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The Strong Limit Theorem for Nonhomogeneous N-Bifurcating Markov Chains Indexed by a N-Branch Cayley Tree

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Abstract: In this paper, by constructing a class of random variables with a parameter and the mean value of one, We establish a strong limit theorem for delayed sums of the multivariate functions of such chains using the Borel-Cantelli lemma.Meanwhile, we will give a strong law of large numbers in the form of moving average for this Markov model with finite state space.

Keywords: N-branch Cayley tree; Nonhomogeneous N-bifurcating Markov chains; Strong law of large numbers.

1 Introduction

Let $G = (V, E)$ be an undirected graph, wehre V is the set of vertices and E is the set of edges. We assume that there is no loop in G, i.e., there is no edge in G which connects a vertex to itself. A tree is a graph T which is connected and contains no circuits. Given any two vertices $\alpha \neq \beta \in T$. Let $\overline{\alpha\beta}$ be the unique path connecting α and β. Define the distance $d(\alpha, \beta)$ to be the number of edges contained in the path $\overline{\alpha\beta}$. Select a vertex as the root (denoted by o). For any two vertices σ and t of tree T, we write $\sigma \leq t$ if σ is on the unique path from the root o to t. We denote by $\sigma \wedge t$ the vertex farthest from o satisfying $\sigma \wedge t \leq t$ and $\sigma \wedge t \leq \sigma$. The set of all vertices with distance n from the root o is called the n-th level of T . We denote by L_n the set of all vertices on level n ($L_o = \{o\}$). We denote by L_m^n to be the set of all vertices from the mth to nth level of T, specially by $T^{(n)}$ to be the set of all vertices on level 0 (the root o) to level n. Let T be any tree and $t \in T\backslash\{o\}$.If a vertex in this tree is on the unique path from the root o to t and is the nearest to t, we call it the predecessor of t and denote it by 1_t, we also call t a successor of 1_t. If the root of a tree has N neighboring vertices and other vertices have $N + 1$ neighboring vertices, we call this type of tree a Cayley tree and denote it by $T_{C,N}$. That is, for any vertex t of Cayley tree $T_{C,N}$, it has N successors on the next level. In this paper,for simplicity, we denote $T_{C,N}$ by T_N (see Fig.1). For any vertex t of the tree T_N, we denote by $t^1, t^2, \cdots, t^N$ the N successors of t, and call them the first successor, the second successor, and so on, until the Nth successor of t, respectively. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and T be any tree,$\{X_t, t \in T\}$ be tree indexed stochastic processes defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Let A be the subgraph of T, $X^A = \{X_t, t \in A\}$.We denote by $|A|$ the number of vertices of A, x^A the realization of X^A.

Benjamini and Peres (1994) gave the definition of tree indexed Markov chain and studied the recurrence and ray-recurrence for them.Yang and Liu (2000) established the strong law of large numbers for frequency of state occurrence on Markov chains indexed by a homogenous tree (in fact,it is special case of tree-indexed Markov chains and PPG-invariant random field). Yang and Liu (2002) studied the strong law of large numbers and entropy ergodic theorem for Markov chain fields on trees (a particular case of tree-indexed Markov chains and PPG-invariant random field).Yang and Ye (2007) established the strong law of large numbers and the entropy ergodic theorem for finite -level non homogeneous Markov chains indexed by a homogeneous tree. Shi and Yang (2010) studied the AEP for high order nonhomogeneous Markov chains indexed by a tree.Dang, Yang and Shi (2015) studied the strong law of large numbers and the entropy ergodic theorem for non homogeneous bifurcating Markov chains indexed by a binary tree.Dang(2018) studied the strong law of large numbers for non homogeneous M-bifurcating Markov chains indexed by a M-branch Cayley tree. Shi,Wang,Zhong,and

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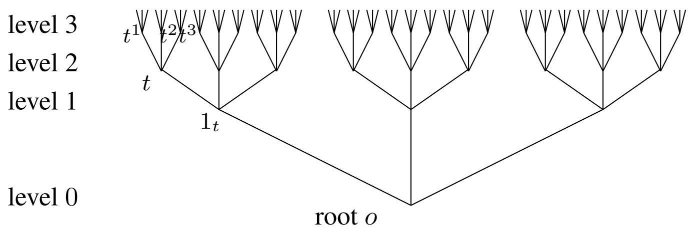

Figure 1: Binary tree $T_{C,3}$

Fan(2021) studied the generalized entropy ergodic theorem for non homogeneous bifurcating Markov chains indexed by a binary tree.

Inspired by Shi,Wang,et.al(2021) a strong law of large numbers for nonhomogeneous bifurcating Markov chains indexed by a binary branch tree.I study nonhomogeneous N-bifurcating Markov chains indexed by a N-branch tree by using a similar thought,and generalize the relate results of Shi,Wang,et.al(2021) and Dang(2018).

2 Definitions and Main Results

Definition 1 *(Dang, 2018) Let T_N be a N branch Cayley tree, S be a countable state pace, and $\{X_t, t \in T_N\}$ be a collection of S-valued random variables defined on probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Let*

$$p = \{p(x), x \in S\} \tag{1}$$

be a distribution on S, and

$$P_t = \big\{P_t(y_1, y_2, ..., y_N \mid x)\big\}, \ \ x, y_1, y_2, ..., y_N \in S, \ \ t \in T_N \tag{2}$$

be a collection of stochastic matrices (that is, $P_t(y_1, \cdots, y_N \mid x) \geq 0, \ \forall y_1, ..., y_N, x \in S$ and $\sum\limits_{(y_1,\cdots,y_N)\in S^N} P_t(y_1, ..., y_N \mid x) = 1, \ \forall x \in S$.) on $S \times S^N$. If $\forall n \geq 1$,

$$\mathbf{P}\left(X^{L_n} = x^{L_n} \mid X^{T^{(n-1)}} = x^{T^{(n-1)}}\right) = \prod_{t \in L_{n-1}} P_t\left(x_{t^1}, x_{t^2}, ..., x_{t^N} \mid x_t\right), \tag{3}$$

and

$$\mathbf{P}\left(X_o = x\right) = p\left(x\right), \ \ \forall x \in S. \tag{4}$$

$\{X_t, t \in T_N\}$ will be called S-valued nonhomogeneous N bifurcating Markov chains indexed by a N branch Cayley tree T_N with the initial distribution (1) *and transition matrices* (2).

Let $S = \{0, 1, \cdots, b-1\}$ be a finite state space, $\{X_t, t \in T_N\}$ be a S-valued nonhomogeneous N bifurcating Markov chains indexed by a N branch Cayley tree defined as before. For $\forall k \in S$, let $S_k(L_{a_n}^{a_n+\phi(n)})$ be the number of k in set of random variables $\{X_t, t \in L_{a_n}^{a_n+\phi(n)}\}$, and $S_k(L_{a_n})$ be the number of k in set of random variables $\{X_t, t \in L_{a_n}\}$, $S_k^i(L_{a_n}^{a_n+\phi(n)-1})$ be the number of k in set of random variables $\{X_{t^i} = k, t \in L_{a_n}^{a_n+\phi(n)-1}\}, i = 1, 2, \cdots, N$, that is,

$$S_k(L_{a_n}^{a_n+\phi(n)}) = |\{t \in L_{a_n}^{a_n+\phi(n)} : X_t = k\}|,$$
$$S_k(L_{a_n}) = |\{t \in L_{a_n} : X_t = k\}|,$$

and

$$S_k^i(L_{a_n}^{a_n+\phi(n)-1}) = |\{t \in L_{a_n}^{a_n+\phi(n)-1} : X_{t^i} = k\}|, \;\; i = 1, 2, \cdots, N.$$

It is easy to see that

$$S_k(L_{a_n}^{a_n+\phi(n)}) = \sum_{t \in L_{a_n}^{a_n+\phi(n)}} I_k(X_t), \tag{5}$$

$$S_k(L_{a_n}) = \sum_{t \in L_{a_n}} I_k(X_t), \tag{6}$$

$$S_k^i(L_{a_n}^{a_n+\phi(n)-1}) = \sum_{t \in L_{a_n}^{a_n+\phi(n)-1}} I_k(X_{t^i}), \quad i = 1, 2, \cdots, N, \tag{7}$$

and

$$S_k(L_{a_n}^{a_n+\phi(n)}) = \sum_{i=1}^{N} S_k^i(L_{a_n}^{a_n+\phi(n)-1}) + S_k(L_{a_n}), \tag{8}$$

where

$$I_k(i) = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases} \tag{9}$$

We will give a strong law of large numbers in the form of moving average for this Markov model with finite state space.

Theorem 1 *Let $S = \{0, 1, \cdots, b-1\}$ be a finite state space, and $\{X_t, t \in T_N\}$ be a S-valued N bifurcating Markov chains indexed by a N branch Cayley tree T_N with stochastic matrices $\{P_t, t \in T_N\}$ defined by Definition 1, $S_k(L_{a_n}^{a_n+\phi(n)})$ be defined by (5). Let $P = \{P(y_1, y_2, \cdots, y_N|x)\}, x, y_1, y_2, \cdots, y_N \in S$ be another stochastic matrix, and let $P_i(y_i|x) = \sum_{y_j: j \neq i} P(y_1, y_2, \cdots, y_N|x)$, $P_i = \{P_i(y|x)\}$, $i = 1, 2, \cdots, N$. Let $Q = \frac{1}{N}\sum_{i=1}^{N} P_i$, and assume that the transition matrix Q is ergodic. Let $\{a_n, n \geq 0\}$ and $\{\phi(n), n \geq 0\}$ be two nonnegative integer sequences such that for any positive integers n, m,*

$$\phi(m+n) - \phi(n) \geq m. \tag{10}$$

If $\forall x, y_1, y_2, \cdots, y_N \in S$,

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n-1)}} \left| P_t(y_1, y_2, \cdots, y_N|x) - P(y_1, y_2, \cdots, y_N|x) \right| = 0, \tag{11}$$

then

$$\lim_{n\to\infty} \frac{S_k(L_{a_n}^{a_n+\phi(n)})}{|L_{a_n}^{a_n+\phi(n)}|} = \pi(k) \;\; a.e. \;\; \forall k \in S, \tag{12}$$

where $\pi = \{\pi(0), \pi(1), \cdots, \pi(b-1)\}$ is the unique stationary distribution determined by the transition matrix Q.

The proof of the above theorem will be given in Section 3.

3 The Proofs

Before providing the proofs of the main results in Section 2, we begin with some lemmas.

Lemma 1 *Let T_N be a N branch Cayley tree, and S be a countable state space. Assuming that $\{X_t, t \in T_N\}$ be a S-valued N bifurcating nonhomogeneous Markov chain indexed by a N branch Cayley tree T_N defined by Definition 1, and $\{g_t(x, y_1, y_2, \cdots, y_N), t \in T_N\}$ be a collection of functions defined on G^{N+1}. Suppose that $\exists\alpha > 0$, s.t. $E[e^{\alpha|g_t(X_t, X_{t^1}, X_{t^2}, \cdots, X_{t^N})|}] < \infty, \forall t \in T_N$. Let $\{a_n, n \geq 0\}$ and $\{\phi(n), n \geq 0\}$ be two sequences of nonnegative integers such that $\phi(n)$ converges to infinity as $n \to \infty$. Assume that for some $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \exp(-|L_{a_n}^{a_n+\phi(n)}|\varepsilon) < \infty. \tag{13}$$

Let

$$H_{a_n,\phi(n)}(\omega) = \sum_{t \in L_{a_n}^{a_n+\phi(n)-1}} g_t(X_t, X_{t^1}, X_{t^2}, \cdots, X_{t^N}), \tag{14}$$

and

$$G_{a_n,\phi(n)}(\omega) = \sum_{t \in L_{a_n}^{a_n+\phi(n)-1}} E\big[g_t(X_t, X_{t^1}, X_{t^2}, \cdots, X_{t^N})|X_t\big]. \tag{15}$$

Let $\alpha > 0$, and set

$$D(\alpha) = \Bigg\{\omega : \limsup_{n\to\infty} \frac{1}{|L_{a_n}^{a_n+\phi(n)}|} \sum_{t \in L_{a_n+1}^{a_n+\phi(n)}} E\Big[g_t^2(X_t, X_{t^1}, X_{t^2}, \cdots, X_{t^N}) \cdot e^{\alpha|g_t(X_t, X_{t^1}, X_{t^2}, \cdots, X_{t^N})|}|X_t\Big] = M(\alpha;\omega) < \infty\Bigg\}. \tag{16}$$

Then we have

$$\lim_{n\to\infty} \frac{H_{a_n,\phi(n)}(\omega) - G_{a_n,\phi(n)}(\omega)}{|L_{a_n}^{a_n+\phi(n)}|} = 0 \quad a.e. \quad \omega \in D(\alpha). \tag{17}$$

Proof The proof of this theorem is similar to that of Lemma 1 of Shi,Wang,et.al(2021),so it is omitted here.

Lemma 2 *(Shi,Wang,et.al,2021) Let T_N be a N branch Cayley tree, $\{a_n, n \geq 0\}$ and $\{\phi(n), n \geq 0\}$ defined as in Lemma 1. Let $\{a_t, t \in T\}$ be a collection of real numbers, and a be a real number. If*

$$\lim_{n\to\infty} \frac{1}{|T_N^{(n)}|} \sum_{t \in T_N^{(n-1)}} |a_t - a| = 0, \tag{18}$$

then

$$\lim_{n\to\infty} \frac{1}{|L_{a_n}^{a_n+\phi(n)}|} \sum_{t \in L_{a_n}^{a_n+\phi(n)-1}} |a_t - a| = 0. \tag{19}$$

Now, we present the proof of Theorem 1 as follows.

Proof of Theorem 1 It is easy to see from (11) that $\lim_{n\to\infty}\phi(n)=\infty$ and (13) is satisfied. By (12) and Lemma 2, we have

$$\lim_{n\to\infty}\frac{1}{|L_{a_n}^{a_n+\phi(n)}|}\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\left|P_t(y_1,y_2,\cdots,y_{t^N}|x)-P(y_1,y_2,\cdots,y_{t^N}|x)\right|=0. \tag{20}$$

Let $g_t(x,y_1,y_2,\cdots,y_N)=I_k(y_1)$ in Lemma 1. Obviously, $\{g_t(x,y_1,y_2,\cdots,y_N),t\in T_N\}$ are uniformly bounded. Since

$$H_{a_n,\phi(n)}(\omega)=\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}I_k(X_{t^1})=S_k^1(L_{a_n}^{a_n+\phi(n)-1}), \tag{21}$$

and

$$\begin{aligned}G_{a_n,\phi(n)}(\omega)&=\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}E[g_t(X_t,X_{t^1},X_{t^2},\cdots,X_{t^N})|X_t]\\&=\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{(x_{t^1},x_{t^2},\cdots,x_{t^N})\in S^N}g_t(X_t,x_{t^1},x_{t^2},\cdots,x_{t^N})P_t(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t)\\&=\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{(x_{t^1},x_{t^2},\cdots,x_{t^N})\in S^N}I_k(x_{t^1})P_t(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t).\end{aligned} \tag{22}$$

From Lemma 1, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{|L_{a_n}^{a_n+\phi(n)}|}\Big\{S_k^1(L_{a_n}^{a_n+\phi(n)-1})\\&-\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{(x_{t^1},x_{t^2},\cdots,x_{t^N})\in S^N}I_k(x_{t^1})P_t(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t)\Big\}=0\ \ a.e..\end{aligned} \tag{23}$$

From (20), it can be easily verified that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{|L_{a_n}^{a_n+\phi(n)}|}\Big\{&\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{(x_{t^1},x_{t^2},\cdots,x_{t^N})\in S^N}I_k(x_{t^1})P_t(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t)\\&-\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{(x_{t^1},x_{t^2},\cdots,x_{t^N})\in S^N}I_k(x_{t^1})P(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t)\Big\}=0.\end{aligned} \tag{24}$$

Since $\sum_{(x_{t^2},\cdots,x_{t^N})\in S^{N-1}}P(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t)=P_1(x_{t^1}|X_t)$, so

$$\begin{aligned}&\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{(x_{t^1},x_{t^2},\cdots,x_{t^N})\in S^N}I_k(x_{t^1})P(x_{t^1},x_{t^2},\cdots,x_{t^N}|X_t)\\&=\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}P_1(k|X_t)\\&=\sum_{t\in L_{a_n}^{a_n+\phi(n)-1}}\sum_{l=0}^{b-1}I_l(X_t)P_1(k|l)\\&=\sum_{l=0}^{b-1}P_1(k|l)S_l(L_{a_n}^{a_n+\phi(n)-1}).\end{aligned} \tag{25}$$

By (23), (24) and (25), we have

$$\lim_{n\to\infty}\frac{1}{|L_{a_n}^{a_n+\phi(n)}|}\Big\{S_k^1(L_{a_n}^{a_n+\phi(n)-1})-\sum_{l=0}^{b-1}P_1(k|l)S_l(L_{a_n}^{a_n+\phi(n)-1})\Big\}=0 \quad a.e.. \tag{26}$$

Let $g_t(x,y_1,y_2,\cdots,y_N)=I_k(y_j), j=2,\cdots,N$ in Lemma 1, similarly, we obtain that

$$\lim_{n\to\infty}\frac{1}{|L_{a_n}^{a_n+\phi(n)}|}\Big\{S_k^j(L_{a_n}^{a_n+\phi(n)-1})-\sum_{l=0}^{b-1}P_j(k|l)S_l(L_{a_n}^{a_n+\phi(n)-1})\Big\}=0 \quad a.e., \quad j=2,\cdots,N. \tag{27}$$

Adding (26) and (27), and noticing that

$$0\le\lim_{n\to\infty}\frac{S_k(L_{a_n})}{|L_{a_n}^{a_n+\phi(n)}|}\le\lim_{n\to\infty}\frac{|L_{a_n}|}{|L_{a_n}^{a_n+\phi(n)}|}=\lim_{n\to\infty}\frac{N^{a_n}}{N^{a_n}(N^{\phi(n)}-1)}=0,$$

$\lim\limits_{n\to\infty}\frac{|L_{a_n}^{a_n+\phi n)}|}{|L_{a_n}^{a_n+\phi(n)-1}|}=N$, and $Q=\frac{1}{N}(P_1+P_2+\cdots+P_N)$. By (10), we have

$$\lim_{n\to\infty}\left\{\frac{S_k(L_{a_n}^{a_n+\phi(n)})}{|L_{a_n}^{a_n+\phi(n)}|}-\sum_{l=0}^{b-1}Q(k|l)\frac{S_l(L_{a_n}^{a_n+\phi(n)-1})}{|L_{a_n}^{a_n+\phi(n)-1}|}\right\}=0 \quad a.e.. \tag{28}$$

Letting $\phi'(n)=\phi(n)-1$, it is easy to see that $\{\phi'(n),n\ge0\}$ also satisfies (13). Using the same argument as that used to derive (28), we can prove that

$$\lim_{n\to\infty}\left\{\frac{S_k(L_{a_n}^{a_n+\phi(n)-1})}{|L_{a_n}^{a_n+\phi(n)-1}|}-\sum_{l=0}^{b-1}Q(k|l)\frac{S_l(L_{a_n}^{a_n+\phi(n)-2})}{|L_{a_n}^{a_n+\phi(n)-2}|}\right\}=0 \quad a.e.. \tag{29}$$

Multiplying the k-th equality of (29) by $Q(j|k)$, adding them together and using (28), we have

$$\begin{aligned}
0 &= \lim_{n\to\infty}\left[\sum_{k=0}^{b-1}\frac{S_k(L_{a_n}^{a_n+\phi(n)-1})}{|L_{a_n}^{a_n+\phi(n)-1}|}Q(j|k)-\sum_{k=0}^{b-1}\sum_{l=0}^{b-1}\frac{S_l(L_{a_n}^{a_n+\phi(n)-2})}{|L_{a_n}^{a_n+\phi(n)-2}|}Q(k|l)Q(j|k)\right]\\
&= \lim_{n\to\infty}\left\{\left[\sum_{k=0}^{b-1}\frac{S_k(L_{a_n}^{a_n+\phi(n)-1})}{|L_{a_n}^{a_n+\phi(n)-1}|}Q(j|k)-\frac{S_j(L_{a_n}^{a_n+\phi(n)})}{|L_{a_n}^{a_n+\phi(n)}|}\right]\right.\\
&\quad\left.+\left[\frac{S_j(L_{a_n}^{a_n+\phi(n)})}{|L_{a_n}^{a_n+\phi(n)}|}-\sum_{k=0}^{b-1}\sum_{l=0}^{b-1}\frac{S_l(L_{a_n}^{a_n+\phi(n)-2})}{|L_{a_n}^{a_n+\phi(n)-2}|}Q(k|l)Q(j|k)\right]\right\}\\
&= \lim_{n\to\infty}\left[\frac{S_j(L_{a_n}^{a_n+\phi(n)})}{|L_{a_n}^{a_n+\phi(n)}|}-\sum_{l=0}^{b-1}\frac{S_l(L_{a_n}^{a_n+\phi(n)-2})}{|L_{a_n}^{a_n+\phi(n)-2}|}Q^{(2)}(j|l)\right] \quad a.e..
\end{aligned} \tag{30}$$

where $Q^{(h)}(j|l)$ is the h-step transition probability determined by Q. By induction, we have

$$\lim_{n\to\infty}\left[\frac{S_j(L_{a_n}^{a_n+\phi(n)})}{|L_{a_n}^{a_n+\phi(n)}|}-\sum_{l=0}^{b-1}\frac{S_l(L_{a_n}^{a_n+\phi(n)-h})}{|L_{a_n}^{a_n+\phi(n)-h}|}Q^{(h)}(j|l)\right]=0 \quad a.e.. \tag{31}$$

Noticing that

$$\frac{1}{|L_{a_n}^{a_n+\phi(n)-h}|}\sum_{l=0}^{b-1}S_l(L_{a_n}^{a_n+\phi(n)-h})=1, \tag{32}$$

and

$$\lim_{N\to\infty}Q^{(h)}(j|l)=\pi(j), \quad j\in S. \tag{33}$$

(12) follows from (31), (32) and (33).

This completes the proof of Theorem 1. □

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