ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.35(2023) No.1,pp.13-23

The Blow-up of Solutions of Parabolic Equation System with Critical Exponential Nonlinearity

Bobing Wang*
Institute of Applied System Analysis, Jiangsu University, Zhenjiang, Jiangsu 212013, P. R. China.
(Received December 13 2022, accepted January 29 2023)

Abstract: We consider the Cauchy problem:

$$\begin{cases} u_t - \Delta u + \lambda_1 u = \mu_1 u \left(e^{u^2} - 1\right) + \beta v \left(e^{uv} - 1\right) & \text{in } \Omega \times T, \\ v_t - \Delta v + \lambda_2 v = \mu_2 v \left(e^{v^2} - 1\right) + \beta u \left(e^{uv} - 1\right) & \text{in } \Omega \times T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain, $\lambda_1, \lambda_2 > -\Lambda_1$ (the first eigenvalue of $\left(-\Delta, H_0^1(\Omega)\right)$), $\mu_1, \mu_2 > 0$ and β is large. The nonlinear term has a critical growth at infinity in the energy space $H^1(\Omega)$ in view of the Moser-Trudinger embedding. Our goal is to investigate from the initial data $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\}$ whether the solution blows up in finite time. We prove that for initial data with energies below or equal to the ground state level, the dichotomy between finite time blow-up can be determined by means of a potential well argument.

Keywords: Moser-Trudinger inequality; Critical growth; Parabolic system

1 Introduction

We consider the Cauchy problem for a two-space dimensional parabolic equation system with exponential-type nonlinearity; more precisely, we focus the attention on the following model problem:

$$\begin{cases} u_t - \Delta u + \lambda_1 u = \mu_1 u \left(e^{u^2} - 1\right) + \beta v \left(e^{uv} - 1\right) & \text{in } \Omega \times T, \\ v_t - \Delta v + \lambda_2 v = \mu_2 v \left(e^{v^2} - 1\right) + \beta u \left(e^{uv} - 1\right) & \text{in } \Omega \times T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently smooth boundary, $\lambda_1, \lambda_2 > -\Lambda_1$ with $\Lambda_1 = \Lambda_1(\Omega)$ the first eigenvalue of $\left(-\Delta, H_0^1(\Omega)\right)$, and we consider initial data in the energy space $H_0^1(\Omega)$, i.e.,

$$(u_0, v_0) \in X.$$

where $X := H_0^1(\Omega) \times H_0^1(\Omega)$

In this framework, energy refers to the functional associated with the stationary problem:

$$I(u,v) = \frac{1}{2} \int_\Omega \left(\lambda_1 u^2 + |\nabla u|^2 + \lambda_2 v^2 + |\nabla v|^2\right) dx - \int_\Omega H(u,v) dx$$

***Corresponding author**. *E-mail address*: wangbobing0718@163.com

Copyright©World Academic Press, World Academic Union
IJNS.2023.03.15

where

$$H(u,v)=\frac{\mu_1}{2}G(u,u)+\beta G(u,v)+\frac{\mu_2}{2}G(v,v) \quad \text{and} \quad G(u,v)=e^{uv}-1-uv.$$

$$\|(u,v)\|_{H^1}:=\left(\|\nabla u\|_{L^2}^2+\lambda_1\|u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\lambda_1\|v\|_{L^2}^2\right)^{\frac{1}{2}}$$

and

$$\|(u,v)\|_{L^2}^2:=\left(\|u\|_{L^2}^2+\|v\|_{L^2}^2\right) \text{ and } \|\nabla(u,v)\|_{L^2}^2=\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2\right).$$

The above functional is well defined in $H^1(\Omega)$, and the nonlinear term $\mu_1 u\left(e^{u^2}-1\right)$ and $\mu_2 v\left(e^{v^2}-1\right)$ that we are considering have critical growth in the energy space in view of the Trudinger-Moser embedding [2].

We define the maximal existence time T_* of the solution

$$T_*:=\sup\left\{T>0: \text{the problem (1.1) admits a solution } (u,v)\in C\left([0,T];H^1(\Omega)\times H^1(\Omega)\right)\right\}\in(0,+\infty].$$

Polynomial case. Let $\Omega\subset\mathbb{R}^N$, $N\geq 3$, be a smooth bounded domain, and let us consider

$$\begin{cases} \partial_t w=\Delta w+|w|^{p-1}w & \text{in } \Omega\times(0,T), \\ w(x,t)=0 & \text{in } \partial\Omega\times(0,T), \\ w(x,0)=w_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

with $1<p\leq 2^*-1$, and $2^*=\frac{2N}{N-2}$. For any initial data in the energy space $H_0^1(\Omega)$, there exists some finite time $T>0$ and a local in time solution w belonging to $C\left([0,T];H_0^1(\Omega)\right)$ (this is a consequence of the L^p+1-existence result in [3] for any $1<p\leq 2^*-1$ and of the smoothing effect of the heat kernel). In this framework, the energy funcional is given by

$$I_p(u):=\frac{1}{2}\|\nabla u\|_{L^2}^2-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

Let $u\in H_0^1(\Omega)\setminus\{0\}$, and let us analyze the energy of the function σu for any $\sigma\geq 0$. By an easy computation, one can show that

$$I_p(\sigma u):=\frac{\sigma^2}{2}\|\nabla u\|_{L^2}^2-\frac{\sigma^p+1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

attains its unique maximum at a point $\bar\sigma=\bar\sigma(u)>0$, and $\bar u:=\bar\sigma u$ satisfies $\|\nabla\bar u\|_{L^2}^2-\|\bar u\|_{L^{p+1}}^{p+1}=0$.

There, the energy $I(\sigma u)$ has the structure of a potential well, and every ray σu, for any $\sigma>0$ and for $u\in H_0^1(\Omega)\setminus\{0\}$, has a unique intersection with the $Nehari\ manifold$

$$N=\left\{u\in H_0^1(\Omega)\setminus\{0\}:\|\nabla u\|_{L^2}^2-\|u\|_{L^{p+1}}^{p+1}=0\right\}.$$

The depth of the well is given by the lowest pass over the ridge define by all possible maps $\sigma\mapsto I_p(\sigma u)$ as u ranges over $H_0^1(\Omega)\setminus\{0\}$, namely

$$d_p:=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\max_{\sigma\geq 0}I_p(\sigma u).$$

It is well known that d_p can be characterized as

$$d_p=\inf_{u\in N}I_p(u), \text{ and also } d_p=\frac{p-1}{2(p+1)}\Lambda^{\frac{2(p+1)}{p-1}},$$

where $\Lambda=\Lambda_{p+1}(\Omega)$ is the best constant in the Sobolev embedding $H_0^1(\Omega)\subset L^{p+1}(\Omega)$, i.e.,

$$\Lambda=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\|\nabla u\|_{L^2}}{\|u\|_{L^{p+1}}}.$$

If $1<p<2^*-1$, then d_p is the energy level of ground state solutions.

The potential well associated with the Cauchy problem (2) is the set

$$W_p:=\left\{u\in H_0^1(\Omega):I_p(u)<d_p,\|\nabla u\|_{L^2}^2-\|u\|_{L^{p+1}}^{p+1}>0\right\}\cup\{0\},$$

and the exterior of the potential well is

$$V_p := \left\{ u \in H_0^1(\Omega) : I_p(u) < d_p, \|\nabla u\|_{L^2}^2 - \|u\|_{L^{p+1}}^{p+1} < 0 \right\}.$$

The sets V_p and W_p are both invariant under the flow associated with (2). Concerning the stable set, if $1 < p < 2^* - 1$, any solution which enters the stable set W_p exists globally in time. This result is a direct consequence of the fact that, in the subcritical case, the time T of local existence of the solution to (2) depends only on the size of the norm of the initial data in $H_0^1(\Omega)$, and for any $u \in W_p$ the Dirichlet norm $\|\nabla u\|_{L^2}$ is uniformly bounded (see[4]).

Definition 1 *We say that (u,v) is a semitrivial solution of (1) if (u,v) satisfies (1) and $u \not\equiv 0, v \equiv 0$ or $u \equiv 0, v \not\equiv 0$. On the other hand, (u,v) is called a (nontrivial) vector solution if (u,v) is a solution of (1) with $u \not\equiv 0, v \not\equiv 0$. A pair (u,v) is called positive if $u > 0, v > 0$ in Ω.*

The stationary problem. For $i = 1,2$, consider the following single equation

$$-\Delta u + \lambda_i u = \mu_i u\left(e^{u^2} - 1\right) \text{ in } \Omega,\ u \in H_0^1(\Omega), \tag{3}$$

whose corresponding functional is given by $J_{\lambda_i,\mu_i} : H_0^1(\Omega) \to \mathbb{R}$,

$$J_{\lambda_i,\mu_i}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda_i u^2\right) dx - \frac{\mu_i}{2}\int_\Omega \left(e^{u^2} - 1 - u^2\right) dx.$$

Notice that if u_1, u_2 solve the single equation (3) for $i = 1,2$ respectively, then $(u_1, 0)$ and $(0, u_2)$ solve systen (1).

By [5, Theorem 1.3], problem (3) admits a mountain pass solution u_{λ_i,μ_i} with minimal energy $E_i < 2\pi$, where

$$E_i = \inf\left\{J_{\lambda_i,\mu_i}(u) : u \in H_0^1(\Omega)\setminus\{0\}, J'_{\lambda_i,\mu_i}(u) = 0\right\}, \tag{4}$$

namely a ground state solution.

Set

$$\beta_1 := \mu_1 \frac{\int_\Omega u_1^2\left(e^{u_1^2} - 1\right) dx}{\int_\Omega u_1^4 dx} \text{ and } \beta_2 := \mu_2 \frac{\int_\Omega u_2^2\left(e^{u_2^2} - 1\right) dx}{\int_\Omega u_2^4 dx}.$$

It is not difficult to show that the stationary problem associated with (1), i.e.,

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u\left(e^{u^2} - 1\right) + \beta v\left(e^{uv} - 1\right) & \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v\left(e^{v^2} - 1\right) + \beta u\left(e^{uv} - 1\right) & \text{in } \Omega, \\ u, v \in H_0^1(\Omega) \end{cases} \tag{5}$$

Therefore, from now on, we will assume $\lambda_1, \lambda_2 > -\Lambda_1$ and $\beta > \bar{\beta}_0$ where

$$\bar{\beta}_0 = 4\frac{\max\{E_1\beta_1, E_2\beta_2\}}{\min\{E_1, E_2\}} > 0.$$

The existence of ground state solutions for (5) is proved in [6].From [6, Lemma 4.2], we also know that the mountain pass level

$$d_\beta^{MP} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),\ \ \Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, I(\gamma(1)) < 0\}. \tag{6}$$

Moreover,

$$0 < d_\beta^{MP} < 2\pi. \tag{7}$$

Another useful characterization of the mountain pass level c can be obtained by means of the Nehari functional

$$J(u,v) := \langle dI(u,v), (u,v)\rangle = \int_\Omega \left(\lambda_1 u^2 + |\nabla u|^2 + \lambda_2 v^2 + |\nabla v|^2\right) dx - \int_\Omega [uH_u(u,v) + vH_v(u,v)]\, dx. \tag{8}$$

Let

$$d_\beta := \inf\left\{I(u,v) : (u,v) \in X \setminus \{(0,0)\}, J(u,v) = 0\right\},$$

where $X = H_0^1(\Omega) \times H_0^1(\Omega)$, then the existence of a mountain pass solution $(u_\beta, v_\beta) \in X \setminus \{(0,0)\}$ to (5) implies $I(u_\beta, v_\beta) = d_\beta^{MP}$ and $dI(u_\beta, v_\beta) \equiv 0$; therefore, $d_\beta \le I(u_\beta, v_\beta) = d_\beta^{MP}$. And in [6] we also know that $d_\beta^{MP} \le d_\beta$; hence,

$$d_\beta = d_\beta^{MP}, \tag{9}$$

and this can be deduced from the geometry of J and I in the energy space. In particular, (9) is a consequence of the following property which gives also the potential well structure of the energy functional I.

Proposition 1 *For any $(u,v) \in X \setminus \{(0,0)\}$, there exists a unique $\bar{\sigma} = \bar{\sigma}(u,v) > 0$ such that*

$$J(\sigma u, \sigma v) = \begin{cases} > 0 & if\ 0 < \sigma < \bar{\sigma}, \\ = 0 & if\ \sigma = \bar{\sigma}, \\ < 0 & if\ \sigma > \bar{\sigma}. \end{cases} \tag{10}$$

We will proof Proposition 1 in Section 2 by simple computations.
Stable and unstable set. According to Proposition 1, for any fixed $(u,v) \in X \setminus \{(0,0)\}$, the function $\sigma \mapsto I(\sigma u, \sigma v)$ has the shape of a potential well. The idea of the potential well approach is to constrain the solution to (1) in the well to the left of $\bar{\sigma}(u,v)$ to ensure global existence. To ensure that the solution of (1) is constrained, we must find the lowest passage of the ridge defined by all possible $I(\sigma u, \sigma v)$ as (u,v) ranges over $X \setminus \{(0,0)\}$. The height of the lower pass over the ridge is the mountain pass level $\tilde{c}$ and $\tilde{c} = d_\beta$

Therefore, the potential well argument suggests to consider the splitting of the d-sublevel set of the energy I determined by the Nehari functional J. More precisely, we consider the unstable set V and the stable set W defined, respectively, by

$$V := \left\{(u,v) \in X \setminus \{(0,0)\} : I(u,v) < d_\beta, J(u,v) < 0\right\},$$

and

$$W := \left\{(u,v) \in X \setminus \{(0,0)\} : I(u,v) < d_\beta, J(u,v) > 0\right\} \cup \{0\}.$$

Theorem 2 *Let $(u,v) \in C([0,T_*); X\setminus\{(0,0)\})$ be the maximal solution to* (1)*, and $(u_0,v_0) \in X\setminus\{(0,0)\}$. If $(u(t_0), v(t_0)) \in V$ for some $t_0 \in [0,T_*)$, then $T_* < +\infty$.*

2 Preliminaries

Let$\Omega \subset \mathbb{R}^2$ be any smooth domain, and let us consider the more general Cauchy problem

$$\begin{cases} \partial_t u = \Delta u + g_1(u,v) & \text{in } \Omega \times (0,T), \\ \partial_t v = \Delta v + g_2(u,v) & \text{in } \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{11}$$

where $(u_0,v_0) \in X \setminus \{(0,0)\}$, and $g_i \in C^1(\mathbb{R},\mathbb{R})$ satisfies
(g1) $g(0) = 0$, and
(g2) for any $\varepsilon > 0$ we have

$$|g(s_1) - g(s_2)| \le C_\varepsilon |s_1 - s_2| \left(e^{(1+\varepsilon)s_1^2} + e^{(1+e)s_2^2}\right),\ s_1,\ s_2 \in \mathbb{R}.$$

for some positive constant C_ε.

Definition 2 *Let $(u_0,v_0) \in X \setminus \{(0,0)\}$. We say that (u,v) is a solution to* (11) *if $(u,v) \in C([0,T]; X\setminus\{(0,0)\})$, and (u,v) verifies the integral equation*

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} g_1(u(s), v(s))\, ds \\ e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} g_2(u(s), v(s))\, ds \end{pmatrix}.$$

Theorem 3 *([5]). If $\alpha \in (0, 4\pi)$, then there exists a constant $C_\alpha > 0$ such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha v^2} - 1\right) dx \le C_\alpha \|v\|_{L^2}^2, \quad for\ any\ v \in H^1\left(\mathbb{R}^2\right)\ with\ \|\nabla v\|_{L^2} \le 1, \tag{12}$$

and the above inequality fails if $\alpha \ge 4\pi$.

Theorem 4 *Let $0 < m < 4\pi$, and $M > 0$. There exists $T = T(m, M) > 0$ such that, for any $(u_0, v_0) \in X \setminus \{(0,0)\}$ with*

$$\|\nabla (u_0, v_0)\|_{L^2}^2 \le m\ \ and\ \ \|(u_0, v_0)\|_{L^2}^2 \le M, \tag{13}$$

the Cauchy problem (1) *has a unique solution $(u, v) \in C([0,T]; X \setminus \{(0,0)\})$.*

Proof. In order to prove the existence of a unique solution $(u, v) \in C([0,T]; X\setminus\{(0,0)\})$, let us first write the equation in (1) in the equivalent integral formulation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(-\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right) + \beta v\left(e^{uv}-1\right)\right) ds \\ e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\left(-\lambda_2 v + \mu_2 v\left(e^{v^2}-1\right) + \beta u\left(e^{uv}-1\right)\right) ds \end{pmatrix}. \tag{14}$$

Since $0 < m < 4\pi$ there exists $\varepsilon \in (0,1)$ such that $m = 4\pi(1-\varepsilon)$. Let us consider the set

$$\begin{aligned} X &= X(m, M) \\ &= \left\{(u,v) \in L^\infty((0,T), X\setminus\{(0,0)\}) : \sup_{t\in[0,T]} \|\nabla(u(t), v(t))\|_{L^2}^2 \le 4\pi\left(1-\frac{\varepsilon}{2}\right);\ \sup_{t\in[0,T]} \|(u,v)\|_{L^2}^2 \le 2M\right\}. \end{aligned}$$

This set endowed with the distance

$$\begin{aligned} d((u_1,v_1),(u_2,v_2)) = &\sup_{t\in[0,1]} \|\nabla(u_1(t), v_1(t)) - \nabla(u_2(t), v_2(t))\|_{L^2} \\ &+ \sup_{t\in[0,1]} \|(u_1(t), v_1(t)) - (u_2(t), v_2(t))\|_{L^2} \end{aligned}$$

is a complete metric space. We show that if $T > 0$ is small enough the map

$$\Phi(u,v) = \begin{pmatrix} \Phi_1 u \\ \Phi_2 v \end{pmatrix} = \begin{pmatrix} e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(-\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right) + \beta v\left(e^{uv}-1\right)\right) ds \\ e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}\left(-\lambda_2 v + \mu_2 v\left(e^{v^2}-1\right) + \beta u\left(e^{uv}-1\right)\right) ds \end{pmatrix}$$

is a contraction form X into itself. We remark that (u_0, v_0) satisfies

$$\|\nabla(u_0,v_0)\|_{L^2} \le \sqrt{m} = \sqrt{4\pi(1-\varepsilon)} < \sqrt{4\pi\left(1-\frac{\varepsilon}{2}\right)}, \quad \|(u_0,v_0)\|_{L^2} \le \sqrt{M}.$$

Let us first prove that Φ maps X into itself.

$$\begin{aligned} \|\Phi_1(u,v)(t)\|_{L^2} &\le \left\|e^{t\Delta}(u_0,v_0)\right\|_{L^2} + \int_0^t \left\|e^{(t-s)\Delta}\left(-\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right) + \beta v\left(e^{uv}-1\right)\right)\right\|_{L^2} ds \\ &\le \|(u_0,v_0)\|_{L^2} + \int_0^t \left\|e^{(t-s)\Delta}\left(-\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right) + \beta v\left(e^{uv}-1\right)\right)\right\|_{L^2} ds \\ &\le \|(u_0,v_0)\|_{L^2} + \int_0^t \left\|e^{(t-s)\Delta}\left(-\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right)\right)\right\|_{L^2} ds + \int_0^t \left\|e^{(t-s)\Delta}\beta v\left(e^{uv}-1\right)\right\|_{L^2} ds \end{aligned}$$

we have

$$\int_0^t \left\|e^{(t-s)\Delta}\beta v\left(e^{uv}-1\right)\right\|_{L^2} ds \le C\beta \int_0^t \frac{1}{(t-s)^{\frac{1}{r}-\frac{1}{s}}} \|v(s)\left(e^{uv}-1\right)\|_{L^r} ds$$

where $1 < r < 2$ will be chosen later. By holder inequality we have

$$\begin{aligned}\|v(s)(e^{uv}-1)\|_r^r &= \int_\Omega |v(s)(e^{uv}-1)|^r dx \\ &\le \int_\Omega |v(s)|^r (e^{ruv}-1)\,dx \\ &\le \left(\int_\Omega |v(s)|^{rp} dx\right)^{\frac{1}{p}} \left(\int_\Omega (e^{rquv}-1)\,dx\right)^{\frac{1}{q}} \\ &\le C\,\|u\|_2^r \left(\int_\Omega (e^{rquv}-1)\,dx\right)^{\frac{1}{q}} \\ &\le C\Big(\|u\|_2^2+\|v\|_2^2\Big)^{\frac{r}{2}} \left(\int_\Omega (e^{rquv}-1)\,dx\right)^{\frac{1}{q}} \\ &\le C\,\|(u,v)\|_{H^1}^r \left(\int_\Omega (e^{rquv}-1)\,dx\right)^{\frac{1}{q}} \\ &\le M\end{aligned}$$

where $rp \ge 2$, and p, q will be chosen later. By [6, Lemma 2.1 and Lemma 2.3] and young inequality we have

$$\begin{aligned}\int_\Omega (e^{rquv}-1)\,dx &= \int_\Omega \left(e^{\sqrt{rq}u\cdot\sqrt{rq}v}-1\right)^{2\cdot\frac{1}{2}} dx \\ &\le \int_\Omega \left[\left(e^{rqu^2}-1\right)\left(e^{rqv^2}-1\right)\right]^{\frac{1}{2}} dx \\ &\le \int_\Omega \left(e^{2rqu^2}+erqv^2\right)^{\frac{1}{2}} dx \\ &< +\infty\end{aligned}$$

Now choosing $q = 1+\varepsilon^2, r = 1+\varepsilon^4$, we can estimate

$$\int_0^t \left\|e^{(t-s)\Delta}\beta v(e^{uv}-1)\right\|_{L^2} ds \le C(\beta, M, \varepsilon)\int_0^t \frac{1}{(t-s)^{\frac{1}{r}-\frac{1}{2}}} ds = \tilde{C} t^{\frac{3}{2}-\frac{1}{r}}.$$

Now, we estimate $\int_0^t \left\|e^{(t-s)\Delta}\left(-\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right)\right)\right\|_{L^2} ds$. We set $\bar{g}(u) = -\lambda_1 u + \mu_1 u\left(e^{u^2}-1\right)$ and by [7] $\bar{g}$ satisfy (g2), so we have

$$\begin{aligned}&\int_0^t \left\|e^{(t-s)\Delta} g(u(s))\right\|_{L^2} ds \le C\int_0^t \left\|e^{(t-s)\Delta}|u(s)|\left(e^{(1+\varepsilon)u^2(s)}+1\right)\right\|_{L^2} ds \\ &\le 2C\int_0^t \left\|e^{(t-s)\Delta}|u(s)|\right\|_{L^2} ds + C\int_0^t \left\|e^{(t-s)\Delta}|u(s)|\left(e^{(1+\varepsilon)u^2(s)}-1\right)\right\|_{L^2} ds \\ &\le 2Ct\sqrt{2M} + C\int_0^t \frac{1}{(t-s)^{\frac{1}{r}-\frac{1}{2}}}\left\||u(s)|\left(e^{(1+\varepsilon)u^2(s)}-1\right)\right\|_{L^r} ds\end{aligned}$$

The values of r and q are the same above. Since we have

$$\begin{aligned}\left\||u(s)|\left(e^{(1+\varepsilon)u^2(s)}-1\right)\right\|_{L^r}^r &\le \int_\Omega |u(s)|^r\left(e^{r(1+\varepsilon)u^2(s)}-1\right) dx \\ &\le \left(\int_\Omega |u(s)|^{rp} dx\right)^{\frac{1}{p}}\left(\int_\Omega \left(e^{rq(1+\varepsilon)u^2(s)}-1\right) dx\right)^{\frac{1}{q}} \\ &\le C\,\|(u,v)\|_{H^1}^r \left(\int_\Omega \left(e^{rq(1+\varepsilon)u^2(s)}-1\right) dx\right)^{\frac{1}{q}}\end{aligned}$$

by Moser-Trudinger inequality and $m < 4\pi\left(1-\frac{\varepsilon}{2}\right)$ we have

$$\begin{aligned}\left(\int_\Omega \left(e^{rq(1+\varepsilon)u^2(s)}-1\right)dx\right)^{\frac{1}{q}} &= \left(\int_\Omega \left(e^{(1+\varepsilon)^2(1+\varepsilon)^4(1+\varepsilon)m\left(\frac{u(s)}{\sqrt{m}}\right)^2}-1\right)dx\right)^{\frac{1}{q}}\\ &= \left(\int_\Omega \left(e^{4\pi(1-\varepsilon)^8\left(\frac{u(s)}{\sqrt{m}}\right)^2}-1\right)dx\right)^{\frac{1}{q}}\\ &\le \left(\frac{1}{m}\|u(s)\|_{L^2}^2\right)^{\frac{1}{q}}\\ &\le C\frac{\|(u,v)\|_{L^2}^2}{m}\\ &\le C\left(\frac{2M}{m}\right)^{\frac{1}{q}}\end{aligned}$$

Therefore, if T is sufficiently small depending only on m, M then

$$\|\Phi_1(u,v)(t)\|_{L^2} \le \sqrt{2M}.$$

Next for any $(u,v)\in X$ and for any $t\in[0,T]$, thanks to $\|v(s)(e^{uv}-1)\|_{L^r}^r \le C$, we obtain

$$\begin{aligned}\|\nabla(\Phi_1(u,v)(t))\|_{L^2} &\le \left\|e^{t\Delta}\nabla(u_0,v_0)\right\|_{L^2} + \int_0^t \left\|\nabla e^{(t-s)\Delta}\left(-\lambda_1 u+\mu_1 u\left(e^{u^2}-1\right)+\beta v\left(e^{uv}-1\right)\right)\right\|_{L^2} ds\\ &\le \left\|e^{t\Delta}\nabla(u_0,v_0)\right\|_{L^2} + \int_0^t \left\|\nabla e^{(t-s)\Delta}\left(-\lambda_1 u+\mu_1 u\left(e^{u^2}-1\right)\right)\right\|_{L^2} ds\\ &\quad + \int_0^t \left\|\nabla e^{(t-s)\Delta}\beta v\left(e^{uv}-1\right)\right\|_{L^2} ds\\ &\le \sqrt{m}+C_\varepsilon\int_0^t \frac{1}{\sqrt{t-s}}\left\|\sqrt{t-s}\nabla e^{(t-s)\Delta}\left[|u(s)|\left(e^{(1+e)u^2}+1\right)\right]\right\|_{L^2} ds\\ &\quad + C\beta\int_0^t \frac{1}{(t-s)^{\frac{1}{r}}}\|v(s)(e^{uv}-1)\|_{L^2} ds\end{aligned}$$

since

$$\begin{aligned}&C_\varepsilon\int_0^t \frac{1}{\sqrt{t-s}}\left\|\sqrt{t-s}\nabla e^{(t-s)\Delta}\left[|u(s)|\left(e^{(1+e)u^2}+1\right)\right]\right\|_{L^2} ds\\ &\le 2C_\varepsilon\int_0^t \frac{1}{\sqrt{t-s}}\left\|\sqrt{t-s}\nabla e^{(t-s)\Delta}\left[|u(s)|\left(e^{(1+e)u^2}+1\right)\right]\right\|_{L^2} ds\\ &\quad + C_\varepsilon\int_0^t \frac{1}{\sqrt{t-s}}\left\|\sqrt{t-s}\nabla e^{(t-s)\Delta}\left[|u(s)|\left(e^{(1+e)u^2}-1\right)\right]\right\|_{L^2} ds\\ &\le 2C_\varepsilon\sqrt{2Mt}+C_{\varepsilon,r}\int_0^t \frac{1}{(t-s)^{\frac{1}{r}}}\left\||u(s)|\left(e^{(1+e)u^2}-1\right)\right\|_{L^r} ds\\ &\le 2C_\varepsilon\sqrt{2Mt}+\tilde{C}_{\varepsilon,r}t^{1-\frac{1}{r}}\end{aligned}$$

with the same $1<r<2$ chosen above. There, if T is sufficiently small depending only on m, M then

$$\sup_{t\in[0,T]}\|\nabla(\Phi_1 u)(t)\|_{L^2} \le \sqrt{4\pi\left(1-\frac{\varepsilon}{2}\right)}.$$

In a similar way it is possible to prove that for $T=T(m,M)$ small enough the map Φ_1 is a contraction on X and also Φ_2. Finally, by using the standard regularizing properties of the heat kernel it is possible to prove that the fixed point $u\in X$ of Φ satisfies $u\in C\left([0,T],H^1(\Omega)\right)$. ■

3 Basic properties of the solution to the model problem (1)

Proposition 5 *For any $t \in (0,T)$, we have*

$$-\frac{d}{dt} I(u,v) = \|\partial_t u(t)\|_{L^2} + \|\partial_t v(t)\|_{L^2}, \tag{15}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2\right) = -\|(u(t), v(t))\|_{H^1}^2 + \int_\Omega [uH_u(u,v) + vH_v(u,v)]\,dx \tag{16}$$

and

$$|\langle dI(u,v), (\varphi,\psi)\rangle| \le \|\partial_t u(t)\|_{L^2}\|\varphi\|_{L^2} + \|\partial_t v(t)\|_{L^2}\|\psi\|_{L^2},\ for\ any\ \varphi,\psi \in H^1(\Omega) \tag{17}$$

Proof. *The monotonicity of the energy* (15) *follows by Green formula and direct calculation $\frac{d}{dt}I(u,v)$. And* (16) *follows by multiplying the equation in* (1) *by (u,v) and integrating over Ω. Finally, to deduce* (17) *we multiply the equation in* (1) *by $(\varphi,\psi) \in H^1(\Omega) \times H^1(\Omega)$, and we integrate over Ω, obtaining* (17). ∎

We complete this section with the following continuity result that can be proved arguing as in [7, Proposition 3.6] with minor modifications.

Lemma 6 *If $T > 0$ and $(u,v) \in C([0,T], H^1(\Omega)) \times C([0,T], H^1(\Omega))$, then*

$$J(u,v) \in C([0,T], \mathbb{R}) \times C([0,T], \mathbb{R}).$$

4 Blow-up in V

If $(u,v) \in V$, and $\bar\sigma = \bar\sigma(u,v) > 0$ is given by Proposition 1, then $\bar\sigma \in (0,1)$, and hence,

$$2d_\beta \le 2I(\bar\sigma u, \bar\sigma v) \le \|(\bar\sigma u, \bar\sigma v)\|_{H^1}^2 < \|(u,v)\|_{H^1}^2. \tag{18}$$

To prove the invariance of the set V under the flow associated with (1), it is crucial to recall that, from (7) and (9),we know that

$$d_\beta > 0. \tag{19}$$

Lemma 7 *Let $(u,v) \in C((0,T_*]; H^1(\mathbb{R})) \times C((0,T_*]; H^1(\mathbb{R}))$ be the maximal solution to* (1), *and $(u_0,v_0) \in X$. If $(u(t_0), v(t_0)) \in V$ for some $t_0 \in [0,T_*)$, then $(u,v) \in V$ for any $t \in [t_0,T_*)$.*

Proof. In view of the monotonicity of the energy (15), and by Lemma 6, it is enough to prove that $J(u(t),v(t)) \neq 0$ for any $t \in (t_0,T_*)$. If $J(u(t),v(t)) = 0$ for some $t \in (t_0,T_*)$, then there exists $t_1 \in (t_0,T_*)$ such that

$$J(u(t),v(t)) < 0 \text{ for any } t \in [t_0,t_1)\,, \text{ and } J(u(t_1),v(t_1)) = 0.$$

Therefore, $(u(t),v(t)) \in V$ for any $t \in [t_0,t_1)$, and

- either $(u(t_1),v(t_1)) \neq 0$. Hence $d_\beta \le I(u(t_1),v(t_1))$, which is not possible due to the monotonicity of the energy (15)
- or $(u(t_1),v(t_1)) = 0$ which yields

$$\lim_{t\to t_1^-} \|u(t),v(t)\|_{H^1} = \|u(t_1),v(t_1)\|_{H^1} = 0,$$

and this contradicts (18) and (19).

∎

In order to prove that solutions entering V blow up in finite time, we will apply the following blow-up Lemma containing the classical idea of the concavity method due to Levine [8].

Lemma 8 *([8]). There exists no non-negative and increasing function $y \in C^2(\bar{t}, +\infty)$, with $\bar{t} \in \mathbb{R}$, such that, for some $\gamma > 0$,*

$$y(t)\, y''(t) \geq (\gamma + 1)\left[y'(t)\right]^2 \text{ on } (\bar{t}, +\infty),$$

and

$$\lim_{t \to +\infty} y(t) = +\infty. \tag{20}$$

Proof. We can prove Lemma 8 by applying the arguments in [2, lemma 4.2] or [8]. In view of (20), $h(t) := y^{-\gamma}(t)$ is well defined on the half-line $(t', +\infty)$, for some $t' \geq \bar{t}$ sufficiently large. Moreover,

$$\lim_{t \to +\infty} h(t) = 0. \tag{21}$$

■

The concavity method works in our setting due to the fact the Nehari functional along solutions entering V is bounded away from zero by a strictly negative constant.

Proposition 9 *Let $(u, v) \in C\left([0, T_*); H^1(\Omega)\right) \times C\left([0, T_*); H^1(\Omega)\right)$ be the maximal solution to* (1) *with $\beta > \bar{\beta}_0$, and $(u_0, v_0) \in X$. If $(u(t_0), v(t_0)) \in V$ for some $t_0 \in [0, T_*)$ then there exists $\varepsilon > 0$ such that $J(u(t), v(t)) < -\varepsilon$ for any $t \in [t_0, T_*)$.*

Proof. Let

$$d'_\beta := \inf\left\{K(u, v) : (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}, J(u, v) \leq 0\right\},$$

where

$$K(u, v) := I(u, v) - \frac{1}{2} J(u, v) = \int_\Omega \frac{1}{2}\left[u H_u(u, v) + v H_v(u, v)\right] - H(u, v)\, dx \tag{22}$$

Then, $d_\beta = d'_\beta$.In fact, clearly $d'_\beta \leq d_\beta$, and in order to deduce that $d_\beta \leq d'_\beta$, it is enough to show that

$$d_\beta \leq K(u, v) \text{ for any } (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\} \text{ with } J(u, v) < 0.$$

Let $(u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}$, and let $\bar{\sigma} = \bar{\sigma}(u, v) > 0$ be as in Proposition 1. If $J(u, v) < 0$, then $\bar{\sigma}(u, v) \in (0, 1)$, and by [2, Proposition 4.3] with a minor modifications we can get

$$d_\beta \leq I(\bar{\sigma} u, \bar{\sigma} v) = K(u, v).$$

With the above characterization of d_β, it is easy to show that for any $\varepsilon > 0$

$$d_{\beta\varepsilon} := \inf\left\{I(u, v) : (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}, J(u, v) = -\varepsilon\right\} \geq d - \frac{\varepsilon}{2} \tag{23}$$

In fact, by direct computations

$$\begin{aligned}
d_{\beta\varepsilon} &= \inf\left\{I(u, v) + \frac{\varepsilon}{2} - \frac{\varepsilon}{2} : (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}, J(u, v) = -\varepsilon\right\} \\
&= \inf\left\{I(u, v) - \frac{1}{2} J(u, v) - \frac{\varepsilon}{2} : (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}, J(u, v) = -\varepsilon\right\} \\
&= \inf\left\{I(u, v) - \frac{1}{2} J(u, v) : (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}, J(u, v) = -\varepsilon\right\} - \frac{\varepsilon}{2} \\
&= \inf\left\{K(u, v) : (u, v) \in H^1(\Omega) \times H^1(\Omega) \setminus \{(0, 0)\}, J(u, v) = -\varepsilon\right\} - \frac{\varepsilon}{2} \\
&\geq d_\beta - \frac{\varepsilon}{2}.
\end{aligned}$$

Next, we assume that the maximal solution (u, v) to (1) satisfies $(u(t_0), v(t_0)) \in V$ for some $t_0 \in [0, T_*)$. Then, there exists $\varepsilon > 0$ such that

$$\min\left\{d_\beta - I(u(t_0), v(t_0)), -J(u(t_0), v(t_0))\right\} > \varepsilon$$

In view of (23) and the monotonicity of the energy (15), we get

$$d_{\beta\varepsilon} \geq d_\beta - \frac{\varepsilon}{2} > I(u(t_0), v(t_0)) \geq I(u(t), v(t)), \text{ for any } t \in [t_0, T_*). \tag{24}$$

Assume that $J\left(u\left(t_1\right), v\left(t_1\right)\right)=-\varepsilon$ for some $t_1 \in\left(t_0, T_*\right)$. Then, $d_{\beta\varepsilon} \leq I\left(u\left(t_1\right), v\left(t_1\right)\right)$, which contradicts (24)

Summarizing, we have $J\left(u\left(t_0\right), v\left(t_0\right)\right)<-\varepsilon$, and $J\left(u\left(t\right), v\left(t\right)\right) \neq-\varepsilon$ for any $t \in\left[t_0, T_*\right)$. Therefore, the proof is complete in view of the continuity of J along the solution, see Lemma 6. ■

Proof of Theorem 2

Proof. We argue by contradiction assuming that the solution (u, v) is global, i.e., $T_*=\infty$, and we apply the blow-up Lemma 8 to the non-negative and increasing C^2-function defined by

$$y(t):=\frac{1}{2} \int_{t_0}^t\|u(s)\|_{L^2}^2+\|v(s)\|_{L^2}^2 d s,\ t \in\left[t_0,+\infty\right). \tag{25}$$

In view of (16), we have

$$y^{\prime \prime}(t)=\frac{1}{2} \frac{d}{d t}\left(\|u(s)\|_{L^2}^2+\|v(s)\|_{L^2}^2\right)=-J(u(t), v(t))>\varepsilon,\ t \in\left(t_0,+\infty\right), \tag{26}$$

where $\varepsilon>0$ is given by Proposition 9. From (26), we deduce that

$$\lim _{t \rightarrow+\infty} y^{\prime}(t)=\lim _{t \rightarrow+\infty} y(t)=+\infty . \tag{27}$$

Since there exists $\theta>2$ such that

$$\theta H(a, b) \leq a H_a(a, b)+b H_b(a, b),\ \text{ for any } a, b \in \mathbb{R}$$

we can esimate

$$\begin{aligned}
y^{\prime \prime}(t)= & -J(u(t), v(t)) \\
= & -\left(\lambda_1\|u(t)\|_{L^2}^2+\lambda_2\|v(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2\right)+\int_{\Omega}\left[u H_u(u, v)+v H_v(u, v)\right] d x \\
\geq & -\left(\lambda_1\|u(t)\|_{L^2}^2+\lambda_2\|v(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2\right)+\theta \int_{\Omega} H(u(t), v(t)) d x \\
= & -\theta I(u(t), v(t))+(\theta-1)\left(\lambda_1\|u(t)\|_{L^2}^2+\lambda_2\|v(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2\right) \\
= & -\theta I(u(t), v(t))+C y^{\prime}(t)
\end{aligned}$$

where $C=C(\theta):=(\theta-2)>0$.

Using (15), we get

$$-I(u(t), v(t))=\int_{t_0}^t\left\|\partial_s u(s)\right\|_{L^2}^2+\left\|\partial_s v(s)\right\|_{L^2}^2 d s-I\left(u\left(t_0\right), v\left(t_0\right)\right), \tag{28}$$

and hence,

$$\begin{aligned}
& y(t) y^{\prime \prime}(t) \\
& \geq \frac{\theta}{2}\left(\int_{t_0}^t\|u(s)\|_{L^2}^2+\|v(s)\|_{L^2}^2 d s\right)\left(\int_{t_0}^t\left\|\partial_s u(s)\right\|_{L^2}^2+\left\|\partial_s v(s)\right\|_{L^2}^2 d s\right) \\
& \quad+y(t)\left(C y^{\prime}(t)-\theta I\left(u\left(t_0\right), v\left(t_0\right)\right)\right) \\
& \geq \frac{\theta}{2}\left(\int_{t_0}^t\left(\int_{\Omega} u(s) \partial_s u(s) d x\right) d s\right)^2+y(t)\left(C y^{\prime}(t)-\theta I\left(u\left(t_0\right), v\left(t_0\right)\right)\right) \\
& \geq \frac{\theta}{2}\left(\int_{t_0}^t \frac{1}{2} \frac{d}{d s}\|u(s)\|_{L^2}^2 d s\right)^2+y(t)\left(C y^{\prime}(t)-\theta I\left(u\left(t_0\right), v\left(t_0\right)\right)\right) \\
& =\frac{\theta}{2}\left(y^{\prime}(t)-y^{\prime}\left(t_0\right)\right)^2+y(t)\left(C y^{\prime}(t)-\theta I\left(u\left(t_0\right), v\left(t_0\right)\right)\right) .
\end{aligned}$$

In view of (27), for any $\gamma \in(0,1)$ there exists $t_\gamma>1$ such that

$$y(t) y^{\prime \prime}(t) \geq \frac{\theta}{2} \gamma\left[y^{\prime}(t)\right]^2,\ \text{ for any } t \geq t_\gamma.$$

If we choose $\gamma>0$ such that $\frac{2}{\theta}<\beta<1$, then we are in the framework of the blow-up Lemma 8, and we reach a contradiction. Thus we prove Theorem 2. ■

References

[1] Y. Fujishima. Blow-up set for a superlinear heat equation and pointedness of the initial data. *Discrete Contin. Dyn. Syst., Ser. A.*, 34 (2014) 4617-4645.

[2] Ishiwata, M., Ruf, B., Sani, F. Asymptotics for a parabolic equation with critical exponential nonlinearity. *J. Evol. Equ.* 21, 1677-1716 (2021).

[3] H. Brezis, T. Cazenave. A nonlinear heat equation with singular initial data. *J. Anal. Math.*, 68 (1996), 277-304.

[4] M. Tsutsumi. On solutions of semilinear differential equations in a Hilbert space. Math. Japan 17 (1972), 173-193.

[5] D.G. de Figueiredo, O.H. Miyagaki, B. Ruf. Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. *Calc. Var. Partial Differ. Equ.* 3 (1995) 139-153.

[6] D. Cassani, H. Tavares, J. Zhang. Bose fluids and positive solutions to weakly coupled systems with critical growth in dimension two. arXiv:1810.04524.

[7] S. Ibrahim, R. Jrad, M. Majdoub, T. Saanouni. Local well posedness of a 2D semilinear heat equation. *Bull. Belg. Math. Soc. Simon Stevin.* 21 (2014), 535-551.

[8] A. H. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Am. Math. Soc.* 192 (1974), 1-21.

[9] A. Haraux, F.B. Weissler. Nonuniqueness for a semilinear initial value problem. *Indiana Univ. Math. J.* 31 (1982) 167-189.

[10] T. Suzuki Semilinear parabolic equation on bounded domain with critical Sobolev exponent. *Indiana Univ. Math. J.* 57 (2008), 3365-3395.

[11] E. Timmermans Phase separation of Bose-Einstein condensates. Phys. Rev. Lett. 81 (1998) 5718-5721.

[12] M. Tsutsumi On solutions of semilinear differential equations in a Hilbert space. Math. Japan 17 (1972), 173-193.

[13] M. Ishiwata. Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in $\mathbb{R}^N$, Math. Ann. 351 (2011) 781-804.

[14] S. Kim. On vector solutions for coupled nonlinear Schrodinger equations with critical exponents. *Commun. Pure Appl. Anal.* 12 (2013) 1259-1277.

[15] M. Struwe. Variational Methods. Application to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer-Verlag, 1990.