(x-y,t-s)(\mu_2v^p+\beta v^{r+1}v^s)(y,s)\,\mathrm{d}y\mathrm{d}s\\&-\int_0^t\int_{\partial B_n}\frac{\partial G_n}{\partial\nu}(x-y,s)(\mu_2v^p+\beta v^{r+1}v^s)(y,s)\,\mathrm{d}y\mathrm{d}s,\end{aligned}$$

where $\nu=\nu(y)$ is the outward unit normal vector to ∂B_n. By the Hopf lemma, there holds $\frac{\partial G_n}{\partial\nu(y)}(x-y,s)\leq 0$. $G_n(x-y,t)$ increases to Green function $G(x-y,t)$ in [19]. Letting $n\longrightarrow\infty$ in the above inequality, then we get

$$u(x,t)\geq\int_{\mathbb{R}^N}G(x-y,t)u(y,0)\,\mathrm{d}y+\int_0^t\int_{\mathbb{R}^N}G(x-y,t-s)(\mu_1u^p+\beta u^rv^{s+1})(y,s)\,\mathrm{d}y\mathrm{d}s\tag{19}$$

and

$$v(x,t) \geq \int_{\mathbb{R}^N} G(x-y,t)v(y,0)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)(\mu_2 v^p + \beta u^{r+1}v^s)(y,s)\,\mathrm{d}y\mathrm{d}s. \tag{20}$$

Since $p = r+s+1$, r, $s > 0$, $\beta < 0$, we infer from the Young's inequality that

$$\beta u^r v^{s+1} \geq \frac{\beta r}{p}u^p + \frac{\beta(s+1)}{p}v^p,\ \beta u^{r+1}v^s \geq \frac{\beta(r+1)}{p}u^p + \frac{\beta s}{p}v^p.$$

As in [9], we infer from (19) and (20) that

$$\begin{aligned}(u+v)(x,t) &\geq \int_{\mathbb{R}^N} G(x-y,t)(u(y,0)+v(y,0))\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)(\mu_1 u^p + \beta u^r v^{s+1} + \mu_2 v^p + \beta u^{r+1}v^s)(y,s)\,\mathrm{d}y\mathrm{d}s \\ &\geq \int_{\mathbb{R}^N} G(x-y,t)(u(y,0)+v(y,0))\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\left[\left(\mu_1 + \frac{\beta(2r+1)}{p}\right)u^p + \left(\mu_2 + \frac{\beta(2s+1)}{p}\right)v^p\right]\mathrm{d}y\mathrm{d}s \\ &\geq C_1 \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)(u^p+v^p)\,\mathrm{d}y\mathrm{d}s + \int_{\mathbb{R}^N} G(x-y,t)(u(y,0)+v(y,0))\,\mathrm{d}y,\end{aligned} \tag{21}$$

where $C_1 = \min\left\{\mu_1 + \frac{\beta(2r+1)}{p}, \mu_2 + \frac{\beta(2s+1)}{p}\right\}$. Since β satisfy (4), it follows that $\mu_1 + \frac{\beta(2r+1)}{p} > 0$ and $\mu_2 + \frac{\beta(2s+1)}{p} > 0$. Thus we know that $C_1 > 0$. On the other hand, we infer from (21), $-|x-y|^2 \geq -2(|x|^2+|y|^2)$ and $u(y,0) > 0$, $v(y,0) > 0$, that for $\forall\, t \geq 1$, $x \in \mathbb{R}^N$,

$$\begin{aligned}(u+v)(x,t) &\geq \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}(u(y,0)+v(y,0))\,\mathrm{d}y \\ &\geq (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \int_{\mathbb{R}^N} (u(y,0)+v(y,0))e^{-\frac{|y|^2}{2t}}\,\mathrm{d}y \\ &= C_2 t^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}}.\end{aligned} \tag{22}$$

Combining (21)-(22), we have

$$(u+v)(x,t) \geq C_3 \int_1^t \int_{\mathbb{R}^N} G(x-y,t-s)s^{-\frac{Np}{2}} e^{-\frac{p|y|^2}{2s}}\,\mathrm{d}y\mathrm{d}s. \tag{23}$$

Let $x = 2(1+4\zeta\eta)y$, $\eta = t-s$, $\zeta = \frac{p}{2s}$. Then we have

$$\int_{\mathbb{R}^N} G(x-y,\eta)e^{-\zeta|y|^2}\,\mathrm{d}y = (1+4\zeta\eta)^{-\frac{N}{2}} e^{-\frac{\zeta|x|^2}{1+4\zeta\eta}}.$$

Hence (23) becomes

$$\begin{aligned}(u+v)(x,t) &\geq C_3 \int_1^t \int_{\mathbb{R}^N} G(x-y,t-s)s^{-\frac{Np}{2}} e^{-\frac{p|y|^2}{2s}}\,\mathrm{d}y\mathrm{d}s \\ &= C_3 \int_1^t s^{-\frac{Np}{2}}(1+4\zeta\eta)^{-\frac{N}{2}} e^{-\frac{\zeta|x|^2}{1+4\zeta\eta}}\,\mathrm{d}s \\ &= C_3 \int_1^t s^{-\frac{Np}{2}}\left[\frac{s}{s+2p(t-s)}\right]^{\frac{N}{2}} e^{-\frac{|x|^2}{2t}\frac{tp}{s+2p(t-s)}}\,\mathrm{d}s.\end{aligned}$$

For all $1 < s < \frac{t}{2}$, let $\tau = \frac{s}{t}$. Since $tp + s - 2ps > 2ps + s - 2ps = s > 0$, then $0 < \frac{tp}{2tp+s-2ps} < 1$. Let $\tau = \frac{s}{t}$, $0 < \frac{p}{\tau+2p(1-\tau)} < 1$, then $\exists\, C_4 > 0$ such that $\left(\frac{1}{\tau+2p(1-\tau)}\right)^{\frac{N}{2}} > C_4 > 0$.

Hence for all $t \geq 2$, we have

$$\begin{aligned}(u+v)(x,t) &\geq C_3 t^{1-\frac{pN}{2}} e^{-\frac{|x|^2}{2t}} \int_{\frac{1}{t}}^{\frac{1}{2}} \tau^{\frac{N}{2}(1-p)} \left[\frac{1}{\tau+2p(1-\tau)}\right]^{\frac{N}{2}} \mathrm{d}\tau \\ &\geq C_3 C_4 t^{1-\frac{pN}{2}} e^{-\frac{|x|^2}{2t}} \int_{\frac{1}{t}}^{\frac{1}{2}} \tau^{\frac{N}{2}(1-p)} \mathrm{d}\tau.\end{aligned} \tag{24}$$

Then we divide into the following two cases to get our conclusion.

(1) If $\frac{N}{2}(1-p) = -1$, then (24) yields

$$(u+v)(x,t) \geq C_3 C_4 t^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \int_{\frac{1}{t}}^{\frac{1}{2}} \tau^{-1} \mathrm{d}\tau \geq C t^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \log t.$$

(2) If $\frac{N}{2}(1-p) > -1$, then (24) yields

$$(u+v)(x,t) \geq C t^{1-\frac{pN}{2}} e^{-\frac{|x|^2}{2t}} \int_{0}^{\frac{1}{2}} \tau^{\frac{N}{2}(1-p)} \mathrm{d}\tau \geq C t^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \log t.$$

By combining (1)-(2), we obtain that for t large,

$$(u+v)(x,t) \geq C t^{-\frac{N}{2}} e^{-\frac{|x|^2}{2t}} \log(t+1).$$

This finishes the proof. ■

Next we prove the upper estimate for solution of (1).

Lemma 10 *Assume that $1 < p \leq 1 + \frac{2}{N}$, $p = r+s+1$ and β satisfies* (4). *Let (u,v) be a positive solution of* (1) *in $\mathbb{R}^N \times [0,\infty)$. Then for each $R > 0$, there holds*

$$\int_{U_R} (u+v)^{\frac{1}{p}} \,\mathrm{d}x\mathrm{d}t \leq C R^{N+2+\frac{2}{p}-\frac{2}{p-1}},$$

where $U_R = \{(x,t) : R < |x| < 2R, R^2 < t < 2R^2\}$, and $C > 0$ is a constant independent of R.

Proof. We set $w = (u+v)^{\frac{1}{p}}$ for $1 < p \leq 1 + \frac{2}{N}$. Then we have

$$\begin{aligned} w_t &= \frac{1}{p}(u+v)^{\frac{1-p}{p}}(u_t+v_t), \nabla w = \frac{1}{p}(u+v)^{\frac{1-p}{p}}(\nabla u + \nabla v), \\ \Delta w &= \frac{1-p}{p^2}(u+v)^{\frac{1-2p}{p}}(\nabla u+\nabla v)^2 + \frac{1}{p}(u+v)^{\frac{1-p}{p}}(\Delta u + \Delta v).\end{aligned}$$

Since $\beta < 0$, it follows from the Young's inequality that

$$\begin{aligned} w_t - \Delta w &\geq \frac{1}{p}(u+v)^{\frac{1-p}{p}}(u_t - \Delta u + v_t - \Delta v) \\ &\geq \frac{1}{p}(u+v)^{\frac{1-p}{p}}(\mu_1 u^p + \beta u^r v^{s+1} + \mu_2 v^p + \beta u^{r+1} v^s) \\ &\geq \frac{1}{p}(u+v)^{\frac{1-p}{p}}\left[(\mu_1 + \frac{\beta(2r+1)}{p})u^p + (\mu_2 + \frac{\beta(2s+1)}{p})v^p\right] \\ &= \frac{C_*}{p}(u+v)^{\frac{1-p}{p}}(u^p+v^p),\end{aligned} \tag{25}$$

where $C_* := \min\left\{\mu_1 + \frac{\beta(2r+1)}{p}, \mu_2 + \frac{\beta(2s+1)}{p}\right\}$. Let ϕ be a smooth cut-off function in $C_c^\infty([0,\infty);[0,1])$ such that $\phi(r) = 1$ if $1 \leq r \leq 2$, $\phi(r) = 0$ if $r \leq \frac{1}{3}$ or $r > 4$. For $R > 0$, we define the functions $\psi(x,t) = \phi(|x|)\cdot\phi(t)$ and $\chi_R(x,t) = \psi^m(\frac{x}{R}, \frac{t}{R^2})$. Then there exist a positive constant C such that

$$\begin{aligned} |\partial_t \chi_R| &= |mR^{-2}\psi^{m-1}\psi_t| \leq CR^{-2}\chi_R^{\frac{m-2}{m}}, \\ |\Delta \chi_R| &= |m(m-1)R^{-2}\psi^{m-2}|\nabla\psi|^2 + mR^{-2}\psi^{m-1}\Delta\psi| \leq CR^{-2}\chi_R^{\frac{m-2}{m}}.\end{aligned}$$

Multiplying (25) by χ_R and integrating over $\mathbb{R}^N\times[0,\infty)$, we infer from the reverse Hölder inequality(see [4, Corollary 2.2] or [6]) that for $\chi_R\le\chi_R^{\frac{m-2}{m}}$

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\left(|\partial_t\chi_R|+|\Delta\chi_R|\right)\mathrm{d}x\mathrm{d}t\\
&\ge\frac{C_*}{p}\int_0^\infty\int_{\mathbb{R}^N}\left[(u^p+v^p)^{\frac{1}{p}}\chi_R\right]^p\left[(u+v)^{\frac{1}{p}}\chi_R\right]^{1-p}\mathrm{d}x\mathrm{d}t\\
&\ge C_5\left(\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)^{\frac{1}{p}}\chi_R\,\mathrm{d}x\mathrm{d}t\right)^p\left(\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\chi_R\,\mathrm{d}x\mathrm{d}t\right)^{1-p}\\
&\ge C_5\left(\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)^{\frac{1}{p}}\chi_R\,\mathrm{d}x\mathrm{d}t\right)^p\left(\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\chi_R^{\frac{m-2}{m}}\,\mathrm{d}x\mathrm{d}t\right)^{1-p}.
\end{aligned}$$

On the other hand, a direct computation shows that

$$\begin{aligned}
&CR^{-2}\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\chi_R^{\frac{m-2}{m}}\,\mathrm{d}x\mathrm{d}t\\
&\ge\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\left(|\partial_t\chi_R|+|\Delta\chi_R|\right)\mathrm{d}x\mathrm{d}t\\
&\ge C_5\left(\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)^{\frac{1}{p}}\chi_R\,\mathrm{d}x\mathrm{d}t\right)^p\left(\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\chi_R^{\frac{m-2}{m}}\,\mathrm{d}x\mathrm{d}t\right)^{1-p}.
\end{aligned}$$

Hence we have

$$R^{-\frac{2}{p}}\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\chi_R^{\frac{m-2}{m}}\,\mathrm{d}x\mathrm{d}t\ge\left(\frac{C_5}{C}\right)^{\frac{1}{p}}\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)^{\frac{1}{p}}\chi_R\,\mathrm{d}x\mathrm{d}t. \tag{26}$$

A simple computation yields

$$\begin{aligned}
&|\partial_t\chi_R^{\frac{m+2}{m}}|=|(m+2)R^{-2}\psi^{m+1}\psi_t|\le\frac{C_6}{2}R^{-2}\chi_R,\\
&|\Delta\chi_R^{\frac{m+2}{m}}|=|(m+2)(m+1)R^{-2}\psi^m|\nabla\psi|^2+(m+2)R^{-2}\psi^{m+1}\Delta\psi|\le\frac{C_6}{2}R^{-2}\chi_R.
\end{aligned}$$

Multiplying the inequalities of (1) by $\chi_R^{\frac{m+2}{m}}$ and integrating in $\mathbb{R}^N\times[0,\infty)$, we can get

$$\begin{aligned}
C_*\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)\chi_R^{\frac{m+2}{m}}\,\mathrm{d}x\mathrm{d}t&\le\int_0^\infty\int_{\mathbb{R}^N}(\mu_1u^p+\beta u^rv^{s+1}+\mu_2v^p+\beta u^{r+1}v^s)\chi_R^{\frac{m+2}{m}}\,\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\infty\int_{\mathbb{R}^N}(u_t-\Delta u+v_t-\Delta v)\chi_R^{\frac{m+2}{m}}\,\mathrm{d}x\mathrm{d}t\\
&\le\int_0^\infty\int_{\mathbb{R}^N}(u+v)(|\partial_t\chi_R^{\frac{m+2}{m}}|+|\Delta\chi_R^{\frac{m+2}{m}}|)\,\mathrm{d}x\mathrm{d}t\\
&\le C_6R^{-2}\int_0^\infty\int_{\mathbb{R}^N}(u+v)\chi_R\,\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{27}$$

Combining (26) and (27), we deduce from the C_p inequality that

$$\begin{aligned}
R^{-\frac{2}{p}}\int_0^\infty\int_{\mathbb{R}^N}(u+v)^{\frac{1}{p}}\chi_R^{\frac{m-2}{m}}\,\mathrm{d}x\mathrm{d}t&\ge\left(\frac{C_5}{C}\right)^{\frac{1}{p}}\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)^{\frac{1}{p}}\chi_R\,\mathrm{d}x\mathrm{d}t\\
&\ge\left(\frac{C_5}{C}\right)^{\frac{1}{p}}2^{\frac{1}{p}-1}\int_0^\infty\int_{\mathbb{R}^N}(u+v)\chi_R\,\mathrm{d}x\mathrm{d}t\\
&\ge\left(\frac{C_5}{C}\right)^{\frac{1}{p}}\frac{C_*}{C_6}2^{\frac{1}{p}-1}R^2\int_0^\infty\int_{\mathbb{R}^N}(u^p+v^p)\chi_R^{\frac{m+2}{m}}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{28}$$

Let us choose $\frac{m-2}{m}p^2 = \frac{m+2}{m}$. From Remark 2 and the Hölder inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} (u+v)^{\frac{1}{p}} \chi_R^{\frac{m-2}{m}} \,\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^\infty \int_{\mathbb{R}^N} (u^{\frac{1}{p}} + v^{\frac{1}{p}}) \chi_R^{\frac{m-2}{m}} \,\mathrm{d}x\mathrm{d}t \\
&\leq C_7 R^{\frac{(N+2)(p^2-1)}{p^2}} \left[\left(\int_0^\infty \int_{\mathbb{R}^N} u^p \chi_R^{\frac{m+2}{m}} \,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p^2}} + \left(\int_0^\infty \int_{\mathbb{R}^N} v^p \chi_R^{\frac{m+2}{m}} \,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p^2}} \right] \\
&\leq C_7 2^{\frac{p^2-1}{p^2}} R^{\frac{(N+2)(p^2-1)}{p^2}} \left(\int_0^\infty \int_{\mathbb{R}^N} (u^p + v^p) \chi_R^{\frac{m+2}{m}} \,\mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p^2}}.
\end{aligned} \tag{29}$$

Substituting (29) into (28), we obtain

$$\begin{aligned}
R^{-\frac{2}{p}} \int_0^\infty \int_{\mathbb{R}^N} (u+v)^{\frac{1}{p}} \chi_R^{\frac{m-2}{m}} \,\mathrm{d}x\mathrm{d}t &\geq R^2 \left(\frac{C_5}{C} \right)^{\frac{1}{p}} \frac{C_*}{C_6} 2^{\frac{1}{p}-1} \int_0^\infty \int_{\mathbb{R}^N} (u^p + v^p) \chi_R^{\frac{m+2}{m}} \,\mathrm{d}x\mathrm{d}t \\
&\geq C R^{2-(N+2)(p^2-1)} \left(\int_0^\infty \int_{\mathbb{R}^N} (u+v)^{\frac{1}{p}} \chi_R^{\frac{m-2}{m}} \,\mathrm{d}x\mathrm{d}t \right)^{p^2},
\end{aligned}$$

where $C := \left(\frac{C_5}{C}\right)^{\frac{1}{p}} \frac{C_*}{C_6} \frac{2^{\frac{1}{p}-p^2}}{C_7^{p^2}}$. Thus, we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (u+v)^{\frac{1}{p}} \chi_R^{\frac{m-2}{m}} \,\mathrm{d}x\mathrm{d}t \leq C R^{N+2+\frac{2}{p}-\frac{2}{p-1}}.$$

From the definition of χ_R, we deduce that

$$\int_{U_R} (u+v)^{\frac{1}{p}} \,\mathrm{d}x\mathrm{d}t \leq C R^{N+2+\frac{2}{p}-\frac{2}{p-1}}.$$

This finishes the proof. ■

Now we are ready to give the proof of Theorem 3.

Proof of Theorem 3. (1) If $\beta > 0$, $1 < p < \frac{N+2}{N}$, we know that any positive solution (u, v) of (1) satisfies $w_t - \Delta w \geq w^p$ in $\mathbb{R}^N \times (0, \infty)$. Since there is no positive solutions of $w_t - \Delta w \geq w^p$, it follows that (1) has no positive solutions in $\mathbb{R}^N \times (0, \infty)$.

(2) If $\beta < 0$ satisfies (4), $1 < p < \frac{N+2}{N}$ and $p = r + s + 1$, we shall use the contradiction arguments to get the results. Suppose that (u, v) is a nontrivial nonnegative solution of (1) in $\mathbb{R}^N \times (0, \infty)$. According to the strong maximum principle, there exist a positive number $\overline{t} > 0$, such that when $t \geq \overline{t}$, $u, v > 0$. We define

$$\overline{u}(x,t) = u(x, t+\overline{t}),\ \overline{v}(x,t) = u(x, t+\overline{t}).$$

Then $(\overline{u}, \overline{v})$ is a positive solution of (1) in $\mathbb{R}^N \times [0, \infty)$. Recall that for any $R > 0$, we define $U_R = \{(x,t) : R < |x| < 2R, R^2 < t < 2R^2\}$. From Lemma 9, we can get

$$\begin{aligned}
\int_{U_R} (\overline{u} + \overline{v})^{\frac{1}{p}}(x,t) \,\mathrm{d}x\mathrm{d}t &\geq C \int_{U_R} (t+\overline{t})^{-\frac{N}{2p}} e^{-\frac{|x|^2}{2p(t+\overline{t})}} \log^{\frac{1}{p}}(1+t+\overline{t}) \,\mathrm{d}x\mathrm{d}t \\
&\geq C \int_{U_R} (2R^2 + 2R^2)^{-\frac{N}{2p}} e^{-\frac{|x|^2}{2pt}} \log^{\frac{1}{p}}(1+R^2) \,\mathrm{d}x\mathrm{d}t \\
&= C R^{-\frac{N}{p}} \log^{\frac{1}{p}}(1+R^2) \int_{U_R} e^{-\frac{|x|^2}{2tp}} \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Let $x = \kappa\xi$, $t = \kappa^2\tau$, $\kappa = \max\{|x|, |t|^{\frac{1}{2}}\} \geq \min\{|x|, |t|^{\frac{1}{2}}\} = R$. We define the set

$$U_R' = \left\{ (\xi, \tau) : \frac{R}{\kappa} < \xi < \frac{2R}{\kappa}, \left(\frac{R}{\kappa}\right)^2 < \tau < 2\left(\frac{R}{\kappa}\right)^2 \right\}.$$

Then we have

$$\int_{U_R} (\overline{u}+\overline{v})^{\frac{1}{p}}\,\mathrm{d}x\mathrm{d}t \geq CR^{-\frac{N}{p}} \log^{\frac{1}{p}}(1+R^2)\int_{U'_R} e^{-\frac{|\xi|^2}{2\tau p}}\kappa^{N+2}\,\mathrm{d}\xi\mathrm{d}\tau$$
$$\geq CR^{-\frac{N}{p}+N+2}\log^{\frac{1}{p}}(1+R^2)\int_{U'_R} e^{-\frac{|\xi|^2}{2\tau p}}\,\mathrm{d}\xi\mathrm{d}\tau$$
$$\geq CR^{-\frac{N}{p}+N+2}\log^{\frac{1}{p}}(1+R^2)\int_{(\frac{R}{\kappa})^2}^{2(\frac{R}{\kappa})^2}\sqrt{2\tau p}\int_{\frac{R}{\kappa}}^{\frac{2R}{\kappa}} e^{-\left(\frac{|\xi|}{\sqrt{2\tau p}}\right)^2}\,\mathrm{d}\left(\frac{|\xi|}{\sqrt{2\tau p}}\right)\mathrm{d}\tau.$$

From Gauss integral, we know that $I=\int_0^\infty e^{-x^2}\,\mathrm{d}x$ and $I^2=\frac{1}{2}\int_0^\infty \frac{1}{1+\xi^2}\,\mathrm{d}\xi=\frac{1}{2}\arctan\xi|_0^\infty=\frac{\pi}{4}$. Hence we deduce that

$$\int_{\frac{R}{\kappa}}^{\frac{2R}{\kappa}} e^{-\left(\frac{|\xi|}{\sqrt{2\tau p}}\right)^2}\,\mathrm{d}\left(\frac{|\xi|}{\sqrt{2\tau p}}\right)=\left(\frac{1}{2}\arctan\frac{|\xi|}{\sqrt{2\tau p}}\Big|_{\frac{R}{\kappa}}^{\frac{2R}{\kappa}}\right)^{\frac{1}{2}}=C_8.$$

Hence for R sufficiently large, one has

$$\int_{U_R}(\bar{u}+\bar{v})^{\frac{1}{p}}\,\mathrm{d}x\mathrm{d}t \geq C_8R^{-\frac{N}{p}+N+2}\log^{\frac{1}{p}}(1+R^2). \tag{30}$$

The Lemma 10 implies that

$$\int_{U_R}(\bar{u}+\bar{v})^{\frac{1}{p}}\,\mathrm{d}x\mathrm{d}t \leq CR^{N+2+\frac{2}{p}-\frac{2}{p-1}}. \tag{31}$$

Combining (30) and (31), we have $p>\frac{N+2}{N}$. This contradicts $1<p\leq 1+\frac{2}{N}$. This finishes the proof. ■

3 Proof of Theorem 4

In this section, we prove the existence of positive solutions of (1) in $\mathbb{R}^N\times\mathbb{R}$. The heat kernel and parabolic norm play an important role in this process. For any $(x,t)\in\mathbb{R}^N\times\mathbb{R}$, we define the parabolic norm by

$$|(x,t)|=\max\{|x|,|t|^{\frac{1}{2}}\}.$$

We first have the following result for heat kernel(see [8, Lemma3.3]).

Lemma 11 *For $\alpha\in(2,N+2)$, the function*

$$\phi(x,t)=\int\int_{\mathbb{R}^N\times\mathbb{R}} G(x-y,t-s)|(y,s)|^{-\phi}\,\mathrm{d}y\mathrm{d}s$$

is C^∞ in $\{(x,t):|(x,t)|>\frac{1}{2}\}$.

Now we give the proof of Theorem 4.

Proof of Theorem 4. (1) If $\beta>0$, $p=r+s+1>\frac{N+2}{N}$, $r,s>0$, we prove that (1) has a positive solution in $\mathbb{R}^N\times\mathbb{R}$. By the Young's inequality, we have

$$\left[\left(\mu_1+\frac{\beta r}{p}\right)^{\frac{1}{p}}u+\left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}}v\right]^p \geq \left(\mu_1+\frac{\beta r}{p}\right)u^p+\frac{\beta(s+1)}{p}v^p\geq \mu_1u^p+\beta u^rv^{s+1} \tag{32}$$

and

$$\left[\left(\mu_2+\frac{\beta s}{p}\right)^{\frac{1}{p}}v+\left(\frac{\beta(r+1)}{p}\right)^{\frac{1}{p}}u\right]^p \geq \left(\mu_2+\frac{\beta s}{p}\right)v^p+\frac{\beta(r+1)}{p}u^p\geq \mu_2v^p+\beta u^{r+1}v^{s}. \tag{33}$$

It follows that if the system

$$\begin{cases} u_t-\Delta u\geq \left[\left(\mu_1+\frac{\beta r}{p}\right)^{\frac{1}{p}}u+\left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}}v\right]^p \\ v_t-\Delta v\geq \left[\left(\mu_2+\frac{\beta s}{p}\right)^{\frac{1}{p}}v+\left(\frac{\beta(r+1)}{p}\right)^{\frac{1}{p}}u\right]^p \end{cases} \tag{34}$$

has a solution, then (1) also has a solution. In the following we prove the existence of solution of (34). To accomplish this, we let $\sigma = \frac{2}{p-1}$, $\lambda_1, \lambda_2 > 0$ and define the functions

$$U(x,t) = \lambda_1(1+|x|^4+t^2)^{-\frac{\sigma}{4}} \quad \text{and} \quad V(x,t) = \lambda_2(1+|x|^4+t^2)^{-\frac{\sigma}{4}}$$

It is clear that

$$u(x,t) = \int\int_{\mathbb{R}^N\times\mathbb{R}} G(x-y,t-s)U^p(y,s)\,\mathrm{d}y\mathrm{d}s$$

and

$$v(x,t) = \int\int_{\mathbb{R}^N\times\mathbb{R}} G(x-y,t-s)V^p(y,s)\,\mathrm{d}y\mathrm{d}s$$

satisfy(see [28, Lemma1])

$$u_t - \Delta u = U^p \quad \text{and} \quad v_t - \Delta v = V^p.$$

We claim that there exists a constant $C > 0$ such that

$$U \geq C\left[\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v\right] \quad \text{and} \quad V \geq C\left[\left(\mu_2 + \frac{\beta s}{p}\right)^{\frac{1}{p}} v + \left(\frac{\beta(r+1)}{p}\right)^{\frac{1}{p}} u\right].$$

For any $(\tilde{x},\tilde{t})$ such that $|(\tilde{x},\tilde{t})| \geq 1$, we have

$$\begin{aligned}\left(\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v\right)(\tilde{x},\tilde{t}) &= \left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} \int\int_{\mathbb{R}^N\times\mathbb{R}} G(\tilde{x}-y,\tilde{t}-s)U^p(y,s)\,\mathrm{d}y\mathrm{d}s \\ &\quad + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} \int\int_{\mathbb{R}^N\times\mathbb{R}} G(\tilde{x}-y,\tilde{t}-s)V^p(y,s)\,\mathrm{d}y\mathrm{d}s.\end{aligned} \tag{35}$$

Let

$$\tilde{x} = \rho\xi,\ y = \rho\eta,\ \tilde{t} = \rho^2\tau,\ s = \rho^2\varsigma,\ \rho = |(\tilde{x},\tilde{t})|,\ |(\xi,\tau)| = \rho^{-1}|(\tilde{x},\tilde{t})| = 1.$$

We define the function

$$\psi(x,t) = \int\int_{\mathbb{R}^N\times\mathbb{R}} G(x-y,t-s)|(y,s)|^{-p\sigma}\,\mathrm{d}y\mathrm{d}s.$$

Since

$$U(x,t) \leq \lambda_1|(x,t)|^{-\sigma} \quad \text{and} \quad V(x,t) \leq \lambda_2|(x,t)|^{-\sigma},$$

it follows from Lemma 11 that ψ is C^∞ in the domain $\{(x,t) : |(x,t)| > \frac{1}{2}\}$. Hence we have

$$\begin{aligned}&\left(\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v\right)(\tilde{x},\tilde{t}) \\ &\leq \left[\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} \lambda_1^p + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} \lambda_2^p\right] \int\int_{\mathbb{R}^N\times\mathbb{R}} G(\tilde{x}-y,\tilde{t}-s)|(y,s)|^{-p\sigma}\,\mathrm{d}y\mathrm{d}s \\ &= C_9 \int\int_{\mathbb{R}^N\times\mathbb{R}} G(\rho\xi-\rho\eta,\rho^2\tau-\rho^2\varsigma)|(\rho\eta,\rho^2\varsigma)|^{-p\sigma}\rho^{N+2}\,\mathrm{d}\eta\mathrm{d}\varsigma \\ &\leq C_9|(\tilde{x},\tilde{t})|^{-\sigma} \max_{\{(\xi,\tau):|(\xi,\tau)|=1\}} \psi(\xi,\tau).\end{aligned} \tag{36}$$

where $C_9 = \left[\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} \lambda_1^p + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} \lambda_2^p\right]$. Thus, for any $(\tilde{x},\tilde{t})$ such that $|(\tilde{x},\tilde{t})| \geq 1$, we have

$$\left(\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v\right)(\tilde{x},\tilde{t}) \leq \tilde{C}|(\tilde{x},\tilde{t})|^{-\sigma} \leq C_{10}U(\tilde{x},\tilde{t}),$$

where $\tilde{C} = \max_{\{(\xi,\tau):|(\xi,\tau)|=1\}} \psi(\xi,\tau)$. On the other hand, for all $|(\tilde{x},\tilde{t})| \le 1$, we infer from (35) that there exists $C_{11} > 0$ such that

$$\left(\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v\right)(\tilde{x},\tilde{t}) \le C_{11} U(x,t).$$

Then we arrive

$$U \ge C\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v$$

and

$$U^p \ge C^p\left[\left(\mu_1 + \frac{\beta r}{p}\right)^{\frac{1}{p}} u + \left(\frac{\beta(s+1)}{p}\right)^{\frac{1}{p}} v\right]^p \ge C^p\left[\left(\mu_1 + \frac{\beta r}{p}\right) u^p + \frac{\beta(s+1)}{p} v^p\right].$$

Similarly, we have in $\mathbb{R}^N \times \mathbb{R}$,

$$V^p \ge C^p\left[(\mu_2 + \frac{\beta s}{p}) u^p + \frac{\beta(r+1)}{p} v^p\right].$$

So we have

$$\begin{aligned} u_t - \Delta u = U^p \ge C^p\left[\left(\mu_1 + \frac{\beta r}{p}\right) u^p + \frac{\beta(s+1)}{p} v^p\right] \ge C^p(\mu_1 u^p + \beta u^r v^{s+1}), \\ v_t - \Delta v = V^p \ge C^p\left[\left(\mu_1 + \frac{\beta s}{p}\right) v^p + \frac{\beta(r+1)}{p} u^p\right] \ge C^p(\mu_2 v^p + \beta u^{r+1} v^s). \end{aligned} \tag{37}$$

By making the dilation

$$u(x,t) = u_k(kx, k^2 t) \quad \text{and} \quad v(x,t) = v_k(kx, k^2 t),$$

we arrive at

$$\partial_t u_k - \Delta u_k \ge k^{-2} C^p (\mu_1 u_k^p + \beta u_k^r v_k^{s+1})$$

and

$$\partial_t v_k - \Delta v_k \ge k^{-2} C^p (\mu_2 v_k^p + \beta u_k^{r+1} v_k^s).$$

Therefore, (u_k, v_k) is a positive solution of (1) in $\mathbb{R}^N \times \mathbb{R}$, when $k > 0$ is sufficiently small.

(2) Similarly, if $\beta < 0$ satisfies (4) and $p > 1 + \frac{2}{N}$, we infer from [8, Proposition4.3] that there exists (u,v) in $\mathbb{R}^N \times \mathbb{R}$ such that

$$u_t - \Delta u = U^p \ge c^p u^p \quad \text{and} \quad v_t - \Delta v = V^p \ge c^p v^p.$$

Hence we choose $k > 0$ such that

$$\partial_t u_k - \Delta u_k \ge k^{-2} C^p u_k^p \ge \mu_1 u_k^p + \beta u_k^r v_k^{s+1}$$

and

$$\partial_t v_k - \Delta v_k \ge k^{-2} C^p v_k^p \ge \mu_2 v_k^p + \beta u_k^{r+1} v_k^s.$$

Therefore, (u_k, v_k) is a positive solution of (1) in $\mathbb{R}^N \times \mathbb{R}$. ■

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