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Traveling Wave Solutions for a Nonlocal Dispersal and Spatiotemporal Delayed Epidemic Model

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Abstract: In this paper, we investigate the existence and nonexistence of traveling waves in a nonlocal diffusion and spatiotemporal delayed epidemic model. It is shown that the existence and nonexistence of traveling waves of the model are depend on both critical wave speed c^* and the basic reproduction number R_0 of the corresponding kinetic system. Specifically, when $R_0 > 1$ and $c > c^*$, the traveling waves of the model exist; when $R_0 \leq 1$ and $c \in \mathbb{R}$ or $R_0 > 1$ and $c < c^*$, the traveling waves of the model does not exist. The obtained results may help people predict how fast an epidemic spread geographically.

Keywords: Traveling waves; Epidemic model; Nonlocal diffusion; Spatio-temporal delay.

1 Introduction

In this paper, we studied existence and nonexistence of traveling wave solutions in the following nonlocal dispersal epidemic model with spatiotemporal delay

$$\partial_t S(x,t) = d_1\big[J * S(x,t) - S(x,t)\big] - \frac{\beta S(x,t) K * I(x,t)}{1 + \alpha K * I(x,t)}, \tag{1.1}$$

$$\partial_t I(x,t) = d_2\big[J * I(x,t) - I(x,t)\big] + \frac{\beta S(x,t) K * I(x,t)}{1 + \alpha K * I(x,t)} - \gamma I(x,t), \tag{1.2}$$

$$\partial_t R(x,t) = d_3\big[J * R(x,t) - R(x,t)\big] + \gamma I(x,t), \tag{1.3}$$

where

$$J * U(x,t) = \int_{-\infty}^{\infty} J(x-y) U(y,t) dy \quad (U \text{ can be } S, I \text{ or } R)$$

and

$$K * I(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} K(x-y, t-s) I(y,s) dy ds.$$

In (1.1)-(1.3), $S(x,t)$, $I(x,t)$ and $R(x,t)$ refer to the densities of susceptible, infected and recovery individuals at location x and time t, respectively. The coefficients $d_i > 0$ $(i = 1, 2)$ denote the diffusion rates of each class, $\beta > 0$ represents the transmission rate, $\gamma > 0$ is the recovery rate. The convolution operators $J * S(x,t) - S(x,t)$, $J * I(x,t) - I(x,t)$ and $J * R(x,t) - R(x,t)$ describe that the rate of susceptible, infected and recovery individuals at position x and time t depend on the influence of neighboring S, I and R in all other positions y [1–6]. The kernel $K(x-y, t-s) \geq 0$ depicts the interaction between the infected and susceptible individuals at location x and the present time t which occurred at location y and earlier time s, so the nonlocal delayed term $K * I$ reflects that the infected individuals have the ability of movement and infectiousness during the latent period. The assumptions on the kernel functions $J(x)$ and $K(x,t)$ are

(H1) $J(x) \in C(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(x)dx = 1$, J is compactly supported and R_1 is the radius of supp$J$;

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(H2) $K(x,t)$ is Lipschitz continuous and compactly supported with respect to space variable x, and R_2 is the radius of supp$K$ for variable x; $K(x,t)$ is compactly supported with respect to time variable t and T is the length of supp$K$ for variable t;

(H3) $\int_0^\infty \int_{\mathbb{R}} K(x,t)dxdt = 1$ and $K(x,t) = K(-x,t) \geq 0$ for $(x,t) \in (-\infty,\infty) \times [0,\infty)$.

Since the first two equations in (1.1)-(1.3) form a closed system, we only need to consider (1.1) and (1.2). A traveling wave solution to (1.1)-(1.2) is a special solution with the form $(S(x,t), I(x,t)) = (S(z), I(z))$, $z = x + ct$, which satisfies the wave profile system

$$d_1[J * S(z) - S(z)] - cS'(z) - \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} = 0, \tag{1.4}$$

$$d_2[J * I(z) - I(z)] - cI'(z) + \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} - \gamma I(z) = 0, \tag{1.5}$$

where $J * S(z) = \int_{\mathbb{R}} J(y)S(z-y)dy$, $J * I(z) = \int_{\mathbb{R}} J(y)I(z-y)dy$ and $K * I(z) = \int_0^\infty \int_{\mathbb{R}} K(y,s)I(z-y-cs)dyds$. Clearly, system (1.4)-(1.5) admits infinitely many equilibria $(S, 0)$ with arbitrary constants $S \geq 0$. The aim of the present paper is to look for the positive solutions $(S(z), I(z))$ of (1.4)-(1.5) satisfying the asymptotic boundary conditions

$$\lim_{z \to -\infty} S(z) := S(-\infty) = S_1 > \lim_{z \to \infty} S(z) := S(\infty) = S_2 \geq 0, \quad \lim_{z \to \pm\infty} I(z) := I(\pm\infty) = 0, \tag{1.6}$$

where S_1 is the size of susceptible individuals at the beginning of the epidemic and S_2 is the size of susceptible individuals after the epidemic.

The rest of the paper is organized as follows. In Section 2, we state our main results. In Section 3, we establish the nonexistence of nontrivial and positive traveling wave solutions for (1.1)-(1.2). In Sections 4, we prove the existence of super-critical traveling wave solutions for (1.1)-(1.2), respectively. In Section 5, we make a brief discussion.

2 Main results

For $(\lambda, c) \in [0,\infty) \times [0,\infty)$, define a function

$$\Theta(\lambda, c) := d_2 \int_{-\infty}^{\infty} J(y)e^{-\lambda y}dy - d_2 - c\lambda + \beta S_1 \int_0^\infty \int_{-\infty}^{\infty} K(y,s)e^{-\lambda(y+cs)}dyds - \gamma. \tag{2.1}$$

Then we state some properties related to the function $\Theta(\lambda, c)$ in the following lemma.

Lemma 1 *Assume that $R_0 := \beta S_1/\gamma > 1$. Then there exists a positive pair (λ^*, c^*) such that*

$$\Theta(\lambda^*, c^*) = d_2 \int_{-\infty}^{\infty} J(y)e^{-\lambda^* y}dy - d_2 - c^*\lambda^* + \beta S_1 \int_0^\infty \int_{-\infty}^{\infty} K(y,s)e^{-\lambda^*(y+c^*s)}dyds - \gamma = 0 \tag{2.2}$$

and

$$\Theta_\lambda(\lambda^*, c^*) = -d_2 \int_{-\infty}^{\infty} yJ(y)e^{-\lambda^* y}dy - c^* - \beta S_1 \int_0^\infty \int_{-\infty}^{\infty} (y + c^*s)K(y,s)e^{-\lambda^*(y+c^*s)}dyds = 0. \tag{2.3}$$

Furthermore, (i) if $0 < c < c^*$, then $\Theta(\lambda, c) > 0$ for $\lambda \in [0,\infty)$; (ii) if $c > c^*$, then the equation $\Theta(\lambda, c) = 0$ has two positive roots $\lambda_1(c) := \lambda_1$ and $\lambda_2(c) := \lambda_2$ with $\lambda_1 < \lambda^* < \lambda_2$ such that $\Theta(\lambda, c) > 0$ for $\lambda \in [0, \lambda_1) \cup (\lambda_2, \infty)$ and $\Theta(\lambda, c) < 0$ for $\lambda \in (\lambda_1, \lambda_2)$.

Proof. By (H1)-(H3), we have $\Theta(\infty, c) = \infty$ for each fixed $c > 0$ and $\Theta(\lambda, \infty) = -\infty$ for each fixed $\lambda > 0$. Moreover, from (H1)-(H3) and $R_0 > 1$, we derive that $\Theta(0, c) = \beta S_1 - \gamma > 0$,

$$
\begin{aligned}
\Theta_c(\lambda, c) &= -\lambda - \lambda\beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty} sK(y,s)e^{-\lambda(y+cs)}dyds < 0, \quad \forall \lambda > 0, \\
\Theta(\lambda, 0) &= d_2\int_{-\infty}^{\infty} J(y)(e^{-\lambda y} - 1)dy + \beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\lambda y}dyds - \gamma \\
&\geq -\lambda d_2 \int_{-\infty}^{\infty} yJ(y)dy + \beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)(1-\lambda y)dyds - \gamma \\
&= \beta S_1 - \gamma > 0, \quad \forall \lambda > 0, \\
\Theta_\lambda(0, c) &= \left[-d_2\int_{-\infty}^{\infty} yJ(y)e^{-\lambda y}dy - c - \beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty}(y+cs)K(y,s)e^{-\lambda(y+cs)}dyds \right]\Bigg|_{\lambda=0} \\
&= -c - \beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty}(y+cs)K(y,s)dyds \\
&= -c - c\beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty} sK(y,s)dyds < 0, \quad \forall c > 0, \\
\Theta_{\lambda\lambda}(\lambda, c) &= d_2\int_{-\infty}^{\infty} y^2J(y)e^{-\lambda y}dy + \beta S_1 \int_0^{\infty}\int_{-\infty}^{\infty}(y+cs)^2K(y,s)e^{-\lambda(y+cs)}dyds > 0.
\end{aligned}
$$

The assertions of this lemma follow immediately from the above computations. ■

Our main results are stated as follows.

Theorem 2 *Suppose that $R_0 \leq 1$ and $c \in \mathbb{R}$ or $R_0 > 1$ and $c < c^*$. Then system (1.4)-(1.5) admits no nontrivial and positive solutions $(S(z), I(z))$ satisfying $I(\pm\infty) = 0$, $S(-\infty) = S_1$ and $\sup_{z\in\mathbb{R}} S(z) \leq S_1$.*

Theorem 3 *Suppose that $R_0 > 1$ and $c \geq c^*$. Then system (1.4)-(1.6) admits a solution $(S(z), I(z))$ satisfying:*

(i) $0 < S(z) < S_1$, $0 < I(z) < \frac{\beta S_1 - \gamma}{\alpha\gamma}$ *for* $z \in \mathbb{R}$;

(ii) If $z \to -\infty$, $I(z) = O(e^{\lambda_1 z})$ *for* $c > c^*$ *and* $I(z) = O(-ze^{\lambda^* z})$ *for* $c = c^*$;

(iii) $S'(\pm\infty) = I'(\pm\infty) = 0$;

(iv) $\gamma\int_{\mathbb{R}} I(z)dz = \beta\int_{\mathbb{R}} \frac{S(z)K*I(z)}{1+\alpha K*I(z)}dz = c(S_1 - S_2)$.

3 Nonexistence of traveling waves

This section is devoted to establishing the nonexistence of traveling wave solutions. Before proving Theorem 2, we present the following lemma.

Lemma 4 *Assume that a nontrivial and positive function pair $(S(z), I(z)) \in C^1(\mathbb{R}, \mathbb{R}^2)$ is a solution of (1.4)-(1.5) satisfying*

$$I(\pm\infty) = 0, \quad S(-\infty) = S_1, \quad \sup_{z\in\mathbb{R}} S(z) \leq S_1. \tag{3.1}$$

Then

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{S(z)K*I(z)}{1+\alpha K*I(z)}dz < \infty, \quad \int_{-\infty}^{\infty} I(z)dz < \infty, \\
&\int_{-\infty}^{\infty} [J*I(z) - I(z)]dz < \infty, \quad \int_{-\infty}^{\infty} K*I(z)dz < \infty.
\end{aligned} \tag{3.2}
$$

Proof. Integrating (1.4) from ξ to η ($\eta > \xi$) yields

$$\begin{aligned}
&\int_\xi^\eta \frac{S(z)K*I(z)}{1+\alpha K*I(z)}dz \\
&= d_1\int_\xi^\eta\int_{-\infty}^\infty J(y)[S(z-y)dy-S(z)]dydz + cS(\xi) - cS(\eta) \\
&= -d_1\int_\xi^\eta\int_{-R_1}^{R_1} yJ(y)\int_0^1 S'(z-\theta y)d\theta dydz + cS(\xi) - cS(\eta) \quad \text{(by (H1))} \\
&= d_1\int_{-R_1}^{R_1} yJ(y)\int_0^1 [S(\xi-\theta y)-S(\eta-\theta y)]d\theta dy + cS(\xi) - cS(\eta) \quad \text{(by Fubini theorem)} \\
&\le 2d_1S_1\int_{-R_1}^{R_1}|y|J(y)dy + 2|c|S_1, \quad \text{(by (3.1))}
\end{aligned}$$

for any $\xi,\eta\in\mathbb{R}$. This implies that

$$\int_{-\infty}^\infty \frac{S(z)K*I(z)}{1+\alpha K*I(z)}dz < \infty. \tag{3.3}$$

Since $I(\pm\infty)=0$ and $I(z)\in C^1(\mathbb{R})$ is nontrivial and positive, we can denote that $M := \sup_{z\in\mathbb{R}} I(z)$, where M is a finite positive constant. Integrating (1.5) from τ to ζ ($\zeta > \tau$) gives

$$\begin{aligned}
\gamma\int_\tau^\zeta I(z)dz &= d_2\int_{-R_1}^{R_1} yJ(y)\int_0^1 [I(\tau-\theta y)-I(\zeta-\theta y)]d\theta dy + \int_\tau^\zeta \frac{\beta S(z)K*I(z)}{1+\alpha K*I(z)}dz + cI(\tau) - cI(\zeta) \\
&\le 2d_2M\int_{-R_1}^{R_1}|y|J(y)dy + \int_{-\infty}^\infty \frac{\beta S(z)K*I(z)}{1+\alpha K*I(z)}dz + 2|c|M
\end{aligned}$$

for any $\tau,\zeta\in\mathbb{R}$. This together with (3.3) implies that

$$\int_{-\infty}^\infty I(z)dz < \infty. \tag{3.4}$$

It follows from (3.3), (3.4), (1.5) and $I(\pm\infty)=0$ that

$$\int_{-\infty}^\infty [J*I(z)-I(z)]dz < \infty. \tag{3.5}$$

Note that

$$\begin{aligned}
&\int_\vartheta^\varsigma [K*I(z)-I(z)]dz \\
&= \int_\vartheta^\varsigma\int_0^\infty\int_{-\infty}^\infty K(y,s)[I(z-y-cs)-I(z)]dydsdz \\
&= -\int_\vartheta^\varsigma\int_0^T\int_{-R_2}^{R_2}(y+cs)K(y,s)\int_0^1 I'(z-\theta(y+cs))d\theta dydsdz \quad \text{(by (H2))} \\
&= \int_0^T\int_{-R_2}^{R_2}(y+cs)K(y,s)\int_0^1 [I(\vartheta-\theta(y+cs))-I(\varsigma-\theta(y+cs))]d\theta dyds \quad \text{(by Fubini theorem)} \\
&\le 2M\int_0^T\int_{-R_2}^{R_2}|y+cs|K(y,s)dyds
\end{aligned}$$

for any $\vartheta,\varsigma\in\mathbb{R}$, which implies that $\int_{\mathbb{R}}[K*I(z)-I(z)]dz<\infty$. Then this together with (3.4) gives that

$$\int_{-\infty}^\infty K*I(z)dz < \infty.$$

The proof of this lemma is finished. ■

In the following, we will divide the proof of Theorem 2 into four cases: $R_0 < 1$ and $c \in \mathbb{R}$; $R_0 = 1$ and $c \in \mathbb{R}$; $R_0 > 1$ and $0 < c < c^*$; $R_0 > 1$ and $c \leq 0$.

Case 1: $R_0 < 1$ and $c \in \mathbb{R}$. Integrating (1.5) over $\mathbb{R}$, we have from Lemma 4 that

$$
\begin{aligned}
&\gamma \int_{-\infty}^{\infty} I(z)dz \\
&= d_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(y)I(z-y)dydz - d_2\int_{-\infty}^{\infty} I(z)dz - c\int_{-\infty}^{\infty} I'(z)dz + \int_{-\infty}^{\infty} \frac{\beta S(z)K*I(z)}{1+\alpha K*I(z)}dz \\
&= \int_{-\infty}^{\infty} \frac{\beta S(z)K*I(z)}{1+\alpha K*I(z)}dz \quad \text{(by Fubini theorem and } I(\pm\infty)=0) \\
&\leq \beta S_1 \int_{-\infty}^{\infty} K*I(z)dz \quad (\text{since} \sup_{z\in\mathbb{R}} S(z) \leq S_1) \\
&< \gamma \int_{-\infty}^{\infty} I(z)dz, \quad \text{(by Fubini theorem, (H3) and } R_0 < 1),
\end{aligned}
$$

which leads to a contradiction.

Case 2: $R_0 = 1$ and $c \in \mathbb{R}$. Integrating (1.5) over $\mathbb{R}$, we obtain from Lemma 4 that

$$
\begin{aligned}
&\beta S_1 \int_{-\infty}^{\infty} I(z)dz \\
&= d_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(y)[I(z-y)-I(z)]dydz - c\int_{-\infty}^{\infty} I'(z)dz + \int_{-\infty}^{\infty} \frac{\beta S(z)K*I(z)}{1+\alpha K*I(z)}dz \quad (\text{by } R_0 = 1) \qquad (3.6)\\
&= \int_{-\infty}^{\infty} \frac{\beta S(z)K*I(z)}{1+\alpha K*I(z)}dz, \quad \text{(by Fubini theorem and } I(\pm\infty)=0),
\end{aligned}
$$

which is equivalent to

$$
\int_{-\infty}^{\infty} \left[\frac{S(z)}{1+\alpha K*I(z)} - S_1\right] K*I(z)dz = 0, \tag{3.7}
$$

since $\int_{\mathbb{R}} I(z)dz = \int_{\mathbb{R}} K*I(z)dz$. Notice that $\left[\frac{S(z)}{1+\alpha K*I(z)} - S_1\right]K*I(z) \leq 0$. By the continuity of $S(z)$ and $K*I(z)$ in $\mathbb{R}$, we deduce from (3.7) that

$$
\left[\frac{S(z)}{1+\alpha K*I(z)} - S_1\right] K*I(z) = 0.
$$

Since Int(supp$K*I$) is nonempty, we get for $z \in$ Int(supp K*I) that

$$
\frac{S(z)}{1+\alpha K*I(z)} - S_1 = 0,
$$

a contradiction appears.

Case 3: $R_0 > 1$ and $0 < c < c^*$. From (3.1) and (H2), we have

$$
\lim_{z\to-\infty} \frac{\beta S(z)}{1+\alpha K*I(z)} = \beta S_1. \tag{3.8}
$$

Due to $R_0 > 1$ and (3.8), there exists a constant $z^* \ll 0$ such that

$$
\frac{\beta S(z)}{1+\alpha K*I(z)} > \frac{\beta S_1+\gamma}{2} \quad \text{for } z < z^*. \tag{3.9}
$$

Then it follows from (1.5) and (3.9) that

$$
cI'(z) \geq d_2[J*I(z)-I(z)] + \frac{\beta S_1+\gamma}{2}[K*I(z)-I(z)] + \frac{\beta S_1-\gamma}{2}I(z) \tag{3.10}
$$

for $z < z^*$. By (3.2), we define $Q(z) := \int_{-\infty}^{z} I(\eta)d\eta$ in $\mathbb{R}$. Integrating (3.10) from $-\infty$ to z with $z < z^*$, we get

$$
\frac{\beta S_1-\gamma}{2}Q(z) \leq cI(z) - d_2[J*Q(z)-Q(z)] - \frac{\beta S_1+\gamma}{2}[K*Q(z)-Q(z)], \tag{3.11}
$$

where we have used Lemma 4, Fubini theorem and $I(-\infty) = 0$. By Lebesgue dominated convergence theorem, Fubini theorem and $Q(-\infty) = 0$, we derive

$$\begin{aligned}
&\int_{-\infty}^{z} [J * Q(\eta) - Q(\eta)]d\eta \\
&= \lim_{\xi\to-\infty} \int_{\xi}^{z} \int_{-\infty}^{\infty} J(y)[Q(\eta - y) - Q(\eta)]dyd\eta \\
&= \lim_{\xi\to-\infty} -\int_{\xi}^{z} \int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} Q'(\eta - \theta y)d\theta dyd\eta \\
&= \lim_{\xi\to-\infty} -\int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} [Q(z - \theta y) - Q(\xi - \theta y)]d\theta dy \\
&= -\int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} Q(z - \theta y)d\theta dy
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\int_{-\infty}^{z} [K * Q(\eta) - Q(\eta)]d\eta \\
&= \lim_{\xi\to-\infty} \int_{\xi}^{z} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)[Q(\eta - y - cs) - Q(\eta)]dydsd\eta \\
&= \lim_{\xi\to-\infty} -\int_{\xi}^{z} \int_{0}^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s) \int_{0}^{1} Q'(\eta - \theta(y + cs))d\theta dydsd\eta \\
&= \lim_{\xi\to-\infty} -\int_{0}^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s) \int_{0}^{1} [Q(z - \theta(y + cs)) - Q(\xi - \theta(y + cs))]d\theta dyds \\
&= -\int_{0}^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s) \int_{0}^{1} Q(z - \theta(y + cs))d\theta dyds,
\end{aligned} \tag{3.13}$$

which implies that $J * Q(z) - Q(z)$ and $K * Q(z) - Q(z)$ are integrable on $(-\infty, z]$ for any $z \in \mathbb{R}$. Then integrating (3.11) from $-\infty$ to z with $z < z^*$, we have

$$\begin{aligned}
&\frac{\beta S_1 - \gamma}{2} \int_{-\infty}^{z} Q(\eta)d\eta \\
&\le cQ(z) - d_2 \int_{-\infty}^{z} [J * Q(\eta) - Q(\eta)]d\eta - \frac{\beta S_1 + \gamma}{2} \int_{-\infty}^{z} [K * Q(\eta) - Q(\eta)]d\eta \\
&= cQ(z) + d_2 \int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} Q(z - \theta y)d\theta dy \\
&\quad + \frac{\beta S_1 + \gamma}{2} \int_{0}^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s) \int_{0}^{1} Q(z - \theta(y + cs))d\theta dyds.
\end{aligned} \tag{3.14}$$

Since $xQ(z - \theta x)$ is nonincreasing with respect to $\theta \in [0, 1]$, we deduce from (3.14) that

$$\begin{aligned}
&\frac{\beta S_1 - \gamma}{2} \int_{-\infty}^{z} Q(\eta)d\eta \\
&\le \left[c + d_2 \int_{-\infty}^{\infty} yJ(y)dy + \frac{\beta S_1 + \gamma}{2} \int_{0}^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s)dyds\right] Q(z) \\
&= \left[c + c\frac{\beta S_1 + \gamma}{2} \int_{0}^{T} \int_{-R_2}^{R_2} sK(y, s)dyds\right] Q(z) \quad \text{(by (H1)-(H3))} \\
&= C_0 Q(z) \quad \text{for } z < z^*,
\end{aligned}$$

where $C_0 := c + c\frac{\beta S_1+\gamma}{2} \int_0^T \int_{-R_2}^{R_2} sK(y, s)dyds$. Due to $Q(\cdot)$ is nondecreasing, we obtain for $z < z^*$ and any $\eta > 0$ that $\frac{\beta S_1-\gamma}{2} \eta Q(z - \eta) \le C_0 Q(z)$. Hence there exist a large enough η_0 and a constant $\kappa \in (0, 1)$ such that

$$Q(z - \eta_0) \le \kappa Q(z) \quad \text{for } z < z^*.$$

Define $\mu_0 := \frac{1}{\eta_0} \ln \frac{1}{\kappa} > 0$ and $L(z) := Q(z)e^{-\mu_0 z}$. Then it gives

$$L(z-\eta_0) = Q(z-\eta_0)e^{-\mu_0(z-\eta_0)} < \kappa Q(z)e^{-\mu_0 z}e^{\mu_0\eta_0} = L(z) \quad \text{for } z < z^*.$$

This together with $L(z) \geq 0$ in $\mathbb{R}$ implies that $L(-\infty)$ exists. Note from (3.2) that

$$\lim_{z\to\infty} L(z) = \lim_{z\to\infty} Q(z)e^{-\mu_0 z} = 0.$$

So there exists some positive constant L_0 such that

$$\sup_{z\in\mathbb{R}} \left\{Q(z)e^{-\mu_0 z}\right\} \leq L_0. \tag{3.15}$$

Then a direct computation yields for $z \in \mathbb{R}$ that

$$\begin{aligned} [J * Q(z)]e^{-\mu_0 z} &= \int_{-\infty}^{\infty} J(y)e^{-\mu_0 y}Q(z-y)e^{-\mu_0(z-y)}dy \\ &\leq L_0 \int_{-R_1}^{R_1} J(y)e^{-\mu_0 y}dy \quad \text{(by (H1) and (3.15))} \\ &< \infty \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} [K * Q(z)]e^{-\mu_0 z} &= \int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\mu_0(y+cs)}Q(z-y-cs)e^{-\mu_0(z-y-cs)}dyds \\ &\leq L_0 \int_0^{\infty}\int_{-R_2}^{R_2} K(y,s)e^{-\mu_0(y+cs)}dyds \quad \text{(by (H2) and (3.15))} \\ &< \infty. \end{aligned} \tag{3.17}$$

It follows from (1.5) and (3.1) that

$$cI'(z) \leq d_2[J * I(z) - I(z)] + \beta S_1 K * I(z) - \gamma I(z). \tag{3.18}$$

Integration (3.18) from $-\infty$ to z, utilizing Fubini theorem and $I(-\infty) = 0$, we have

$$cI(z) \leq d_2[J * Q(z) - Q(z)] + \beta S_1 K * Q(z) - \gamma Q(z). \tag{3.19}$$

Thus we infer from (3.15)-(3.17) and (3.19) that

$$\sup_{z\in\mathbb{R}} \left\{I(z)e^{-\mu_0 z}\right\} < \infty. \tag{3.20}$$

From (3.20), we deduce

$$\begin{aligned} [J * I(z)]e^{-\mu_0 z} &= \int_{-\infty}^{\infty} J(y)e^{-\mu_0 y}I(z-y)e^{-\mu_0(z-y)}dy \\ &\leq \sup_{z\in\mathbb{R}} \left\{I(z)e^{-\mu_0 z}\right\} \int_{-R_1}^{R_1} J(y)e^{-\mu_0 y}dy \end{aligned}$$

and

$$\begin{aligned} [K * I(z)]e^{-\mu_0 z} &= \int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\mu_0(y+cs)}I(z-y-cs)e^{-\mu_0(z-y-cs)}dy \\ &\leq \sup_{z\in\mathbb{R}} \left\{I(z)e^{-\mu_0 z}\right\} \int_0^{T}\int_{-R_2}^{R_2} K(y,s)e^{-\mu_0(y+cs)}dyds, \end{aligned}$$

which ensures that

$$\sup_{z\in\mathbb{R}}\left\{[J*I(z)]e^{-\mu_0 z}\right\}<\infty \quad\text{and}\quad \sup_{z\in\mathbb{R}}\left\{[K*I(z)]e^{-\mu_0 z}\right\}<\infty. \tag{3.21}$$

By (3.18), (3.20) and (3.21), we obtain

$$\sup_{z\in\mathbb{R}}\left\{I'(z)e^{-\mu_0 z}\right\}<\infty. \tag{3.22}$$

Moreover, (1.5) is equivalent to

$$d_2[J*I(z)-I(z)]-cI'(z)+\beta S_1 K*I(z)-\gamma I(z)=\left[\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}\right]K*I(z). \tag{3.23}$$

Since $S(-\infty)=S_1$ and $K*I(-\infty)=0$, we get that for any $\varepsilon>0$, there exists some constant $z^{**}\ll 0$ such that

$$\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}<\varepsilon \quad \text{for } z<z^{**}.$$

Then this implies that

$$\begin{aligned}\left[\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}\right]K*I(z)&\le\left[\frac{\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}+K*I(z)}{2}\right]^2\\&\le\left[\frac{\varepsilon+K*I(z)}{2}\right]^2\\&\le\left[K*I(z)\right]^2 \quad \text{for } z<z^{**}.\end{aligned} \tag{3.24}$$

It follows from (3.21) and (3.24) that

$$\sup_{z\in\mathbb{R}}\left\{e^{-2\mu_0 z}\left[\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}\right]K*I(z)\right\}<\infty. \tag{3.25}$$

Thus

$$\int_{-\infty}^{\infty}e^{-\lambda z}\left[\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}\right]K*I(z)dz<\infty \tag{3.26}$$

for $0<\text{Re}\lambda<2\mu_0$.

For $\lambda\in\mathbb{C}$ with $0<\text{Re}\lambda<\mu_0$, define the bilateral Laplace transform of $I(z)$ by $\mathcal{L}(\lambda):=\int_{\mathbb{R}}e^{-\lambda z}I(z)dz$. Noting the facts that

$$\int_{-\infty}^{\infty}e^{-\lambda z}J*I(z)dz=\mathcal{L}(\lambda)\int_{-\infty}^{\infty}e^{-\lambda y}J(y)dy,$$

$$\int_{-\infty}^{\infty}e^{-\lambda z}K*I(z)dz=\mathcal{L}(\lambda)\int_0^{\infty}\int_{-\infty}^{\infty}e^{-\lambda(y+cs)}K(y,s)dyds$$

and taking the bilateral Laplace transform on (1.5), we derive

$$\Theta(\lambda,c)\mathcal{L}(\lambda)=\int_{-\infty}^{\infty}e^{-\lambda z}\left[\beta S_1-\frac{\beta S(z)}{1+\alpha K*I(z)}\right]K*I(z)dz \tag{3.27}$$

for $\lambda\in\mathbb{C}$ with $0<\text{Re}\lambda<\mu_0$.

The property of Laplace transform [7] asserts that either there exists a real number μ_0 such that $\mathcal{L}(\lambda)$ is analytic for $\lambda\in\mathbb{C}$ with $0<\text{Re}\lambda<\mu_0$ and $\lambda=\mu_0$ is singular point of $\mathcal{L}(\lambda)$, or for $\lambda\in\mathbb{C}$ with $\text{Re}\lambda>0$, $\mathcal{L}(\lambda)$ is well defined. For (3.27), the two Laplace integrals can be analytically continued to the whole right half-plane. Otherwise the integral on the left-hand side of (3.27) has a singularity at $\lambda=\mu_0$ and it is analytic for all λ with $\text{Re}\lambda<\mu_0$. However, (3.26) implies that the integral on the right-hand side of (3.27) is actually analytic for all λ with $\text{Re}\lambda<2\mu_0$, a contradiction. Thus (3.27) holds for all λ with $\text{Re}\lambda>0$. However, Lemma 1 implies that $\Theta(\infty,c)=\infty$ for $c\in(0,c^*)$. Another contradiction appears.

Case 4: $R_0 > 1$ and $c \le 0$. Integrating (3.10) from $-\infty$ to z ($z < z^*$) twice, we deduce that

$$\begin{aligned}
0 &\ge cQ(z) \quad (\text{since } c \le 0 \text{ and } Q(z) > 0 \text{ on } \mathbb{R}) \\
&\ge d_2 \int_{-\infty}^{z} [J * Q(\eta) - Q(\eta)]d\eta + \frac{\beta S_1 + \gamma}{2} \int_{-\infty}^{z} [K * Q(\eta) - Q(\eta)]d\eta + \frac{\beta S_1 - \gamma}{2} \int_{-\infty}^{z} Q(\eta)d\eta \\
&\quad (\text{by Fubini theorem and } I(-\infty) = 0) \\
&= -d_2 \int_{-\infty}^{\infty} yJ(y) \int_0^1 Q(z - \theta y)d\theta dy - \frac{\beta S_1 + \gamma}{2} \int_0^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s) \int_0^1 Q(z - \theta(y + cs))d\theta dy ds \\
&\quad + \frac{\beta S_1 - \gamma}{2} \int_{-\infty}^{z} Q(\eta)d\eta \quad (\text{by (3.12) and (3.13)}) \\
&\ge \Big[- d_2 \int_{-\infty}^{\infty} yJ(y)dy - \frac{\beta S_1 + \gamma}{2} \int_0^{\infty} \int_{-\infty}^{\infty} (y + cs)K(y, s)dyds \Big] Q(z) + \frac{\beta S_1 - \gamma}{2} \int_{-\infty}^{z} Q(\eta)d\eta \\
&\quad (\text{since } -xQ(z - \theta x) \text{ is nondecreasing with respect to } \theta \in [0, 1]) \\
&= -c \int_0^T \int_{-R_2}^{R_2} sK(y, s)dydsQ(z) + \frac{\beta S_1 - \gamma}{2} \int_{-\infty}^{z} Q(\eta)d\eta \quad (\text{by (H1)-(H3)}) \\
&> 0 \quad (\text{since } Q(z) > 0 \text{ on } \mathbb{R},\ R_0 > 1 \text{ and } c \le 0),
\end{aligned}$$

which leads to a contradiction. Combining the above four cases, we obtain the assertions in Theorem 2.

4 Super-critical traveling waves

This section is devoted to proving the existence result for the case of $R_0 > 1$ and $c > c^*$ in Theorem 3. For this, we define the following nonnegative continuous functions on the real line.

$$\begin{aligned}
S_+(z) &: = S_1, \\
I_+(z) &: = \min\left\{e^{\lambda_1 z}, \frac{\beta S_1 - \gamma}{\alpha\gamma}\right\} = \begin{cases} e^{\lambda_1 z}, & z < z_1, \\ \dfrac{\beta S_1 - \gamma}{\alpha\gamma}, & z \ge z_1, \end{cases} \\
S_-(z) &: = \max\{S_1 - \epsilon_1^{-1} e^{\epsilon_1 z}, 0\} = \begin{cases} S_1 - \epsilon_1^{-1} e^{\epsilon_1 z}, & z < z_2, \\ 0, & z \ge z_2, \end{cases} \\
I_-(z) &: = \max\{e^{\lambda_1 z} - L_1 e^{(\lambda_1 + \epsilon_2)z}), 0\} = \begin{cases} e^{\lambda_1 z} - L_1 e^{(\lambda_1 + \epsilon_2)z}, & z < z_3, \\ 0, & z \ge z_3, \end{cases}
\end{aligned}$$

where

$$z_1 = \frac{1}{\lambda_1} \ln \frac{\beta S_1 - \gamma}{\alpha\gamma}, \quad z_2 = \frac{1}{\epsilon_1} \ln(\epsilon_1 S_1), \quad z_3 = \frac{1}{\epsilon_2} \ln \frac{1}{L_1},$$

λ_1 is defined in Lemma 1, L_1, ϵ_1 and ϵ_2 are positive constants to be determined in the following lemma.

Lemma 5 *For given small enough $\epsilon_1, \epsilon_2 > 0$ and large enough $L_1 > 1$, the functions $S_\pm(z)$ and $I_\pm(z)$ satisfy*

$$d_1[J * S_+(z) - S_+(z)] - cS_+'(z) - \frac{\beta S_+(z)K * I_-(z)}{1 + \alpha K * I_-(z)} \le 0, \quad z \in \mathbb{R}, \tag{4.1}$$

$$d_2[J * I_+(z) - I_+(z)] - cI_+'(z) + \frac{\beta S_+(z)K * I_+(z)}{1 + \alpha K * I_+(z)} - \gamma I_+(z) \le 0, \quad z \ne z_1, \tag{4.2}$$

$$d_1[J * S_-(z) - S_-(z)] - cS_-'(z) - \frac{\beta S_-(z)K * I_+(z)}{1 + \alpha K * I_+(z)} \ge 0, \quad z \ne z_2, \tag{4.3}$$

$$d_2[J * I_-(z) - I_-(z)] - cI_-'(z) + \frac{\beta S_-(z)K * I_-(z)}{1 + \alpha K * I_-(z)} - \gamma I_-(z) \ge 0, \quad z \ne z_3. \tag{4.4}$$

Proof. Proof of (4.1). By the definition of $S_+(z)$ and $I_-(z)$ in $\mathbb{R}$, one can see that (4.1) holds obviously.

Proof of (4.2). From the definition of $I_+(z)$ and the assumptions (H1), (H3), we get for $z \in \mathbb{R}$ that

$$J * I_+(z) \le \min\left\{e^{\lambda_1 z}\int_{-\infty}^{\infty} J(y)e^{-\lambda_1 y}dy, \frac{\beta S_1 - \gamma}{\alpha\gamma}\right\} \tag{4.5}$$

and

$$K * I_+(z) \le \min\left\{e^{\lambda_1 z}\int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\lambda_1(y+cs)}dyds, \frac{\beta S_1 - \gamma}{\alpha\gamma}\right\}. \tag{4.6}$$

If $z < z_1$, then $I_+(z) = e^{\lambda_1 z}$. We have for $z < z_1$ that

$$\begin{aligned}
&d_2[J * I_+(z) - I_+(z)] - cI'_+(z) + \frac{\beta S_+(z) K * I_+(z)}{1 + \alpha K * I_+(z)} - \gamma I_+(z)\\
&\le d_2[J * I_+(z) - I_+(z)] - cI'_+(z) + \beta S_1 K * I_+(z) - \gamma I_+(z)\\
&\le d_2 e^{\lambda_1 z}\left[\int_{-\infty}^{\infty} J(y)e^{-\lambda_1 y}dy - 1\right] - c\lambda_1 e^{\lambda_1 z} + \beta S_1 e^{\lambda_1 z}\int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\lambda_1(y+cs)}dyds - \gamma e^{\lambda_1 z}\\
&\quad \text{(by (4.5) and (4.6))}\\
&= e^{\lambda_1 z}\Theta(\lambda_1, c)\\
&= 0, \quad \text{(by Lemma 1)}.
\end{aligned}$$

If $z > z_1$, then $I_+(z) = \frac{\beta S_1 - \gamma}{\alpha\gamma}$ and $S_+(z) = S_1$. Using (4.5) and (4.6), we get for $z > z_1$ that

$$\begin{aligned}
&d_2[J * I_+(z) - I_+(z)] - cI'_+(z) + \frac{\beta S_+(z) K * I_+(z)}{1 + \alpha K * I_+(z)} - \gamma I_+(z)\\
&\le \frac{\beta S_1 \frac{\beta S_1 - \gamma}{\alpha\gamma}}{1 + \alpha\frac{\beta S_1 - \gamma}{\alpha\gamma}} - \gamma\frac{\beta S_1 - \gamma}{\alpha\gamma}\\
&= 0.
\end{aligned}$$

Proof of (4.3). By the definition of $S_-(z)$ and (H1), we obtain for $z \in \mathbb{R}$ that

$$J * S_-(z) \ge \max\left\{S_1 - \epsilon_1^{-1}e^{\epsilon_1 z}\int_{-\infty}^{\infty} J(y)e^{-\epsilon_1 y}dy, 0\right\}. \tag{4.7}$$

Select $\epsilon_1 \in (0, \lambda_1)$ small enough such that

$$d_1\epsilon_1^{-1}\int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon_1 y})dy + c - \beta S_1 e^{(\lambda_1 - \epsilon_1)z}\int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\lambda_1(y+cs)}dyds \ge 0, \quad z < z_2. \tag{4.8}$$

If $z < z_2$, then

$$S_-(z) = S_1 - \epsilon_1^{-1}e^{\epsilon_1 z} < S_1. \tag{4.9}$$

For $z < z_2$, it follows that

$$\begin{aligned}
&d_1[J * S_-(z) - S_-(z)] - cS'_-(z) - \frac{\beta S_-(z) K * I_+(z)}{1 + \alpha K * I_+(z)}\\
&\ge d_1[J * S_-(z) - S_-(z)] - cS'_-(z) - \beta S_-(z) K * I_+(z)\\
&\ge d_1\left[\epsilon_1^{-1}e^{\epsilon_1 z} - \epsilon_1^{-1}e^{\epsilon_1 z}\int_{-\infty}^{\infty} J(y)e^{-\epsilon_1 y}\right] + ce^{\epsilon_1 z} - \beta S_1 e^{\lambda_1 z}\int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\lambda_1(y+cs)}dyds\\
&\quad \text{(by (4.6), (4.7) and (4.9))}\\
&= e^{\epsilon_1 z}\left[d_1\epsilon_1^{-1}\int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon_1 y})dy + c - \beta S_1 e^{(\lambda_1 - \epsilon_1)z}\int_0^{\infty}\int_{-\infty}^{\infty} K(y,s)e^{-\lambda_1(y+cs)}dyds\right]\\
&\ge 0, \quad \text{(by (4.8))}.
\end{aligned}$$

If $z > z_2$, then $S_-(z) = 0$ and (4.3) holds trivially.

Proof of (4.4). From (H1), (H3) and the definition of $I_-(z)$, we deduce for $z \in \mathbb{R}$ that

$$J * I_-(z) \geq \max\left\{ e^{\lambda_1 z} \int_{-\infty}^{\infty} J(y) e^{-\lambda_1 y} dy - L_1 e^{\lambda_1 + \epsilon_2} \int_{-\infty}^{\infty} J(y) e^{-(\lambda_1+\epsilon_2)y} dy, 0 \right\} \tag{4.10}$$

and

$$K * I_-(z) \leq K * I_+(z) \leq m_0 e^{\lambda_1 z}, \tag{4.11}$$

where $m_0 := \int_0^T \int_{-R_2}^{R_2} K(y,s) e^{-\lambda_1(y+cs)} dyds$. Choose small enough $\epsilon_2 \in (0, \min\{\epsilon_1, \lambda_2 - \lambda_1\})$ and large enough $L_1 > 1$ such that $z_3 < z_2$. Then we have for $z < z_3$ that

$$I_-(z) = e^{\lambda_1 z} - L_1 e^{(\lambda_1+\epsilon_2)z} \quad \text{and} \quad S_-(z) = S_1 - \epsilon_1^{-1} e^{\epsilon_1 z} < S_1. \tag{4.12}$$

Since $\epsilon_2 < \epsilon_1 < \lambda_1$, we get for $z < z_3 < 0$ that

$$e^{(\epsilon_1-\epsilon_2)z} < 1 \quad \text{and} \quad e^{(\lambda_1-\epsilon_2)z} < 1. \tag{4.13}$$

Notice that

$$\Theta(\lambda_1, c) = 0 \quad \text{and} \quad \Theta(\lambda_1+\epsilon_2, c) < 0, \tag{4.14}$$

due to $\lambda_1 < \lambda_1 + \epsilon_2 < \lambda_2$. It follows from (4.11) and (4.12) that

$$\begin{aligned}
& -\beta S_1 K * I_-(z) + \frac{\beta S_-(z) K * I_-(z)}{1 + \alpha K * I_-(z)} \\
& \geq -\beta S_1 K * I_-(z) + \beta S_-(z) K * I_-(z)[1 - \alpha K * I_-(z)] \\
& = -\beta S_1 K * I_-(z) + \beta S_-(z) K * I_-(z) - \alpha\beta S_-(z)[K * I_-(z)]^2 \\
& \geq -\beta S_1 K * I_-(z) + \beta(S_1 - \epsilon_1^{-1} e^{\epsilon_1 z}) K * I_-(z) - \alpha\beta S_1 [K * I_-(z)]^2 \\
& \geq -\beta \epsilon_1^{-1} m_0 e^{(\lambda_1+\epsilon_1)z} - \alpha\beta S_1 m_0^2 e^{2\lambda_1 z}.
\end{aligned} \tag{4.15}$$

We derive for $z < z_3$ that

$$\begin{aligned}
& d_2[J * I_-(z) - I_-(z)] - cI_-'(z) + \frac{\beta S_-(z) K * I_-(z)}{1 + \alpha K * I_-(z)} - \gamma I_-(z) \\
& = d_2[J * I_-(z) - I_-(z)] - cI_-'(z) + \beta S_1 K * I_-(z) - \gamma I_-(z) - \beta S_1 K * I_-(z) + \frac{\beta S_-(z) K * I_-(z)}{1 + \alpha K * I_-(z)} \\
& \geq e^{\lambda_1 z}\Theta(\lambda_1, c) - L_1 e^{(\lambda_1+\epsilon_2)z}\Theta(\lambda_1+\epsilon_2, c) - \beta\epsilon_1^{-1} m_0 e^{(\lambda_1+\epsilon_1)z} - \alpha\beta S_1 m_0^2 e^{2\lambda_1 z} \\
& \quad \text{(by (4.10), (4.12), (4.15) and (2.1))} \\
& = -e^{(\lambda_1+\epsilon_2)z}\Theta(\lambda_1+\epsilon_2, c)\left[L_1 - \frac{\beta\epsilon_1^{-1} m_0 e^{(\epsilon_1-\epsilon_2)z} + \alpha\beta S_1 m_0^2 e^{(\lambda_1-\epsilon_2)z}}{-\Theta(\lambda_1+\epsilon_2, c)}\right] \\
& \geq -e^{(\lambda_1+\epsilon_2)z}\Theta(\lambda_1+\epsilon_2, c)\left[L_1 - \frac{\beta\epsilon_1^{-1} m_0 + \alpha\beta S_1 m_0^2}{-\Theta(\lambda_1+\epsilon_2, c)}\right] \quad \text{(by (4.13))} \\
& \geq 0
\end{aligned}$$

for large enough $L > 1$. If $z > z_3$, then $I_-(z) = 0$ and (4.4) follows immediately. ■

Given a constant σ satisfying $\sigma > \beta/\alpha$, we define

$$\begin{aligned}
H_1(S, I)(z) &:= d_1 J * S(z) + \sigma S(z) - \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)}, \\
H_2(S, I)(z) &:= d_2 J * I(z) + \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)}.
\end{aligned}$$

Then (1.4) and (1.5) are equivalent to

$$\begin{aligned}
cS'(z) + (\sigma + d_1)S(z) &= H_1(S, I)(z), \\
cI'(z) + (\gamma + d_2)I(z) &= H_2(S, I)(z).
\end{aligned}$$

Define a cone

$$\Gamma := \{(S,I) \in C(\mathbb{R},\mathbb{R}^2) : S_-(z) \le S(z) \le S_+(z), I_-(z) \le I(z) \le I_+(z)\}.$$

Obviously, Γ is nonempty, bounded, closed and convex in $C(\mathbb{R},\mathbb{R}^2)$. For any $(S,I) \in \Gamma$, we define an operator $F = (F_1, F_2) : \Gamma \mapsto C(\mathbb{R},\mathbb{R}^2)$ by

$$F_1(S,I)(z) := \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)} H_1(S,I)(\eta)d\eta,$$
$$F_2(S,I)(z) := \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)} H_2(S,I)(\eta)d\eta.$$

It is easy to verify that any fixed point of F is a solution for (1.4) and (1.5).

Lemma 6 *The operator $F = (F_1, F_2)$ maps Γ into itself.*

Proof. By the monotonicity of H_i $(i = 1, 2)$ with respect to its variables, it is sufficient to show for any $(S,I) \in \Gamma$ that

$$S_-(z) \le F_1(S_-, I_+)(z) \le F_1(S,I) \le F_1(S_+, I_-)(z) \le S_+(z) \tag{4.16}$$

and

$$I_-(z) \le F_2(S_-, I_-)(z) \le F_2(S,I) \le F_2(S_+, I_+)(z) \le I_+(z), \quad \forall z \in \mathbb{R}. \tag{4.17}$$

Proof of (4.16). We infer from (4.1) that

$$\begin{aligned} F_1(S_+, I_-)(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)} H_1(S_+, I_-)(\eta)d\eta \\ &\le \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)} [cS_+'(\eta) + (\sigma + d_1)S_+(\eta)]d\eta \\ &= S_+(z) \quad \text{for } z \in \mathbb{R}. \end{aligned}$$

On the other hand, we obtain from (4.3) that

$$\begin{aligned} F_1(S_-, I_+)(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)} H_1(S_-, I_+)(\eta)d\eta \\ &\ge \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)} [cS_-'(\eta) + (\sigma + d_1)S_-(\eta)]d\eta \\ &= S_-(z) \quad \text{for } z \ne z_2. \end{aligned}$$

Then utilizing the continuity of both $F_1(S_-, I_+)(z)$ and $S_-(z)$ on z_2 gives $F_1(S_-, I_+)(z) \ge S_-(z)$ for $z \in \mathbb{R}$.

Proof of (4.17). It follows from (4.3) that

$$\begin{aligned} F_2(S_+, I_+)(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)} H_2(S_+, I_+)(\eta)d\eta \\ &\le \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)} [cI_+'(\eta) + (\gamma + d_2)I_+(\eta)]d\eta \\ &= I_+(z) \quad \text{for } z \ne z_1. \end{aligned}$$

Then using the continuity of $F_2(S_+, I_+)(z)$ and $I_+(z)$ on z_1 yields $F_2(S_+, I_+)(z) \le I_+(z)$ for $z \in \mathbb{R}$. On the other hand, we have from (4.4) that

$$\begin{aligned} F_2(S_-, I_-)(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)} H_2(S_-, I_-)(\eta)d\eta \\ &\ge \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)} [cI_-'(\eta) + (\gamma + d_2)I_-(\eta)]d\eta \\ &= I_-(z) \quad \text{for } z \ne z_3. \end{aligned}$$

Then applying the continuity of both $F_2(S_-, I_-)(z)$ and $I_-(z)$ on z_3, we get $F_2(S_-, I_-)(z) \geq I_-(z)$ for $z \in \mathbb{R}$. The proof of this lemma is completed. ■

Now we introduce a functional space

$$B_\mu(\mathbb{R}, \mathbb{R}^2) := \left\{ \phi = (\phi_1, \phi_2) \in C(\mathbb{R}, \mathbb{R}^2) : \ |\phi_i|_\mu < \infty, \ i = 1, 2 \right\}$$

endowed with the norm

$$|\phi|_\mu := \max\left\{ |\phi_1|_\mu, \ |\phi_2|_\mu \right\},$$

where $0 < \mu < \min\left\{ \frac{\sigma+d_1}{c}, \frac{\gamma+d_2}{c} \right\}$ and $|\phi_i|_\mu := \sup_{z\in\mathbb{R}} |\phi_i(z)|e^{-\mu|z|}$. Clearly, $B_\mu(\mathbb{R}, \mathbb{R}^2)$ is a Banach space with the decay norm $|\cdot|_\mu$.

Lemma 7 *The operator $F = (F_1, F_2) : \Gamma \mapsto \Gamma$ is completely continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$.*

Proof. (i) Continuity. For any $(S_1, I_1) \in \Gamma$ and $(S_2, I_2) \in \Gamma$, we derive that

$$\begin{aligned}
&|H_1(S_1, I_1)(z) - H_1(S_2, I_2)(z)|e^{-\mu|z|} \\
&\leq d_1|J * S_1(z) - J * S_2(z)|e^{-\mu|z|} + \left(\sigma + \frac{\beta}{\alpha}\right)|S_1(z) - S_2(z)|e^{-\mu|z|} + 2\beta|K * I_1(z) - K * I_2(z)|e^{-\mu|z|} \\
&\leq \left(\sigma + \frac{\beta}{\alpha}\right)|S_1 - S_2|_\mu + d_1 e^{-\mu|z|} \int_{-R_1}^{R_1} J(y)|S_1(z-y) - S_2(z-y)|dy \\
&\quad + 2\beta e^{-\mu|z|} \int_0^T \int_{-R_2}^{R_2} K(y,s)|I_1(z-y-cs) - I_2(z-y-cs)|dyds \\
&\leq \left(\sigma + \frac{\beta}{\alpha}\right)|S_1 - S_2|_\mu + d_1|S_1 - S_2|_\mu e^{-\mu|z|}e^{\mu(|z|+R_1)} \int_{-R_1}^{R_1} J(y)dy \\
&\quad + 2\beta|I_1 - I_2|_\mu e^{-\mu|z|}e^{\mu(|z|+R_2+cT)} \int_0^T \int_{-R_2}^{R_2} K(y,s)dyds \\
&= \left(\sigma + \frac{\beta}{\alpha} + d_1 e^{\mu R_1}\right)|S_1 - S_2|_\mu + 2\beta e^{\mu(R_2+cT)}|I_1 - I_2|_\mu
\end{aligned}$$

and

$$\begin{aligned}
&|H_2(S_1, I_1)(z) - H_2(S_2, I_2)(z)|e^{-\mu|z|} \\
&\leq d_2|J * I_1(z) - J * I_2(z)|e^{-\mu|z|} + \frac{\beta}{\alpha}|S_1(z) - S_2(z)|e^{-\mu|z|} + 2\beta|K * I_1(z) - K * I_2(z)|e^{-\mu|z|} \\
&\leq \frac{\beta}{\alpha}|S_1 - S_2|_\mu + (d_2 e^{\mu R_1} + 2\beta e^{\mu(R_2+cT)})|I_1 - I_2|_\mu,
\end{aligned}$$

which implies that there exists a positive constant C_0 such that

$$|H_i(S_1, I_1) - H_i(S_2, I_2)|_\mu \leq C_0(|S_1 - S_2|_\mu + |I_1 - I_2|_\mu), \quad i = 1, 2.$$

Then we obtain that

$$\begin{aligned}
|F_1(S_1, I_1) - F_1(S_2, I_2)|_\mu &= |F_1(S_1, I_1)(z) - F_1(S_2, I_2)(z)|e^{-\mu|z|} \\
&\leq \frac{1}{c}|H_1(S_1, I_1) - H_1(S_2, I_2)|_\mu \int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)e^{\mu|\eta|}e^{-\mu|z|}} d\eta \\
&\leq \frac{1}{c}|H_1(S_1, I_1) - H_1(S_2, I_2)|_\mu \int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)e^{\mu|\eta-z|}} d\eta \\
&= \frac{C_0}{\sigma + d_1 - c\mu}(|S_1 - S_2|_\mu + |I_1 - I_2|_\mu)
\end{aligned}$$

and

$$\begin{aligned}|F_2(S_1,I_1)-F_2(S_2,I_2)|_\mu &= |F_2(S_1,I_1)(z)-F_2(S_2,I_2)(z)|e^{-\mu|z|}\\ &\le \frac{1}{c}|H_2(S_1,I_1)-H_2(S_2,I_2)|_\mu\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)}e^{\mu|\eta|}e^{-\mu|z|}d\eta\\ &\le \frac{1}{c}|H_2(S_1,I_1)-H_2(S_2,I_2)|_\mu\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)}e^{\mu|\eta-z|}d\eta\\ &= \frac{C_0}{\gamma+d_2-c\mu}(|S_1-S_2|_\mu+|I_1-I_2|_\mu),\end{aligned}$$

which proves that F is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$.

(ii) Compactness. For any $(S,I)\in\Gamma$, we get $H_1(S,I)(z)\le(d_1+\sigma)S_1$ and $H_2(S,I)(z)\le\frac{(d_2+\gamma)(\beta S_1-\gamma)}{\alpha\gamma}$ in $\mathbb{R}$. Then for $z\in\mathbb{R}$, we have

$$\begin{aligned}\left|\frac{dF_1(S,I)(z)}{dz}\right| &= \left|-\frac{\sigma+d_1}{c^2}\int_{-\infty}^{z}e^{-\frac{\sigma+d_1}{c}(z-\eta)}H_1(S,I)(\eta)d\eta+\frac{1}{c}H_1(S,I)(z)\right|\\ &\le \frac{2(\sigma+d_1)S_1}{c}\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\left|\frac{dF_2(S,I)(z)}{dz}\right| &= \left|-\frac{\gamma+d_2}{c^2}\int_{-\infty}^{z}e^{-\frac{\gamma+d_2}{c}(z-\eta)}H_2(S,I)(\eta)d\eta+\frac{1}{c}H_2(S,I)(z)\right|\\ &\le \frac{2(\gamma+d_2)(\beta S_1-\gamma)}{c\alpha\gamma}.\end{aligned} \tag{4.19}$$

From Lemma 6, we infer that

$$|F_1(S,I)(z)|+|F_2(S,I)(z)|\le S_1+\frac{\beta S_1-\gamma}{\alpha\gamma}\quad\text{for } z\in\mathbb{R}. \tag{4.20}$$

Then for any $\varepsilon>0$, there exists a sufficiently large number $Z>0$ such that

$$\big[|F_1(S,I)(z)|+|F_2(S,I)(z)|\big]e^{-\mu|z|}\le\left(S_1+\frac{\beta S_1-\gamma}{\alpha\gamma}\right)e^{-\mu Z}<\varepsilon,\quad z>|Z|. \tag{4.21}$$

Using (4.18)-(4.20) and Arzelà-Ascoli theorem, we can select finite elements in $F(\Gamma)$ such that they are a finite ε-net of $F(\Gamma)(z)$ on $[-Z,Z]$ with the supremum norm. Together with (4.21), we conclude that it is also a finite ε-net of $F(\Gamma)(z)$ in $\mathbb{R}$ with the decay norm $|\cdot|_\mu$. Hence F is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$. The proof is finished. ∎

Applying Lemmas 6, 7 and Schauder's fixed point theorem, we obtain the following results.

Proposition 8 *Suppose that $R_0>1$ and $c>c^*$. Then (1.4)-(1.5) has a solution $(S,I)(z)\in B_\mu(\mathbb{R},\mathbb{R}^2)$ satisfying*

$$S_-(z)\le S(z)\le S_+(z)\quad\textit{and}\quad I_-(z)\le I(z)\le I_+(z)\quad\textit{for}\ z\in\mathbb{R}. \tag{4.22}$$

In the remainder of this section, we will investigate some properties concerning the solution (S,I) of (1.4)-(1.5).

Proposition 9 *Let $(S,I)(z)$ be a solution of (1.4)-(1.5) in $\mathbb{R}$. Then the following assertions are valid.*

(i) $0<S(z)<S_1$, $0<I(z)<\frac{\beta S_1-\gamma}{\alpha\gamma}$ *for* $z\in\mathbb{R}$.

(ii) $S(-\infty)=S_1$, $I(-\infty)=0$ *and* $I(z)=O(e^{\lambda_1 z})$ *as* $z\to-\infty$.

(iii) The limit $S(\infty)$ *exists,* $S(\infty):=S_2<S_1$ *and* $I(\infty)=0$.

(iv) $S'(\pm\infty)=I'(\pm\infty)=0$;

(v) $\gamma\int_{\mathbb{R}}I(z)dz=\beta\int_{\mathbb{R}}\frac{S(z)K*I(z)}{1+\alpha K*I(z)}dz=c(S_1-S_2)$.

Proof. (i) Noting that $\sigma > \beta/\alpha$, we infer from (4.22) that

$$\begin{aligned} S(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)}\left[d_1 J * S(\eta) + \sigma S(\eta) - \frac{\beta S(\eta) K * I(\eta)}{1+\alpha K * I(\eta)}\right] d\eta \\ &\geq \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\sigma+d_1}{c}(z-\eta)}\left[d_1 J * S_-(\eta) + \left(\sigma - \frac{\beta}{\alpha}\right) S_-(\eta)\right] d\eta \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} I(z) &= \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)}\left[d_2 J * I(\eta) + \frac{\beta S(\eta) K * I(\eta)}{1+\alpha K * I(\eta)}\right] d\eta \\ &\geq \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\gamma+d_2}{c}(z-\eta)}\left[d_2 J * I_-(\eta) + \frac{\beta S_-(\eta) K * I_-(\eta)}{1+\alpha K * I_-(\eta)}\right] d\eta \\ &> 0 \quad \text{for } z \in \mathbb{R}. \end{aligned}$$

Assume that there exists some $\hat{z} \in \mathbb{R}$ such that $S(\hat{z}) = S_1$, then $S'(\hat{z}) = 0$. Since (H1) and (H3), we deduce from (1.4) that

$$\begin{aligned} 0 &= d_1[J * S(\hat{z}) - S(\hat{z})] - cS'(\hat{z}) - \frac{\beta S(\hat{z}) K * I(\hat{z})}{1+\alpha K * I(\hat{z})} \\ &= d_1[J * S(\hat{z}) - S_1] - \frac{\beta S_1 K * I(\hat{z})}{1+\alpha K * I(\hat{z})} \quad (\text{since } S(\hat{z}) = S_1 \text{ and } S'(\hat{z}) = 0) \\ &\leq -\frac{\beta S_1 K * I(\hat{z})}{1+\alpha K * I(\hat{z})} \quad (\text{since } I(z) > 0 \text{ and } 0 < S(z) \leq S_1 \text{ in } \mathbb{R}) \\ &< 0, \end{aligned}$$

which yields a contradiction. Thus $S(z) < S_1$ for $z \in \mathbb{R}$. Suppose that there is a $\tilde{z} \in \mathbb{R}$ such that $I(\tilde{z}) = \frac{\beta S_1 - \gamma}{\alpha\gamma}$, then $I'(\tilde{z}) = 0$. By (H1) and (H3), we obtain from (1.5) that

$$\begin{aligned} 0 &= d_2[J * I(\tilde{z}) - I(\tilde{z})] - cI'(\tilde{z}) + \frac{\beta S(\tilde{z}) K * I(\tilde{z})}{1+\alpha K * I(\tilde{z})} - \gamma I(\tilde{z}) \\ &< \frac{\beta S_1 \frac{\beta S_1 - \gamma}{\alpha\gamma}}{1+\alpha \frac{\beta S_1 - \gamma}{\alpha\gamma}} - \gamma \frac{\beta S_1 - \gamma}{\alpha\gamma} \\ &\quad \left(\text{since } I(\tilde{z}) = \frac{\beta S_1 - \gamma}{\alpha\gamma}, I'(\tilde{z}) = 0, I(z) \leq \frac{\beta S_1 - \gamma}{\alpha\gamma} \text{ and } S(z) < S_1 \text{ in } \mathbb{R}\right) \\ &= 0, \end{aligned}$$

which leads to a contradiction. Hence $I(z) < \frac{\beta S_1 - \gamma}{\alpha\gamma}$ for $z \in \mathbb{R}$.

(ii) Applying sandwich rule in (4.22) gives that $S(-\infty) = S_1$, $I(-\infty) = 0$ and $I(z) = O(e^{\lambda_1 z})$ as $z \to -\infty$.

(iii) Using (4.22) and (1.5), we have that $|I'(z)|$ is uniformly bounded in $\mathbb{R}$. By the analogous arguments to Lemma 4, one can get that $\int_{\mathbb{R}} I(z)dz < \infty$. Therefore, $I(\infty) = 0$. To derive the existence of $S(\infty)$, on the contrary, we assume that $\limsup_{z\to\infty} S(z) > \liminf_{z\to\infty} S(z)$. By Fluctuation Lemma [8], we have that there exists a sequence $\{\xi_n\}$ satisfying $\xi_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} S(\xi_n) = \limsup_{z\to\infty} S(z) = \rho_1 \quad \text{and} \quad S'(\xi_n) = 0. \tag{4.23}$$

At the same time, there exists a sequence $\{\eta_n\}$ satisfying $\eta_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} S(\eta_n) = \liminf_{z\to\infty} S(z) = \rho_2 < \rho_1 \quad \text{and} \quad S'(\eta_n) = 0. \tag{4.24}$$

Utilizing the similar arguments in [9–11], one can deduce that $S(\xi_n + y) \to \rho_1$ and $S(\eta_n + y) \to \rho_2$ as $n \to \infty$ for arbitrary $y \in [-R_1, R_1]$. In view of (3.2), we have

$$\lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz = 0. \tag{4.25}$$

Integrating (1.4) from η_n to ξ_n gives

$$\begin{aligned}
0 &< c(\rho_1 - \rho_2) \quad \text{(by (4.23) and (4.24))} \\
&= c \lim_{n\to\infty} [S(\xi_n) - S(\eta_n)] \\
&= d_1 \lim_{n\to\infty} \int_{\xi_n}^{\eta_n} \int_{-R_1}^{R_1} J(y)[S(z-y) - S(y)] dy dz - \lim_{n\to\infty} \int_{\xi_n}^{\eta_n} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz \quad \text{(by (H1))} \\
&= d_1 \lim_{n\to\infty} \int_{\xi_n}^{\eta_n} \int_{-R_1}^{R_1} J(y)[S(z-y) - S(y)] dy dz \quad \text{(by (4.25))} \\
&= -d_1 \lim_{n\to\infty} \int_{\xi_n}^{\eta_n} \int_{-R_1}^{R_1} y J(y) \int_0^1 S'(z - \theta y) d\theta dy dz \\
&= d_1 \lim_{n\to\infty} \int_{-R_1}^{R_1} y J(y) \int_0^1 [S(\eta_n - \theta y) - S(\xi_n - \theta y)] d\theta dy \quad \text{(by Fubini theorem)} \\
&= 0, \quad \text{(by Lebesgue dominated convergence theorem)},
\end{aligned}$$

a contradiction appears. This implies that $\limsup_{z\to\infty} S(z) = \liminf_{z\to\infty} S(z)$, i.e., the limit $S(\infty)$ exists ($S(\infty) := S_2$). We are now in a position to show that $S_2 < S_1$. Due to $S(z) < S_1$ in $\mathbb{R}$, we have $S_2 \le S_1$. Assume that $S_2 = S_1$. Then using (3.2) and integrating (1.4) over $\mathbb{R}$, we obtain

$$\begin{aligned}
0 &= c(S_2 - S_1) \\
&= d_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(y)[S(z-y) - S(z)] dy dz - \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz \\
&= -\int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz \quad \text{(by Fubini theorem)} \\
&< 0,
\end{aligned}$$

which yields a contradiction. Thus $S_2 < S_1$.

(iv) Recalling that $S(-\infty) = S_1$, $S(\infty) = S_2$, $I(\pm\infty) = 0$ and applying Lebesgue dominated convergence theorem, (H1) and (H3), we have that

$$\lim_{z\to-\infty} J * S(z) = S_1, \quad \lim_{z\to\infty} J * S(z) = S_2 \quad \text{and} \lim_{\pm\infty} J * I(z) = \lim_{\pm\infty} K * I(z) = 0.$$

Together with (1.4) and (1.5), we conclude that $S'(\pm\infty) = I'(\pm\infty) = 0$.

(v) Integrating (1.4) over $\mathbb{R}$ yields

$$\begin{aligned}
0 &= d_1 \int_{-\infty}^{\infty} [J * S(z) - S(z)] dz - c \int_{-\infty}^{\infty} S'(z) dz - \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz \\
&= c(S_1 - S_2) - \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz, \quad \text{(by Fubini theorem, } S(-\infty) = S_1 \text{ and } S(\infty) = S_2).
\end{aligned} \tag{4.26}$$

Integrating (1.5) over $\mathbb{R}$ gives

$$\begin{aligned}
0 &= d_2 \int_{-\infty}^{\infty} [J * I(z) - I(z)] dz - c \int_{-\infty}^{\infty} I'(z) dz + \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz - \gamma \int_{-\infty}^{\infty} I(z) dz \\
&= \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{1 + \alpha K * I(z)} dz - \gamma \int_{-\infty}^{\infty} I(z) dz, \quad \text{(by Fubini theorem and } I(\pm\infty) = 0).
\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27), we obtain that

$$\gamma \int_{-\infty}^{\infty} I(z)dz = \beta \int_{-\infty}^{\infty} \frac{S(z)K * I(z)}{1 + \alpha K * I(z)} dz = c(S_1 - S_2).$$

The assertions of this proposition are completed. ■

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