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The Regularity of the Solutions for a 2D Stochastic Cahn-Hilliard-Navier-Stokes Model

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Abstract: In this paper, the stochastic 2D Cahn-Hilliard-Navier-Stokes equations have been considered. On the basis of [19], using Itó formula and a priori estimates, the regularity as well as applications to the solutions in a higher regular space of the equations have been obtained.

Keywords: Cahn-Hilliard-Navier-Stokes; A priori estimates; Regularity.

1 Introduction

Many industrial problems involve multi-phase flows, phase field models have developed rapidly in recent years. Multi-phase flows based on the phase field model usually consist of the following two equations: the incompressible Navier-Stokes equations for governing fluid flows and the Cahn-Hilliard equation for the interface evolution. This system describes the evolution of an incompressible isothermal mixture of binary fluids and it has been investigated by many articles and the references therein. We mention some of them, the global existence of weak solutions to nonlocal Cahn-Hilliard-Navier-Stokes (CH-NS) systems was obtained in [3]. The existence of strong solutions and optimal distributed control to two dimensional nonlocal CH-NS systems have been studied in [11, 12, 14]. Frigeri et al. [13] discussed the three dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with shear dependent viscosity. Gal and Grasselli [15] discussed asymptotic behavior of the CH-NS systems. Medjo [20, 21] obtained the existence of pullback attractors for the non-autonomous CH-NS systems. For the compressible case, Feireisl et al. [8] studied the global existence of weak solutions and Chen et al. [2] proved asymptotic stability of solutions for one dimensional compressible Navier-Stokes-Cahn-Hilliard systems.

When the perturbation of stochastic noise is taken into account, the existence of a random attractor of the CH-NS systems driven by the small additive noise was obtained in [18]. Medjo [19] discussed on the existence and uniqueness of solutions to a stochastic 2D Cahn-Hilliard-Navier-Stokes model. Recently, Deugoué and Medjo [4, 6] studied the exponential behavior and convergence of the solutions of the stochastic globally modified CH-NS equations. For the asymptotic properties of the distribution of the solution as the noise coefficient tends to zero, Deugoué and Medjo [5] obtained the large deviation for the 2D CH-NS systems when the first equation of (1) driven by a multiplicative noise. As an improvement of [5], Qiu and Wang [22] established a large deviation for the 2D CH-NS systems when the order parameter ϕ was also disturbed by multiplicative noise.

It is known that the regularity of the solution for a hydrodynamic system is useful not only for proving the ergodicity of the equations, but also for the error estimates of the time discrete scheme of the equations. The regularity analysis and different numerical approximations of the Cahn-Hilliard equation have been developed(e.g.[1, 7, 10, 17]). Furthermore, [9] proved H^1-regularity results of the solution for the determination Cahn-Hilliard-Navier-Stokes equations. [16] extended the H^1-regularity results to the H^2-regularity on the solution and its time derivative in terms of powers of ϵ^{-1} independent of t. In this paper, as the perturbation of stochastic noise is taken into account, on this basis of [19], in which the well-posedness of the weak solution to stochastic 2D CH-NS equations has been proved, we study the well-posedness of the solution in the higher regular space.

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This paper is organized as follows. Firstly, the preliminary knowledge is given in §2. Then, by Itô formula and a priori estimates, the regularity as well as applications of the stochastic 2D Cahn-Hilliard-Navier-Stokes equation have been obtained in §3.

2 Preliminaries

In this paper, we assume the domain $\mathcal{M}$ is a bounded domain in $\mathbb{R}^2$. Then, we consider the following 2D stochastic Cahn-Hilliard-Navier-Stokes model([15, 16, 19])

$$\begin{cases} \dfrac{\partial v}{\partial t} = \nu_1 \Delta v - (v \cdot \nabla)v - \nabla p + k\mu\nabla\phi + g_1(t,v) + g_2(t,v)\dot{W}_t, \\ \nabla \cdot v = 0, \\ \dfrac{\partial \phi}{\partial t} = -v \cdot \nabla\phi + \nu_3\Delta\mu, \\ \mu = -\nu_2\Delta\phi + \alpha f(\phi), \\ v = 0, \partial_\eta\phi = 0, \partial_\eta\Delta\phi = 0, \partial_\eta\mu = 0 \text{ on } \partial\mathcal{M} \times (0,\infty), \\ (v,\phi)(0) = (v_0,\phi_0) \text{ in } \mathcal{M}, \end{cases} \tag{1}$$

where $f(\phi) = F'(\phi)$ and $F(\phi) = \frac{1}{4}(\phi^2-1)^2$. $v = (v_1, v_2)$ is the velocity of the fluid, p is the pressure, ϕ is the phase parameter, $g_1(t,v)$ is an external forcing term, $g_2(t,v)\dot{W}_t$ is a random external force, where $\dot{W}_t$ is a Wiener process, $\nu_1, \nu_2, \nu_3, k, \alpha$ are constant.

From(1), we deduce the conservation of the following quantity(see [6, 15, 19])

$$\langle\phi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \phi(x,t)dx,$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. So we have

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle, \forall t \geq 0.$$

For the sake of convenience, we set

$$\langle\phi(t)\rangle = \langle\phi(0)\rangle = 0, \forall t \geq 0.$$

We set

$$\vartheta = \{u \in C_c^\infty(\mathcal{M}) : \nabla \cdot u = 0, \text{in } \mathcal{M}\},$$

the closure of ϑ in $(L^2(\mathcal{M}))^2$ be denoted by $\mathcal{H}_1$ and the closure of ϑ in $(H_0^1(\mathcal{M}))^2$ be denoted by V_1. $(\cdot,\cdot)_{L^2}$ indicates the scalar product of $\mathcal{H}_1$, and $|\cdot|_{L^2}$ indicates the norm of $\mathcal{H}_1$. The scalar product of V_1 is given by

$$((u,v)) = \sum_{i=1}^{2}(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i})_{L^2}, ||u|| = ((u,u))^{1/2}.$$

We introduce the operator A_0:

$$A_0 v = -P\Delta v, \forall v \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where P is the Leray-Helmotz projector. A_0 is a self-adjoint positive unbounded operator in $\mathcal{H}_1$ and A_0^{-1} is a compact linear operator on $\mathcal{H}_1$.

The linear nonnegative unbounded operator A_1 on $L^2(\mathcal{M})$ is defined by

$$A_1\phi = -\Delta\phi, \forall\phi \in D(A_1) = \{\phi \in H^2(\mathcal{M}), \frac{\partial\phi}{\partial\eta} = 0, \text{ on } \partial\mathcal{M}\},$$

the norm of $D(A_1)$ is given by $|A_1 \cdot|_{L^2} + |\langle\cdot\rangle|$, which is equivalent to the H^2-norm.

As in [15, 19], we introduce the linear positive unbounded operator B on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2-functions with null mean

$$B\phi = -\Delta\phi, \forall\phi \in D(B) = D(A_1) \cap L_0^2(\mathcal{M}),$$

where B^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B^s, for any $s \in R$, $|B^{s/2} \cdot|_{L^2}, s > 0$, which is an equivalent norm to the canonical H^s-norm on $D(B^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Note that $A_1 = B$ on $D(B)$.

Let

$$H_2 = D(B^0) = L_0^2(\mathcal{M}), V_2 = D(B^{1/2}),$$

$|\cdot|_{L^2}$ indicates the norm of H_2, and $||\cdot||$ indicates the norm of V_2, where $||\psi|| = |B^{1/2}\psi|_{L^2}$.

We define the bilinear operators B_0, B_1 as well as the coupling mapping R_0 and set

$$\begin{aligned}
&\langle B_0(u,v), w\rangle = \int_M [(u\cdot\nabla)v]\cdot w dx = b_0(u,v,w), \forall u,v,w \in D(A_0),\\
&\langle B_1(u,\phi), \rho\rangle = \int_M [(u\cdot\nabla)\phi]\rho dx = b_1(u,\phi,\rho), \forall u \in D(A_0), \phi,\rho \in D(A_1),\\
&\langle R_0(\mu,\phi), w\rangle = \int_M \mu[\nabla\phi\cdot w]dx = b_1(w,\phi,\mu),\\
&\quad \forall w \in D(A_0), \phi \in D(A_1)\cap H^3(\mathcal{M}), \mu \in L^2(\mathcal{M}).
\end{aligned}$$

Let

$$\mathbb{Y} = \mathcal{H}_1 \times D(B^{1/2}),$$

the norm of the complete metric space $\mathbb{Y}$ is given by

$$|(v,\phi)|_{\mathbb{Y}}^2 = k^{-1}|v|_{L^2}^2 + \nu_2|\nabla\phi|_{L^2}^2.$$

We define the Hilbert space $\mathbb{V}$ to be $V_1 \times D(B)$, the norm is that

$$||(v,\phi)||_{\mathbb{V}}^2 = ||v||^2 + |B\phi|_{L^2}^2.$$

For simplicity of notation, we set $\nu_1 = \nu_2 = \nu_3 = \alpha = k = 1$. Therefore, we rewrite (1) as

$$\begin{cases}
dv = [-A_0 v - B_0(v,v) + R_0(\mu,\phi) + g_1(t,v)]dt + g_2(t,v)dW\\
d\phi = -[A_1\mu + B_1(v,\phi)]dt\\
\mu = A_1\phi + f(\phi)\\
(v,\phi)(0) = (v_0,\phi_0),
\end{cases} \tag{2}$$

where $R_0(\mu,\phi) = k\mu\nabla\phi$.

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space and we define an increasing and right-continuous family $\mathfrak{F}_{t\in[0,T]}$ of subset of $\mathfrak{F}$, where $\mathfrak{F}_0$ is the set of zero probability in $\mathfrak{F}$. Let $\{\beta_t^j, t \geq 0, j = 1,2,...\}$ ba a sequence of mutually independent standrad real $\mathfrak{F}_t$-Wiener processes, we define $\Re$ is a separable Hilbert space, and let

$$W_t = \sum_{j=1}^{\infty} \beta_t^j e_j,$$

where $\{e_j, j = 1,2,...\}$ be an orthonormal basis of $\Re$.

We define X is a separable Banach space and for $\forall p \in [1,\infty]$, the space $M_{\mathfrak{F}_t}^p(0,T;X)$ is a Banach subspace of $L^p(\Omega\times(0,T), d\mathbb{P}\times dt; X)$, where the space $M_{\mathfrak{F}_t}^p(0,T;X)$ of all processes $\rho \in L^p(\Omega\times(0,T), d\mathbb{P}\times dt; X)$ are $\mathfrak{F}_t$-progressively measureable. For $\forall 1 \leq p < \infty$, the space $L^p(\Omega; C([0,T];X))$ indicates all continuous and $\mathfrak{F}_t$-progressively measurable X-valued processes $\{\rho_t; 0 \leq t \leq T\}$, satisfying

$$\mathbb{E}(\sup_{t\in[0,T]} ||\rho_t||_X^p) < \infty.$$

Let Z is a Hilbert space, we define the separable Hilbert space $\mathfrak{L}^2(\Re; Z)$ is the Hilbert-Schmidt operators from $\Re$ into Z. $((\cdot,\cdot))_{\mathfrak{L}^2(\Re;Z)}$ denotes the scalar product in $\mathfrak{L}^2(\Re;Z)$, and $||\cdot||_{\mathfrak{L}^2(\Re;Z)}$ denotes the associated norm in $\mathfrak{L}^2(\Re;Z)$.

Assumption 1. *We assume that* $\nabla g_1 : \Omega \times (0,T) \times \mathcal{H}_1 \longrightarrow V_1^*$, $\nabla g_2 : \Omega \times (0,T) \times \mathcal{H}_1 \longrightarrow \mathfrak{L}^2(\Re; \mathcal{H}_1)$, $\nabla g_1, \nabla g_2$ *are measurable and satisfy Lipschitz conditions, for any* $v_1, v_2 \in V_1$

$$\|\nabla g_1(t, v_1) - \nabla g_1(t, v_2)\|_{V_1^*} \leq L_1 |\nabla v_1 - \nabla v_2|_{L^2}, \tag{3}$$

$$\nabla g_1(t, 0) \in M^2_{\mathfrak{F}_t}(0, T; V_1^*), \tag{4}$$

$$\|\nabla g_2(t, v_1) - \nabla g_2(t, v_2)\|_{\mathfrak{L}^2(\Re; \mathcal{H}_1)} \leq L_2 |\nabla v_1 - \nabla v_2|_{L^2}, \tag{5}$$

$$\nabla g_2(t, 0) \in M^2_{\mathfrak{F}_t}(0, T; \mathfrak{L}^2(\Re; \mathcal{H}_1)). \tag{6}$$

For $\forall (w, \psi) \in \mathbb{Y}$,

$$\varepsilon(w, \psi) = |w|^2_{L^2} + ||\psi||^2 + 2\langle F(\psi), 1\rangle + c,$$

where c is a constant large enough.

As in [15, 19], there exists a monotone non-decreasing function such that

$$|(w, \psi)|^2_{\mathbb{Y}} \leq \varepsilon(w, \psi) \leq Q(|(w, \psi)|^2_{\mathbb{Y}}), \forall (w, \psi) \in \mathbb{Y}.$$

3 Regularity

The global well-posedness of the weak solution to the stochastic 2D Cahn-Hilliard-Navier-Stokes equations has been proved in [19].

Lemma 1 (see [19]) *We suppose that* $g_1(\cdot, 0) \in L^2(\Omega, (0, T; V_1^*)$, $g_2(\cdot, 0) \in L^2(\Omega, (0, T; \mathfrak{L}^2(\Re; \mathcal{H}_1))$ *and* $(v_0, \phi_0) \in L^2(\Omega, \mathfrak{F}_0, \mathbb{P}; \mathbb{Y})$ *satisfies* $\mathbb{E}[\varepsilon(v_0, \phi_0)] < \infty$. *Then, there exists a unique solution* $(v, \phi) \in L^2(\Omega, C(0, T; \mathbb{Y})) \cap L^2(\Omega, (0, T; \mathbb{V}))$.

Moreover, the following estimate holds:

$$\begin{aligned} &\mathbb{E}(\sup_{[0,T]} |(v, \phi)|^2_{\mathbb{Y}}) + \mathbb{E}\big(\int_0^T ||(v, \phi)||^2_{\mathbb{V}} dt\big) \leq c\mathbb{E}[\varepsilon(v_0, \phi_0)] + c\mathbb{E}(\int_0^T \|g_1(t, 0)\|^2_{V_1^*} dt) \\ &+ c\mathbb{E}(\int_0^T \|g_2(t, 0)\|^2_{\mathfrak{L}^2(\Re; \mathcal{H}_1)} dt). \end{aligned} \tag{7}$$

Next, we prove the regularity of the solution to a stochastic Cahn-Hilliard-Navier-Stokes equations.

Theorem 2 *Based on the Lemma 1, furthermore, we assume* $(v_0, \phi_0) \in L^2(\Omega, \mathfrak{F}_0, \mathbb{P}; \mathbb{V})$ *and Assumption 1 is satisfied, then, there exists a unique solution*

$$(v, \phi) \in L^2(\Omega, C(0, T; \mathbb{V})) \cap L^2(\Omega, (0, T; B^{\frac{1}{2}}(\mathbb{V}))).$$

Moreover, for any $T > 0$, $\exists\, N$, *the following estimate holds:*

$$\begin{aligned} &\mathbb{E} \sup_{[0,T]} (\|v\|^2_{L^2} + |\Delta\phi|^2_{L^2} + |\mu|^2_{L^2}) + \mathbb{E}\big(\int_0^T (|\Delta v|^2_{L^2} + |\phi_t|^2_{L^2} + |\Delta\mu|^2_{L^2}) ds \\ &\leq ce^{2N} \mathbb{E}(\|v_0\|^2_{L^2} + |\Delta\phi_0|^2_{L^2} + |\mu_0|^2_{L^2}) + c\mathbb{E}(\int_0^T \|\nabla g_1(t, 0)\|^2_{V_1^*} dt) \\ &+ c\mathbb{E}(\int_0^T \|\nabla g_2(t, 0)\|^2_{\mathfrak{L}^2(\Re; \mathcal{H}_1)} dt). \end{aligned} \tag{8}$$

Proof. From$(2)_1$ and the Itô formula, we derive that

$$\begin{aligned}&|\nabla v|_{L^2}^2+2\int_0^t|\Delta v|_{L^2}^2ds=|\nabla v_0|_{L^2}^2+2\int_0^t\langle B_0(v,v),\Delta v\rangle ds\\&-2\int_0^t\langle R_0(\mu,\phi),\Delta v\rangle ds+2\int_0^t\langle\nabla g_1(s,v),\nabla v\rangle ds+2\int_0^t\langle\nabla g_2(s,v),\nabla v\rangle dW_s\\&+\int_0^t\|\nabla g_2(s,v)\|_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}^2ds.\end{aligned}\tag{9}$$

Taking the scalar product in $(2)_2$ with $2\phi_t-2\Delta\mu$, we have

$$2|\phi_t|_{L^2}^2-4\langle\phi_t,\Delta\mu\rangle+2|\Delta\mu|_{L^2}^2+2\langle B_1(v,\phi),\phi_t-\Delta\mu\rangle=0.\tag{10}$$

Taking the scalar product in $(2)_3$ with $2\Delta\phi_t$, we obtain that

$$2\langle\nabla\mu,\nabla\phi_t\rangle=\frac{d|\Delta\phi|^2}{dt}+2\langle f'(\phi)\nabla\phi,\nabla\phi_t\rangle.\tag{11}$$

Differentiating $(2)_3$, we deduce

$$\mu_t=A_1\phi_t+f'(\phi)\phi_t.\tag{12}$$

Taking the scalar product in (12) with 2μ, we get

$$\frac{d|\mu|_{L^2}^2}{dt}+2\langle\Delta\mu,\phi_t\rangle-2\langle f'(\phi)\phi_t,\mu\rangle=0.\tag{13}$$

Integrating the (10), (11) and (13) from 0 to t, adding the resulting equalities to (9), we have that

$$\begin{aligned}&|\nabla v|_{L^2}^2+|\Delta\phi|_{L^2}^2+|\mu|_{L^2}^2+2\int_0^t(|\Delta v|_{L^2}^2+|\Delta\mu|_{L^2}^2+|\phi_t|_{L^2}^2)ds\\&=|\nabla v_0|_{L^2}^2+|\Delta\phi_0|_{L^2}^2+|\mu_0|_{L^2}^2+2\int_0^t\langle B_0(v,v),\Delta v\rangle ds-2\int_0^t\langle R_0(\mu,\phi),\Delta v\rangle ds\\&+2\int_0^t\langle\nabla g_1(s,v),\nabla v\rangle ds+2\int_0^t\langle\nabla g_2(s,v),\nabla v\rangle dW_s+\int_0^t\|\nabla g_2(s,v)\|_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}^2ds\\&-2\int_0^t\langle B_1(v,\phi),\phi_s-\Delta\mu\rangle ds-2\int_0^t\langle f'(\phi)\nabla\phi,\nabla\phi_s\rangle ds+2\int_0^t\langle f'(\phi)\phi_s,\mu\rangle ds.\end{aligned}\tag{14}$$

Note that

$$\begin{aligned}|\langle B_0(v,v),\Delta v\rangle|&\le|v|_{L^4}|\nabla v|_{L^4}|\Delta v|_{L^2}\le\frac{1}{8}|\Delta v|_{L^2}^2+c|v|_{L^2}^2\|v\|_{L^2}^4\\&\le\frac{1}{8}|\Delta v|_{L^2}^2+c|v|_{L^2}^2\|v\|_{L^2}^2(\|v\|_{L^2}^2+|\Delta\phi|_{L^2}^2+|\mu|_{L^2}^2),\end{aligned}\tag{15}$$

$$\begin{aligned}|\langle B_1(v,\phi),\phi_t-\Delta\mu\rangle|&\le|v|_{L^4}|\nabla\phi|_{L^4}|\phi_t-\Delta\mu|_{L^2}\\&\le\frac{1}{4}|\phi_t|_{L^2}^2+\frac{1}{2}|\Delta\mu|_{L^2}^2+c|v|_{L^2}^2\|v\|_{L^2}^2\\&+c\|\phi\|_{L^2}^2(|\Delta\phi|_{L^2}^2+\|v\|_{L^2}^2+|\mu|_{L^2}^2),\end{aligned}\tag{16}$$

$$\begin{aligned}|\langle R_0(\mu,\phi),\Delta v\rangle|&\le|\Delta v|_{L^2}|\nabla\phi|_{L^4}|\mu|_{L^4}\\&\le\frac{1}{8}|\Delta v|_{L^2}^2+c(\|\mu\|_{L^2}^2+\|\phi\|_{L^2}^2)(|\Delta\phi|_{L^2}^2+\|v\|_{L^2}^2+|\mu|_{L^2}^2),\end{aligned}\tag{17}$$

$$\begin{aligned}|\langle f'(\phi)\nabla\phi,\nabla\phi_t\rangle|&=|\langle f''(\phi)(\nabla\phi)^2+f'(\phi)\Delta\phi,\phi_t\rangle|\\&\le\frac{1}{8}|\phi_t|_{L^2}^2+c|\phi|_{L^2}^2\|\phi\|^4+\frac{1}{8\gamma}|\Delta\phi|_{L^2}^2\\&\le\frac{1}{8}|\phi_t|_{L^2}^2+c|\phi|_{L^2}^2\|\phi\|^4+c(|\Delta\phi|_{L^2}^2+\|v\|_{L^2}^2+|\mu|_{L^2}^2),\end{aligned}\tag{18}$$

$$\begin{aligned}\|\nabla g_2(s,v)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)} &\le 2cL_2^2|\nabla v|^2_{L^2} + c\|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)} \\ &\le 2cL_2^2(|\Delta\phi|^2_{L^2} + \|v\|^2_{L^2} + |\mu|^2_{L^2}) + c\|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)},\end{aligned} \tag{19}$$

$$\begin{aligned}|\langle\nabla g_1(s,v),\nabla v\rangle| &\le L_1\|v\||\Delta v| + c\|\nabla g_1(s,0)\|_{V_1^*}|\Delta v| \\ &\le \frac{1}{4}|\Delta v|^2_{L^2} + cL_1^2\|v\|^2_{L^2} + c\|\nabla g_1(s,0)\|^2_{V_1^*} \\ &\le \frac{1}{4}|\Delta v|^2_{L^2} + cL_1^2(|\Delta\phi|^2_{L^2} + \|v\|^2_{L^2} + |\mu|^2_{L^2}) + c\|\nabla g_1(s,0)\|^2_{V_1^*},\end{aligned} \tag{20}$$

$$\begin{aligned}|\langle f'(\phi)\phi_t,\mu\rangle| &= |\langle(3\phi^2-1)\phi_t,\mu\rangle| \\ &\le 3|\phi^2-1|_{L^2}|\phi_t|_{L^\infty}|\mu|_{L^2} + 2|\phi_t|_{L^2}|\mu|_{L^2} \\ &\le \frac{1}{8}|\phi_t|^2_{L^2} + c(|\Delta\phi|^2_{L^2} + \|v\|^2_{L^2} + |\mu|^2_{L^2}).\end{aligned} \tag{21}$$

By the Lemma 1, we can define a stopping time τ_N such that

$$\tau_N = inf\{t \le T; \int_0^t (|v|^2\|v\|^2 + \|\mu\|^2_{L^2} + 2\|\phi\|^2_{L^2} + L_1^2 + L_2^2 + 2)ds \ge N\}.$$

Set

$$K(t) = c(|v|^2\|v\|^2 + \|\mu\|^2_{L^2} + 2\|\phi\|^2_{L^2} + L_1^2 + L_2^2 + 2),$$
$$Y(t) = \|v\|^2_{L^2} + |\Delta\phi|^2_{L^2} + |\mu|^2_{L^2},$$
$$\sigma(t) = exp(-2\int_0^t K(s)ds).$$

Applying Itô formula to $\sigma(t)Y(t)$ and from $(15)-(21)$, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t\wedge\tau_N]}\sigma(s)Y(s) + \mathbb{E}\int_0^{t\wedge\tau_N}\sigma(s)[|\Delta v|^2_{L^2} + |\phi_t|^2_{L^2} + |\Delta\mu|^2_{L^2}]ds \\ &\le \mathbb{E}Y(0) + 2c\mathbb{E}\int_0^{t\wedge\tau_N}\sigma(s)\|\nabla g_1(s,0)\|^2_{V_1^*}ds \\ &+ c\mathbb{E}\int_0^{t\wedge\tau_N}\sigma(s)\|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}ds + 2\mathbb{E}\int_0^{t\wedge\tau_N}\sigma(s)\langle\nabla g_2(s,v),\nabla v\rangle dW_s.\end{aligned} \tag{22}$$

By the Burkholder-Davis-Gundy inequality, we can deduce that

$$\begin{aligned}&\mathbb{E}\int_0^{t\wedge\tau_n}\langle\nabla g_2(s,v),\nabla v\rangle dW_s \le c\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}|\nabla v|\int_0^{t\wedge\tau_n}\|\nabla g_2(s,v)\|_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}ds \\ &\le cL_2^2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}|\nabla v|^2_{L^2} + c\mathbb{E}\int_0^{t\wedge\tau_n}\|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}ds \\ &\le cL_2^2\mathbb{E}\sup_{s\in[0,t\wedge\tau_n]}Y(s) + c\mathbb{E}\int_0^{t\wedge\tau_n}\|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}ds.\end{aligned} \tag{23}$$

Then

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t\wedge\tau_N]}\sigma(s)Y(s) + \mathbb{E}\int_0^{t\wedge\tau_N}\sigma(s)[|\Delta v|^2_{L^2} + |\phi_t|^2_{L^2} + |\Delta\mu|^2_{L^2}]ds \le \mathbb{E}Y(0) \\ &+ 2c\mathbb{E}\int_0^{t\wedge\tau_N}\sigma(s)[\|\nabla g_1(s,0)\|^2_{V_1^*} + \|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}]ds.\end{aligned} \tag{24}$$

For any given time T, Let N be large enough, using Lemma 1 and the definition of τ_N, we can get $T\wedge\tau_N = T$, and

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]}Y(s) + \mathbb{E}\int_0^T[|\Delta v|^2_{L^2} + |\phi_t|^2_{L^2} + |\Delta\mu|^2_{L^2}]ds \le ce^{2N}\mathbb{E}Y(0) \\ &+ 2c\mathbb{E}\int_0^T[\|\nabla g_1(s,0)\|^2_{V_1^*} + \|\nabla g_2(s,0)\|^2_{\mathfrak{L}^2(\Re;\mathcal{H}_1)}]ds.\end{aligned} \tag{25}$$

Therefore, the proof of the Theorem 2 has been completed. ■

Remark 3 *Medjo [19] proved* $(v,\phi)\in L^2(\Omega, C(0,T;\mathbb{Y}))\cap L^2(\Omega,(0,T;\mathbb{V}))$, $\mu\in L^2(\Omega,(0,T;D(B^{\frac{1}{2}})))$. *In this paper, we obtain the following regularity*

$$(v,\phi)\in L^2(\Omega, C(0,T;\mathbb{V}))\cap L^2(\Omega,(0,T;B^{\frac{1}{2}}(\mathbb{V}))),$$

and $\mu\in L^2(\Omega, C(0,T;\mathcal{H}_1))\cap L^2(\Omega,(0,T;D(B)))$, $\phi_t\in L^2(\Omega,(0,T,\mathcal{H}_1))$ *and these regularity results are useful for the error estimates of the time discrete scheme of the Cahn-Hilliard-Navier-Stokes equations.*

Remark 4 *From Therem 2, we know that the solution* (v,ϕ) *of the system (2) belongs to* $C([0,T],\mathbb{V})$ *for the initial value* $(v_0,\phi_0)\in\mathbb{V}$. *From [19], there exists a unique weak solution in* $C(0,T;\mathbb{Y})\cap L^2(0,T;\mathbb{V})$ *for the initial value* $(v_0,\phi_0)\in\mathbb{Y}$. *From the uniqueness, we can conclude that for the initial value* $(v_0,\phi_0)\in\mathbb{Y}$, $(v,\phi)\in\mathbb{V}$. *Since* $\mathbb{V}\subset\mathbb{Y}$ *is compact, the standard Kryloff-Bogoliubov procedure yields the existence of an invariant measure on* $\mathbb{Y}$ *which is loaded on* $\mathbb{V}$.

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