Existence of positive solutions for a nonhomogeneous Schrödinger-Poisson system in $\mathbb{R}^3$

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Abstract: In this paper, we study the following problem

$$\begin{aligned}
(P) \quad \left\{ \begin{array}{ll}
-\Delta u + u + K(x)\phi(x)u = a(x)f(u) + h(x), & \text{in } \mathbb{R}^3, \\
-\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3,
\end{array} \right.
\end{aligned}$$

where $K, h \in L^2(\mathbb{R}^3)$, $a(x)$ is a positive bounded function, and $f(s)$ is asymptotically linear with respect to $s$ at infinity, that is $f(s)/s$ goes to a constant as $s \to +\infty$. Under suitable assumptions on $K$, $a$ and $f$, we prove that the problem $(P)$ has at least two positive solutions for $|K|_2$ and $|h|_2$ small by using Ekeland’s variational principle and Mountain pass theorem.

Keywords: Nonhomogeneous; Schrödinger-Poisson system; Asymptotically linear; Variational method

1 Introduction

In this paper, we consider the following nonhomogeneous Schrödinger-Poisson system

$$\begin{aligned}
\left\{ \begin{array}{ll}
-\Delta u + u + K(x)\phi(x)u = a(x)f(u) + h(x), & \text{in } \mathbb{R}^3, \\
-\Delta \phi = K(x)u^2, & \text{in } \mathbb{R}^3,
\end{array} \right.
\end{aligned} \quad (1)$$

where $K \in L^2(\mathbb{R}^3)$, $a$ is a positive bounded function, $h \in L^2(\mathbb{R}^3)$, $h \geq (\not=)0$.

If $h(x) \equiv 0$, system (1) becomes the well known Schrödinger-Possion system. Such a system, also called Schrödinger-Maxwell system, arises in many interesting physical contexts. In fact, according to a classical model, the interaction of a charged particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and Poisson equations (we refer the reader to [1] and the references therein for more details on the physical aspects). Recently, the Schrödinger-Possion system has been widely studied by using the modern variational method and critical point theory, and existence, nonexistence and multiplicity results are obtained in many papers, see e.g. [2-17] and the references therein.

If $K(x) \equiv 0$, system (1) becomes the nonhomogeneous Schrödinger equation which has been extensively investigated, see for example, [18-23] and the references therein. In [21], the authors prove that problem (1) with $K(x) \equiv 0$ have at least two positive solutions when $f(s)$ is either asymptotical linear or superlinear with respect to $s$ at infinity. A natural question is to ask what would happen if $K(x) \not\equiv 0$. To the author’s knowledge, it seems that there are few results about (1) in this case.

On the function $f$ and $a$, we make the following assumptions.

(F1) $f \in C(\mathbb{R}^3, \mathbb{R}^+)$, $f(s) \equiv 0$ for all $s \leq 0$ and $f(s)s^{-1} \to 0$ as $s \to 0^+$.

(F2) There exists $l \in (0, +\infty)$ such that $f(s)s^{-1} \to l$, as $s \to +\infty$.

(A1) $a(x)$ is a positive continuous function and there exists $R_0 > 0$ such that

$$\sup\{f(s)/s : s > 0\} < \inf\{1/a(x) : |x| \geq R_0\}.$$

Example 1.1 Define

$$f(s) = \begin{cases} 
  s^2/(1 + s), & s > 0, \\
  0, & s \leq 0.
\end{cases}$$

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Then, it is easy to see that $f(s)$ satisfies the condition $(F_1)$ and $(F_2)$.

**Remark 1** Condition $(A_1)$ is assumed first in [24].

Our main result is the following:

**Theorem 2** Suppose that $b(x) \in L^2(\mathbb{R}^3)$, $b \geq (\neq) 0$ for all $x \in \mathbb{R}^3$, and $K(x)$ satisfies

$(K_1)$ $K \in L^2(\mathbb{R}^3)$, $K(x) > 0$ for all $x \in \mathbb{R}^3$.

Let $(F_1), (F_2)$ and $(A_1)$ hold and $l > \mu^*$ with

$$
\mu^* = \inf \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} a(x) u^2 \, dx = 1 \right\}. \tag{2}
$$

Then there exists $m > 0$ such that problem (1) has two positive solutions $u_0, u_1 \in H^1(\mathbb{R}^3)$ satisfying $I(u_0) < 0$ and $I(u_1) > 0$ if $|h|_2 < m$ and $|K|_2 < m$.

To obtain our result, we have to overcome several difficulties in using variational method. First, since $f(s)$ is asymptotically linear in $s$ at infinity, there are some difficulties to check the that variational functional has the Mountain Pass geometry. Second, since $f(s)$ is asymptotically linear in $s$ at infinity, it is known that the classical Ambrosetti-Rabinowitz technique condition on $f(s)$ is no longer available. So, we have to find another method to verify the boundedness of a (PS) sequence. Third, since the problem (1) is set on $\mathbb{R}^3$, it is well known that the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is not compact, and then it is usually difficult to prove the compactness of a (PS) sequence. We overcome the first difficulty by using $l > \mu^*$ with $\mu^*$ given by (2). Motivated by [14], we overcome the second difficulty by using $K \in L^2(\mathbb{R}^3)$. Motivated by some techniques used in [25], we establish a compactness lemma to overcome the third difficulty.

The remained of this paper is organized as follows. In Section 2, we outline the variational setting and give a compactness lemma. In Section 3, we give the proof of Theorem 2.

**Notation.** Throughout the paper, we denote by $C$ and $C_i$ various positive constants which may vary from place to place. The usual norm in $L^s(\mathbb{R}^3)$ with $1 \leq s \leq \infty$ is denoted by $|.|_s$. $D^{1,2}(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \}$ denotes the usual Sobolev space endowed with the norm $\| u \|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$. $H^1(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3) \}$ denotes the usual Sobolev space endowed with the norm $\| u \|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx$.

## 2 Variational setting and compactness lemma

In this section, we outline the variational framework for problem (1) and establish a compactness lemma. We can easily reduce (1) to a nonlinear Schrödinger equation with a nonlocal term. Indeed, for any $u \in H^1(\mathbb{R}^3)$, let us consider the linear functional $T_u : D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$
T_u(v) = \int_{\mathbb{R}^3} K(x) u^2 \, v \, dx.
$$

By Hölder’s inequality and Sobolev’s inequality, we have

$$
|T_u(v)| \leq |K|_2 |u|_6^2 |v|_6 \leq C |K|_2 \|u\|^2 \|v\|_{D^{1,2}(\mathbb{R}^3)}, \quad \forall v \in D^{1,2}(\mathbb{R}^3). \tag{3}
$$

Thus $T_u$ is continuous in $D^{1,2}(\mathbb{R}^3)$. Then by the Riesz Representation Theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$
\int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v \, dx = \int_{\mathbb{R}^3} K(x) u^2 \, v \, dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3).
$$

Therefore, $-\Delta \phi_u = u^2$ in a weak sense. Moreover, we can write an integral expression for $\phi_u$ in the form

$$
\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(y) u^2(y)}{4\pi |x-y|} \, dy, \tag{4}
$$

and by (3) we obtain that

$$
\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} = \|T_u\| \leq C |K|_2 \|u\|^2. \tag{5}
$$

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Hence, by Hölder’s inequality and (5), we get
\[ \int_{\mathbb{R}^3} K(x)\phi_n u^2 dx \leq C|K|_2 \|u\|^2 \|\phi_n\|_{H^{1,2}(\mathbb{R}^3)} \leq C|K|_2 \|u\|^4. \] (6)

Then we define a functional \( I : H^1(\mathbb{R}^3) \to \mathbb{R} \) by
\[ I(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2}(|\nabla u|^2 + u^2) + \frac{1}{4} K(x)\phi_n u^2 - a(x)F(u) - h(x)u \right) dx, \] (7)
where \( F(u) = \int_0^u f(s)ds \). By (6), the functional \( I \) is a well defined and \( I \in C^1(H^1(\mathbb{R}^3), \mathbb{R}) \) with derivative by
\[ (I'(u), v) = \int_{\mathbb{R}^3} \left( \nabla u \nabla v + u v + K(x)\phi_n uv - a(x)fu - h(x)v \right) dx, \quad \forall v \in H^1(\mathbb{R}^3). \]

It can be proved that \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a critical point of the functional \( I \) and \( \phi = \phi_n \), see for instance [1].

The functions \( \phi_n \) possess the following properties:

**Lemma 3** Assume that \((K_1)\) holds.
(i) \( \phi_n \geq 0 \), for any \( u \in H^1(\mathbb{R}^3) \). Moreover, \( \phi_n > 0 \), as soon as \( u \neq 0 \).
(ii) If \( u_n \to u \) in \( H^1(\mathbb{R}^3) \), then \( \phi_n \to \phi_u \) in \( D^{1,2}(\mathbb{R}^3) \).

**Proof.** (i) By (4) and \((K_1)\), the conclusion is obvious.
(ii) Let \( u_n \in H^1(\mathbb{R}^3) \) be such that \( u_n \to u \) in \( H^1(\mathbb{R}^3) \). Then \( u_n \) is bounded in \( H^1(\mathbb{R}^3) \) and in \( L^6(\mathbb{R}^3) \). To verify \( \phi_u \to \phi_u \) in \( D^{1,2}(\mathbb{R}^3) \), it is sufficient to show \((\phi_{u_n} - \phi_u, v)_{D^{1,2}(\mathbb{R}^3)} \to 0 \), for all \( v \in D^{1,2}(\mathbb{R}^3) \). Indeed, for any \( v \in D^{1,2}(\mathbb{R}^3) \), by Hölder’s inequality we have
\[ (\phi_{u_n} - \phi_u, v)_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (K(x)u_n^2 - K(x)u^2) v dx \]
\[ \leq |u_n + u|_6^6 \left( \int_{\mathbb{R}^3} K(x)^{3/2}(u_n - u)^{3/2} dx \right)^{2/3} \]
\[ \leq C|v|_6 \left( \int_{\mathbb{R}^3} K(x)^{3/2}(u_n - u)^{3/2} dx \right)^{2/3} \]
\[ \to 0 \]
as \( n \to \infty \), since \((u_n - u)^{3/2} \to 0 \) in \( L^4(\mathbb{R}^3) \) and \( K(x)^{3/2} \in L^{4/3}(\mathbb{R}^3) \). \( \blacksquare \)

Motivated by [25], we have the following compactness lemma for problem (1).

**Lemma 4** Assume that \((F_1), (F_2)\) and \((A_1)\) hold. Let \( h \in L^2(\mathbb{R}^3) \), \( K \) satisfies \((K_1)\), and \( \{u_n\} \subset H^1(\mathbb{R}^3) \) is a bounded \((PS)\) sequence of \( I \). Then \( \{u_n\} \) has a strongly convergent subsequence in \( H^1(\mathbb{R}^3) \).

**Proof.** It is sufficient to prove that for any \( \epsilon > 0 \), there exist \( R(\epsilon) \geq R_0 \) (\( R_0 \) is given by \((A_1)\)) and \( n(\epsilon) > 0 \) such that for any \( R \geq R(\epsilon) \) and \( n \geq n(\epsilon) \),
\[ \int_{|x| \geq R} (|\nabla u|^2 + u^2) dx \leq \epsilon. \] (8)

For any fixed \( R > 0 \), let \( \xi_R \in C^\infty(\mathbb{R}^3, \mathbb{R}) \) be a cut-off function such that \( 0 \leq \xi_R \leq 1 \),
\[ \xi_R(x) = \begin{cases} 0, & 0 \leq |x| \leq R/2, \\ 1, & |x| \geq R, \end{cases} \]
and, for some constant \( C > 0 \) (independent of \( R \)), \( |\nabla \xi_R(x)| \leq \frac{C}{R} \) for all \( x \in \mathbb{R}^3 \). Then for any \( u \in H^1(\mathbb{R}^3) \) and \( R \geq R_0 \), there exists a constant \( C_1 > 0 \) which is independent of \( R \) such that \( \|\xi_R u\| \leq C_1 \|u\| \). Since \( I'(u_n) \to 0 \) in \( H^{-1}(\mathbb{R}^3) \) and \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \), we have that for any \( \epsilon > 0 \), there exists \( n(\epsilon) \geq 0 \) such that
\[ \langle I'(u_n), \xi_R u_n \rangle \leq \|I'(u_n)\|_{H^{-1}(\mathbb{R}^3)} \|\xi_R u_n\| \leq \frac{\epsilon}{4}, \quad \text{for } n \geq n(\epsilon), \]
that is, if \( n \geq n(\epsilon) \), then
\[
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2)\xi_R \, dx + \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \xi_R \, dx + \int_{\mathbb{R}^3} K(x)\phi a_n u_n^2 \xi_R \, dx \\
\leq \int_{\mathbb{R}^3} (a(x)f(u_n) + h(x))u_n \xi_R \, dx + \frac{\epsilon}{4}.
\]
(9)

It follows from \((F_1)\) and \((A_1)\) that there exists \( \theta \in (0, 1) \) such that
\[
a(x)f(u_n)u_n \leq \theta u_n^2 \quad \text{for } |x| \geq R_0.
\]
(10)

Since \( h \in L^2(\mathbb{R}^3) \) and \( u_n \) is a positive solution of problem (1),

\[
\int_{\mathbb{R}^3} h(x)u_n \xi_R \leq |h\xi|_{2}|u_n|_{2} \leq \frac{\epsilon}{4}, \quad \text{for } R \geq R(\epsilon).
\]
(11)

Then for \( R \geq R(\epsilon) \) and \( n \geq n(\epsilon) \), combining (9), (10) and (11) we deduce that
\[
\int_{\mathbb{R}^3} (|\nabla u_n|^2 + (1 - \theta)u_n^2)\xi_R \, dx \leq \frac{C_0}{R} u_n \|u_n\|^2 + \frac{\epsilon}{2} \leq \frac{C_2}{R} + \frac{\epsilon}{2}.
\]
(12)

Noting that the constant \( C_2 \) is independent of \( R \), we can choose \( R > 0 \) large enough such that (8) holds.

3 Proof of Theorem 2

In what follows, we give first Lemma 5 which is required by using Ekeland’s variational principle.

**Lemma 5** If \((F_1), (F_2)\) and \((A_1)\) hold, \( h \in L^2(\mathbb{R}^3) \) and \( K \) satisfies (K1). Then there exist \( \rho, \alpha, m_1 > 0 \) such that \( I(u)\|u\| = \rho \geq \alpha > 0 \) for \( |h|_2 < m_1 \).

**Proof.** For any \( \epsilon > 0 \), it follows from \((F_1)\) and \((F_2)\) that there exists \( C_1(\epsilon) > 0 \) such that
\[
|f(s)| \leq \epsilon|s| + C_1(\epsilon)|s|^5, \quad \text{for all } s \in \mathbb{R},
\]
and then, by (13) and the Mean Value Theorem, we obtain
\[
|F(s)| \leq \frac{\epsilon}{2}|s|^2 + \frac{C_1(\epsilon)}{6}|s|^6, \quad \text{for all } s \in \mathbb{R}.
\]
(14)

By the Sobolev embedding, we have
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi a_n u^2 \, dx - \int_{\mathbb{R}^3} (a(x)f(u) + h(x)u) \, dx \\
\geq \frac{1}{2} \|u\|^2 - C_1 \epsilon \|u\|^2 - C_2(\epsilon)\|u\|^6 - |h|_2 \|u\| \\
= \|u\| \left[ \left( \frac{1}{2} - C_1 \epsilon \right) \|u\| - C_2(\epsilon)\|u\|^5 - |h|_2 \right].
\]
(15)

Taking \( \epsilon = \frac{1}{4C_1} \) and setting \( g(t) = \frac{1}{4} t - C_2 t^5 \) for \( t \geq 0 \), we see that there exists \( \rho > 0 \) such that \( \max_{t \geq 0} g(t) = g(\rho) := m_1 \). Then, it follows from (15) that there exists \( \alpha > 0 \) such that \( I(u)\|u\| = \rho \geq \alpha > 0 \) for \( |h|_2 < m_1 \).

For \( \rho \) given by Lemma 5, we denote \( B_\rho = \{ u \in H^1(\mathbb{R}^3) : \|u\| < \rho \} \). Then by Ekeland’s variational principle ([26]) and Lemma 5 we have the following theorem, which shows that \( I \) has a local minimum if \( |h|_2 \) is small.

**Theorem 6** Assume that \((F_1), (F_2)\) and \((A_1)\) hold, \( \|h\| \in L^2(\mathbb{R}^3) \), \( \|h\| \geq (\neq) 0 \), and \( K \) satisfies (K1). If \( |h|_2 < m_1, m_1 \) is given by Lemma 5, then there exists \( u_0 \in H^1(\mathbb{R}^3) \) such that
\[
I(u_0) = \inf \{ I(u) : u \in B_\rho \} < 0,
\]
and \( u_0 \) is a positive solution of problem (1).
Proof. Since $h(x) \in L^2(\mathbb{R}^3)$, $h \geq (\neq)0$, we can choose a function $\varphi \in H^1(\mathbb{R}^3)$ such that
\[
\int_{\mathbb{R}^3} h(x)\varphi(x)dx > 0.
\]
For $t > 0$, by (6) we have
\[
I(t\varphi) = \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + \varphi^2)dx + \frac{t^4}{4} \int_{\mathbb{R}^3} K(x)\varphi^2dx - \int_{\mathbb{R}^3} a(x)F(t\varphi)dx - t \int_{\mathbb{R}^3} h(x)\varphi(x)dx
\leq \frac{t^2}{2} \|\varphi\|^2 + \frac{Ct^4}{4} \|K\|^2_2\|\varphi\|^4 - t \int_{\mathbb{R}^3} h(x)\varphi(x)dx < 0 \quad \text{for} \ t > 0 \ \text{small enough.}
\]
Hence $c_0 := \inf\{I(u) : u \in B_{\rho}\} < 0$. By the Ekeland’s variational principle, there exists $\{u_n\} \subset B_{\rho}$ such that
\[
I(u_n) \to c_0, \quad I'(u_n) \to 0,
\]
that is, $\{u_n\}$ is a bounded $(PS)$ sequence of $I$. Hence, Lemma 4 implies that there exists $u_0 \in H^1(\mathbb{R}^3)$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$. ■

Next we prove that problem (1) has a mountain pass type solution. For this purpose, we use the Mountain Pass Theorem. The following lemma shows that $I$ defined in (7) has the so-called mountain pass geometry.

Lemma 7 If $(F_1)$, $(F_2)$ and $(A_1)$ hold and $l > \mu^*$ with $\mu^*$ given by (2), $h \in L^2(\mathbb{R}^3)$ and $K$ satisfies $(K_1)$. Then there exist $m_2 > 0$ and $v \in H^1(\mathbb{R}^3)$ with $\|v\| > \rho$, $\rho$ is given by Lemma 5, such that $I(v) < 0$ for $|K|_2 < m_2$.

Proof. Since $l > \mu^*$, we can choose a nonnegative function $\phi \in H^1(\mathbb{R}^3)$ with
\[
\int_{\mathbb{R}^3} a(x)\phi^2dx = 1 \quad \text{such that} \quad \int_{\mathbb{R}^3} (|\nabla\phi|^2 + \phi^2)dx < l.
\]
Then, we define the functional $I_0 : H^1(\mathbb{R}^3) \to \mathbb{R}$ by
\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx - \int_{\mathbb{R}^3} a(x)F(u)dx - \int_{\mathbb{R}^3} h(x)u dx.
\]
Thus, by $(F_2)$ and Fatou’s lemma we deduce that,
\[
\lim_{t \to +\infty} \frac{I_0(t\phi)}{t^2} = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla\phi|^2 + \phi^2)dx - \lim_{t \to +\infty} \int_{\mathbb{R}^3} a(x)\frac{F(t\phi)}{t^2}dx - \lim_{t \to +\infty} \frac{1}{l} \int_{\mathbb{R}^3} h(x)\phi(x)dx
\leq \frac{1}{2} \|\phi\|^2 - l < 0.
\]
So, $I_0(t\phi) \to -\infty$ as $t \to +\infty$, then there exists $v \in H^1(\mathbb{R}^3)$ with $\|v\| > \rho$ such that $I_0(v) < 0$. It follows from (6) that
\[
I(v) = I_0(v) + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_nv^2dx
\leq I_0(v) + C|K|_2^2\|v\|^4.
\]
Hence, taking $m_2 := \sqrt{\frac{I_0(v)}{C\|v\|^4}}$, we see that $I(v) < 0$ for $|K|_2 < m_2$. ■

For the functional defined by (7), $\alpha$ and $v$ given by Lemmas 5 and 7, respectively, we define
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$
where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = v\}$.
Obviously, $c \geq \alpha > 0$. Since Lemmas 5 and 7 hold, the Mountain Pass Theorem [26] implies there exists a $(PS)$ sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that
\[
I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \ \text{in} \ H^{-1}(\mathbb{R}^3),
\]
where $H^{-1}(\mathbb{R}^3)$ denotes the dual space of $H^1(\mathbb{R}^3)$.

The following lemma shows that the $(PS)$ sequence $\{u_n\}$ obtained by (17) is bounded in $H^1(\mathbb{R}^3)$.

\[
LINS \ homepage: \ http://www.nonlinearscience.org.uk/\]
Lemma 8 Assume that \((F_1), (F_2)\) and \((A_1)\) hold, \(h(x) \in L^2(\mathbb{R}^3), \ h \geq (\neq) 0\), and \(K\) satisfies \((K_1)\). Then the \((PS)\) sequence \(\{u_n\}\) obtained by (17) is bounded in \(H^1(\mathbb{R}^3)\).

Proof. By contradiction, after passing to a subsequence, let \(\|u_n\| \to +\infty \) as \(n \to +\infty\). Set \(\omega_n := u_n/\|u_n\|\). Clearly, \(\omega_n\) is bounded in \(H^1(\mathbb{R}^3)\) and there exists \(\omega \in H^1(\mathbb{R}^3)\) such that, up to a subsequence,

\[
\omega_n \to \omega, \quad \text{in } H^1(\mathbb{R}^3),
\]

\[
\omega_n \to \omega, \quad \text{in } L^2(\mathbb{R}^3),
\]

\[
\omega_n \to \omega, \quad \text{a.e. in } \mathbb{R}^3.
\]

Case 1. \(\omega \equiv 0\) in \(H^1(\mathbb{R}^3)\). On one hand, since \(\|u_n\| \to +\infty \) as \(n \to +\infty\), it follows from (17) that

\[
\frac{\langle f'(u_n), u_n \rangle}{\|u_n\|^2} = o(1),
\]

that is,

\[
o(1) = \|\omega_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx
\]

\[
\geq 1 - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx
\]

where, and in what follows, \(o(1)\) denotes a quantity which goes the zero as \(n \to \infty\). Thus,

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx \geq 1.
\]

(21)

On the other hand, By \((A_1)\), there exists a constant \(\theta \in (0, 1)\) such that

\[
\sup\{f(s)/s : s > 0\} \leq \theta \inf\{1/a(x) : |x| \geq R_0\}.
\]

This yields, for any \(n \in \mathbb{N}\),

\[
\int_{|x| \geq R_0} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx \leq \theta \int_{|x| \geq R_0} \omega_n^2 dx \leq \theta \|\omega_n\|^2 = \theta < 1.
\]

(22)

Since the embedding \(H^1(B_{R_0}(0)) \hookrightarrow L^2(B_{R_0}(0))\) is compact, from (18)-(20) we have \(\omega_n \to \omega\) strongly in \(L^2(B_{R_0}(0))\).

By \((F_1)\) and \((F_2)\), we have

\[
\int_{|x| \leq R_0} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx \leq C \int_{|x| \leq R_0} \omega_n^2 dx \to 0, \quad \text{as } n \to \infty.
\]

(23)

So, by (22) and (23) we obtain that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^3} \frac{f(x, u_n)}{u_n}\omega_n^2 dx < 1,
\]

which contradicts (21). Hence, \(\omega \neq 0\).

Case 2. \(\omega \neq 0\) in \(H^1(\mathbb{R}^3)\). Since \(\|u_n\| \to +\infty \) as \(n \to +\infty\), it follows from (17) that

\[
\frac{\langle f'(u_n), u_n \rangle}{\|u_n\|^2} = o(1),
\]

that is,

\[
o(1) = \frac{1}{\|u_n\|^2} + \int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx - \int_{\mathbb{R}^3} a(x)\frac{f(u_n)}{u_n}\omega_n^2 dx.
\]

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Combing this with \((F_1)\) and \((F_2)\), we obtain
\[
\int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx = o(1). \tag{24}
\]

We can easily verify that
\[
\int_{\mathbb{R}^3} K(x)\phi_{\omega_n}\omega_n^2 dx = \int_{\mathbb{R}^3} K(x)\phi_{\omega}\omega^2 dx + o(1). \tag{25}
\]

Indeed, from \(\omega_n \rightharpoonup \omega\) in \(H^1(\mathbb{R}^3)\), we can assume that,
\[
\omega_n \rightarrow \omega \quad \text{in} \quad L^s(\mathbb{R}^3), \quad \text{for} \quad 2 \leq s \leq 6.
\]

First, \(\phi_{\omega_n} \rightharpoonup \phi_{\omega}\) in \(D^{1,2}(\mathbb{R}^3)\) by Lemma 3 implies that
\[
\phi_{\omega_n} \rightarrow \phi_{\omega} \quad \text{in} \quad L^6(\mathbb{R}^3),
\]

and then, since \(K(x)\omega^2 \in L^{6/3}(\mathbb{R}^3)\) by Hölder’s inequality, we obtain that
\[
\int_{\mathbb{R}^3} K(x)(\phi_{\omega_n} - \phi_{\omega})\omega^2 dx = o(1), \tag{26}
\]

Thus by (26) and Hölder’s inequality, we have
\[
\int_{\mathbb{R}^3} (K(x)\phi_{\omega_n}\omega_n^2 - K(x)\phi_{\omega}\omega^2) dx = \int_{\mathbb{R}^3} K(x)\phi_{\omega_n}(\omega_n^2 - \omega^2) dx + \int_{\mathbb{R}^3} K(x)(\phi_{\omega_n} - \phi_{\omega})\omega^2 dx
\]
\[
\leq |\phi_{\omega_n}|_6 |\omega_n + \omega|_6 \int_{\mathbb{R}^3} |K(x)(\omega_n - \omega)|^{3/2} dx + o(1)
\]
\[
\leq C \int_{\mathbb{R}^3} |K(x)(\omega_n - \omega)|^{3/2} dx + o(1)
\]
\[
\rightarrow 0
\]

as \(n \rightarrow \infty\), since \(|\omega_n - \omega|^{3/2} \rightarrow 0\) in \(L^{4}(\mathbb{R}^3)\) and \(K^{3/2} \in L^{4/3}(\mathbb{R}^3)\). Thus, combine (24) and (25), we obtain
\[
\int_{\mathbb{R}^3} K(x)\phi_{\omega}\omega^2 dx = o(1),
\]

which implies that \(\omega \equiv 0\) by \((K_1)\) and Lemma 3. That is a contradiction. \(\blacksquare\)

**Proof of Theorem 2** Setting \(m := \min\{m_1, m_2\}\), then it follows from Lemmas 4, 5, 7 and 8 that problem (1) has a positive solution \(u_1 \in H^1(\mathbb{R}^3)\) with \(I(u_1) > 0\) for \(|h|_2 < m\) and \(|K|_2 < m\). So, the proof is completed by Theorem 6.

**References**


