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Existence and Uniqueness of Solution for Navier Problems with Degenerated Operators in Weighted Sobolev Spaces

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Abstract: In this paper we are interested in the existence of solutions for Navier problem associated with the degenerate nonlinear elliptic equations

$$\begin{cases} \Delta\big[\omega_1\,|\Delta u|^{p-2}\Delta u+\omega_2|\Delta u|^{q-2}\Delta u\big] \\ -\operatorname{div}\big[\mathcal{A}(x,\nabla u)\,\nu_1+\mathcal{B}(x,u,\nabla u)\,\nu_2\big]=f_0(x)-\displaystyle\sum_{j=1}^{n}D_jf_j(x)\ \text{ in }\ \Omega, \\ u(x)=\Delta u(x)=0\ \text{ on }\ \partial\Omega, \end{cases}$$

in the setting of the weighted Sobolev spaces.
Keywords: degenerate nonlinear elliptic equations, weighted Sobolev spaces.

1 Introduction

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $X=W_0^{1,r}(\Omega,\nu_1)\cap W^{2,p}(\Omega,\omega_1)$ (see Definition 3 and Definition 4) for the Navier problem

$$(P)\begin{cases} Lu(x)=f_0(x)-\displaystyle\sum_{j=1}^{n}D_jf_j(x)\ \text{ in }\ \Omega, \\ u(x)=\Delta u(x)=0\ \text{ on }\ \partial\Omega, \end{cases}$$

where L is the partial differential operator

$$Lu(x)=\Delta\big[\omega_1\,|\Delta u|^{p-2}\Delta u+\omega_2|\Delta u|^{q-2}\Delta u\big]-\operatorname{div}\big[\mathcal{A}(x,\nabla u)\,\nu_1+\mathcal{B}(x,u,\nabla u)\,\nu_2\big]$$

where $D_j=\partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, ν_1 and ν_2 are four weight functions, $2\le q<p<\infty$, $2\le s<r<\infty$ and the functions $\mathcal{A}_j:\Omega\times\mathbb{R}^n\to\mathbb{R}$ and $\mathcal{B}_j:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j=1,...,n)$ satisfying the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\xi)$ is measurable on Ω for all $\xi\in\mathbb{R}^n$,
$\xi\mapsto\mathcal{A}_j(x,\xi)$ is continuous on $\mathbb{R}^n$ for almost all $x\in\Omega$;
(H2) There exists a constant $\theta_1>0$ such that

$$\langle\mathcal{A}(x,\xi)-\mathcal{A}(x,\xi'),(\xi-\xi')\rangle\ge\theta_1\,|\xi-\xi'|^r,$$

whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\ne\xi'$, and $\mathcal{A}(x,\xi)=(\mathcal{A}_1(x,\xi),...,\mathcal{A}_n(x,\xi))$ (where $\langle.,.\rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^n$);
(H3) $\langle\mathcal{A}(x,\xi),\xi\rangle\ge\lambda_1|\xi|^r$, where λ_1 is a positive constant;
(H4) $|\mathcal{A}(x,\xi)|\le K_1(x)+h_1(x)|\xi|^{r/r'}$, where K_1 and h_1 are nonnegative functions, with $h_1\in L^\infty(\Omega)$ and $K_1\in L^{r'}(\Omega,\nu_1)$ (with $1/r+1/r'=1$);

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(H5) $x\mapsto\mathcal{B}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n$,
$(\eta,\xi)\mapsto\mathcal{B}_j(x,\eta,\xi)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$ for almost all $x\in\Omega$;
(H6) There exists a constant $\theta_2>0$ such that

$$\langle\mathcal{B}(x,\eta,\xi)-\mathcal{B}(x,\eta',\xi'),(\xi-\xi')\rangle\geq\theta_2\,|\xi-\xi'|^s,$$

whenever $\xi,\xi'\in\mathbb{R}^n$, $\xi\neq\xi'$, where $\mathcal{B}(x,\eta,\xi)=(\mathcal{B}_1(x,\eta,\xi),...,\mathcal{B}_n(x,\eta,\xi))$;
(H7) $\langle\mathcal{B}(x,\eta,\xi),\xi\rangle\geq\lambda_2|\xi|^s+\Lambda_2|\eta|^s$, where $\lambda_2>0$ and $\Lambda_2\geq 0$ are constants;
(H8) $|\mathcal{B}(x,\eta,\xi)|\leq K_2(x)+g_1(x)|\eta|^{s/s'}+g_2(x)|\xi|^{s/s'}$, where K_2,g_1 and g_2 are nonnegative functions, with g_1 and $g_2\in L^\infty(\Omega)$, and $K_2\in L^{s'}(\Omega,\omega_2)$ (with $1/q+1/q'=1$).

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega=\omega(x)$, $x\in\Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus, $\mu_\omega(E)=\int_E\omega(x)\,dx$ for measurable sets $E\subset\mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. In the particular case where $p=q=2$ and $\omega_1=\omega_2\equiv 1$, $\nu_1=0$ and $\nu_2=1$ we have the equation

$$\Delta^2u-\sum_{j=1}^n D_j\mathcal{B}_j(x,u,\nabla u)=f,$$

where Δ^2u is the biharmonic operator. Biharmonic equations appear in the study of mathematical model in several real-life processes as, among others, radar imaging or incompressible flows.

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2], [3], [4] and [7]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1] and [6]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [15]). These classes have found many useful applications in harmonic analysis (see [17]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [13]). There are, in fact, many interesting examples of weights (see [12] for p-admissible weights).

In the non-degenerate case, for all $f\in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u=f(x),\ \text{in}\ \Omega\\ u(x)=0,\ \text{on}\ \partial\Omega\end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ (see [11]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u=f(x),\ \text{in}\ \Omega\\ u(x)=0,\ \text{on}\ \partial\Omega\end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [5]), where $\Delta_pu=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [16] and the references therein), and the degenerated p-Laplacian was studied in [7].

The following theorem will be proved in section 3.

Theorem 1 *Let $2\leq q<p<\infty$, $2\leq s<r<\infty$ and assume (H1)-(H8). If*
(i) $\omega_1\in A_p$, $\omega_2\in\mathcal{W}(\Omega)$, and $\dfrac{\omega_2}{\omega_1}\in L^{p/(p-q)}(\Omega,\omega_1)$;
(ii) $\nu_1\in A_r$, $\nu_2\in A_s$ and $\dfrac{\nu_2}{\nu_1}\in L^{r/(r-s)}(\Omega,\nu_1)$;
(iii) $f_j/\nu_1\in L^{r'}(\Omega,\nu_1)$ $(j=0,1,...,n)$.

Then the problem (P) has a unique solution $u \in X = W_0^{1,r}(\Omega,\nu_1)\cap W^{2,p}(\Omega,\omega_1)$. Moreover, we have

$$\|u\|_X \le \gamma_{p,r}\left(\frac{1}{p'}M^{p'-1}+\frac{1}{r'}\left(\frac{M}{\lambda_1}\right)^{r'-1}\right)$$

where $\gamma_{p,r} = p\,r/(p\,r-p-r)$, $M = C_\Omega\|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)}+\sum_{j=1}^{n}\|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}$ and C_Ω is the constant in Theorem 3.

2 DEFINITIONS AND BASIC RESULTS

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0<\omega<\infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1<p<\infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}\,dx\right)^{p-1}\le C,$$

for all balls $B\subset\mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1<q\le p$, then $A_q\subset A_p$ (see [10], [12] or [17] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x;2r))\le C\,\mu(B(x;r)),$$

for every ball $B = B(x;r)\subset\mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,dx$. If $\omega\in A_p$, then μ is doubling (see Corollary 15.7 in [12]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x\in\mathbb{R}^n$, is in A_p if and only if $-n<\alpha<n(p-1)$ (see Corollary 4.4, Chapter IX in [17]).

If $\omega\in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p\le C\frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [12]). Therefore, if $\mu(E)=0$ then $|E|=0$. The measure μ and the Lebesgue measure $|.|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E)=0$ if and only if $|E|=0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 1 *Let ω be a weight, and let $\Omega\subset\mathbb{R}^n$ be open. For $1<p<\infty$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that*

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p\,\omega\,dx\right)^{1/p}<\infty.$$

If $\omega\in A_p$, $1<p<\infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [18]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2 *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $1<p<\infty$, k be a nonnegative integer and $\omega\in A_p$. We shall denote by $W^{k,p}(\Omega,\omega)$, the weighted Sobolev spaces, the set of all functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$, $1\le|\alpha|\le k$. The norm in the space $W^{k,p}(\Omega,\omega)$ is defined by*

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p\,\omega\,dx+\sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u|^p\,\omega\,dx\right)^{1/p}. \tag{1}$$

If $\omega\in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (1) (see Corollary 2.1.6 in [18]). We also define the space $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (1). We have that the spaces $W^{k,p}(\Omega,\omega)$ and $W_0^{k,p}(\Omega,\omega)$ are Banach spaces.

The space $W_0^{1,p}(\Omega,\omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (1). Equipped with this norm, $W_0^{1,p}(\Omega,\omega)$ is a reflexive Banach space (see [14] for more information about the spaces $W^{1,p}(\Omega,\omega)$). The dual of space $W_0^{1,p}(\Omega,\omega)$ is the space

$$
\begin{aligned}
&[W_0^{1,p}(\Omega,\omega)]^* \\
&= \{T = f_0 - \operatorname{div}(F),\ F = (f_1,...,f_n) :\ \frac{f_j}{\omega} \in L^{p'}(\Omega,\omega), j = 0,1,...,n\}.
\end{aligned}
$$

It is evident that a weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (where c_1 and c_2 are constants), give nothing new (the space $W_0^{1,p}(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following results.

Theorem 2 *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$ a.e. on Ω;
(ii) $|u_{m_k}(x)| \le \Phi(x)$ a.e. on Ω.

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [9]. ■

Theorem 3 *(The weighted Sobolev inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist constants C_Ω and δ positive such that for all $u \in W_0^{1,p}(\Omega,\omega)$ and all k satisfying $1 \le k \le n/(n-1) + \delta$,*

$$
\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}, \tag{2}
$$

where C_Ω depends only on n, p, the A_p-constant $C(p,\omega)$ of ω and the diameter of Ω.

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [8]). To extend the estimates (2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2). ■

Lemma 4 *Let $1 < p < \infty$.*
(a) There exists a constant $\alpha_p > 0$ such that

$$
\left|\, |x|^{p-2}x - |y|^{p-2}y \right| \le \alpha_p\, |x-y|(|x|+|y|)^{p-2}, \forall x, y \in \mathbb{R}^n;
$$

(b) There exist two positive constants β_p, γ_p such that for every $x, y \in \mathbb{R}^n$

$$
\beta_p\,(|x|+|y|)^{p-2}|x-y|^2 \le \langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \le \gamma_p\,(|x|+|y|)^{p-2}|x-y|^2,
$$

where $\langle .,.\rangle$ denote the Euclidian scalar product in $\mathbb{R}^n$.

Proof. See [5], Proposition 17.2 and Proposition 17.3. ■

Definition 3 *We denote by $X = W_0^{1,r}(\Omega,\nu_1) \cap W^{2,p}(\Omega,\omega_1)$ with the norm*

$$
\|u\|_X = \|\Delta u\|_{L^p(\Omega,\omega_1)} + \|\,|\nabla u|\,\|_{L^r(\Omega,\nu_1)}.
$$

Definition 4 *We say that an element $u \in X = W_0^{1,r}(\Omega,\nu_1) \cap W^{2,p}(\Omega,\omega_1)$ is a (weak) solution of problem (P) if*

$$
\begin{aligned}
&\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\, \omega_1\, dx + \int_\Omega |\Delta u|^{q-2}\Delta u\Delta\varphi\,\omega_2\, dx \\
&+ \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle\, \nu_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla\varphi\rangle\,\nu_2\, dx \\
&= \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx,
\end{aligned}
$$

for all $\varphi \in X$.

Remark 5 *(a) If* $\dfrac{\omega_2}{\omega_1} \in L^{p/(p-q)}(\Omega,\omega_1)$ *($2 \le q < p < \infty$) then there exits a constant* $M_1 > 0$ *such that*

$$\|u\|_{L^q(\Omega,\omega_2)} \le M_1\, \|u\|_{L^p(\Omega,\omega_1)},$$

where $M_1 = \|\omega_2/\omega_1\|^{1/q}_{L^{p/(p-q)}(\Omega,\omega_1)}$*. In fact, by Hölder's inequality,*

$$\begin{aligned}
\|u\|^q_{L^q(\Omega,\omega_2)} &= \int_\Omega |u|^q\, \omega_2\, dx = \int_\Omega |u|^q\, \frac{\omega_2}{\omega_1}\, \omega_1\, dx \\
&\le \left(\int_\Omega |u|^{q\,p/q} \omega_1\, dx\right)^{q/p} \left(\int_\Omega \left(\frac{\omega_2}{\omega_1}\right)^{p/(p-q)} \omega_1\, dx\right)^{(p-q)/p} \\
&= \|u\|^q_{L^p(\Omega,\omega_1)}\, \|\omega_2/\omega_1\|_{L^{p/(p-q)}(\Omega,\omega_1)}.
\end{aligned}$$

(b) Analogously, if $\dfrac{\nu_2}{\nu_1} \in L^{r/(r-s)}(\Omega,\nu_1)$ *(with* $2 \le s < r < \infty$*), then there exits a constant* $M_2 = \|\nu_2/\nu_1\|^{1/s}_{L^{r/(r-s)}(\Omega,\nu_1)}$ *such that*

$$\|u\|_{L^s(\Omega,\nu_2)} \le M_2\, \|u\|_{L^r(\Omega,\nu_1)}.$$

3 PROOF OF THEOREM 1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 6 *Let* $A : X \to X^*$ *be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space* X*. Then the following assertions hold:*
(a) For each $T \in X^*$ *the equation* $Au = T$ *has a solution* $u \in X$*;*
(b) If the operator A *is strictly monotone, then equation* $A\,u = T$ *is uniquely solvable in* X*.*

Proof. See Theorem 26.A in [20]. ■

To prove Theorem 1, we define $\mathbf{B}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4} : X \times X \to \mathbb{R}$ and $\mathbf{T} : X \to \mathbb{R}$ by

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B_3}(u,\varphi) + \mathbf{B_4}(u,\varphi),$$

$$\begin{aligned}
\mathbf{B_1}(u,\varphi) &= \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle\, \nu_1\, dx, \\
\mathbf{B_2}(u,\varphi) &= \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\, \nu_2\, dx, \\
\mathbf{B_3}(u,\varphi) &= \int_\Omega |\Delta u|^{p-2} \Delta u \Delta\varphi\, \omega_1\, dx, \\
\mathbf{B_4}(u,\varphi) &= \int_\Omega |\Delta u|^{q-2} \Delta u \Delta\varphi\, \omega_2\, dx, \\
\mathbf{T}(\varphi) &= \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx.
\end{aligned}$$

Then $u \in X$ is a (weak) solution to problem (P) if

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B_3}(u,\varphi) + \mathbf{B_4}(u,\varphi) = \mathbf{T}(\varphi),$$

for all $\varphi \in X$.

Step 1. For $j = 1, ..., n$ we define the operator $F_j : X \to L^{r'}(\Omega,\nu_1)$ as

$$(F_j u)(x) = \mathcal{A}_j(x, \nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4) we obtain

$$\begin{aligned}
\|F_j u\|^{r'}_{L^{r'}(\Omega,\nu_1)} &= \int_\Omega |F_j u(x)|^{r'} \nu_1\, dx \\
&= \int_\Omega |\mathcal{A}_j(x,\nabla u)|^{r'} \nu_1\, dx \\
&\leq \int_\Omega \left(K_1 + h_1|\nabla u|^{r/r'}\right)^{r'} \nu_1\, dx \\
&\leq C_r \int_\Omega \left[(K_1^{r'} + h_1^{r'}|\nabla u|^r)\nu_1\right] dx \\
&= C_r\left[\int_\Omega K_1^{r'} \nu_1\, dx + \int_\Omega h_1^{r'}|\nabla u|^r \nu_1\, dx\right] \\
&\leq C_r\left(\|K_1\|^{r'}_{L^{r'}(\Omega,\nu_1)} + \|h_1\|^{r'}_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^r_{L^r(\Omega,\nu_1)}\right) \\
&\leq C_r\left(\|K_1\|^{r'}_{L^{r'}(\Omega,\nu_1)} + \|h_1\|^{r'}_{L^\infty(\Omega)}\|u\|^r_X\right), \qquad (3)
\end{aligned}$$

where the constant C_r depends only on r. Therefore, in (3) we obtain

$$\|F_j u\|_{L^{r'}(\Omega,\nu_1)} \leq C_r^{1/r'}\left(\|K_1\|_{L^{r'}(\Omega,\nu_1)} + \|h_1\|_{L^\infty(\Omega)})\, \|u\|_X^{r/r'}\right).$$

(ii) Let $u_m \to u$ in X as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{r'}(\Omega,\nu_1)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \to u$ in X, then $|\nabla u_m| \to |\nabla u|$ in $L^r(\Omega,\nu_1)$. Using Theorem 2 (since $\nu_1 \in A_r$), there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_1 \in L^r(\Omega,\nu_1)$ such that

$$\begin{aligned}
&u_{m_k}(x) \to u(x) \text{ a.e. in } \Omega, \\
&D_j u_{m_k}(x) \to D_j u(x) \text{ a.e. in } \Omega, \\
&|\nabla u_{m_k}(x)| \leq \Phi_1(x) \text{ a.e. in } \Omega.
\end{aligned}$$

Next, applying (H4) we obtain

$$\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{r'}_{L^{r'}(\Omega,\nu_1)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{r'} \nu_1\, dx \\
&= \int_\Omega |\mathcal{A}_j(x,\nabla u_{m_k}) - \mathcal{A}_j(x,\nabla u)|^{r'} \nu_1\, dx \\
&\leq C_r \int_\Omega \left(|\mathcal{A}_j(x,\nabla u_{m_k})|^{r'} + |\mathcal{A}_j(x,\nabla u)|^{r'}\right)\nu_1\, dx \\
&\leq C_r\left[\int_\Omega \left(K_1 + h_1|\nabla u_{m_k}|^{r/r'}\right)^{r'} \nu_1\, dx\right. \\
&+ \left.\int_\Omega \left(K_1 + h_1|\nabla u|^{r/r'}\right)^{r'} \nu_1\, dx\right] \\
&\leq 2\,C_r \int_\Omega \left(K_1 + h_1\Phi_1^{r/r'}\right)^{r'} \nu_1\, dx \\
&\leq 2\,C_r\left[\int_\Omega K_1^{r'} \nu_1\, dx + \int_\Omega h_1^{r'}\Phi_1^r \nu_1\, dx\right] \\
&\leq 2\,C_r\left[\|K_1\|^{r'}_{L^{r'}(\Omega,\nu_1)} + \|h_1\|^{r'}_{L^\infty(\Omega)}\int_\Omega \Phi_1^r \nu_1\, dx\right] \\
&= 2\,C_r\left[\|K_1\|^{r'}_{L^{r'}(\Omega,\nu_1)} + \|h_1\|^{r'}_{L^\infty(\Omega)}\|\Phi_1\|^r_{L^r(\Omega,\nu_1)}\right].
\end{aligned}$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, \nabla u_{m_k}(x)) \rightarrow \mathcal{A}_j(x, \nabla u(x)) = F_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{r'}(\Omega,\nu_1)} \rightarrow 0,$$

that is,

$$F_j u_{m_k} \rightarrow F_j u \;\text{ in }\; L^{r'}(\Omega, \nu_1).$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$F_j u_m \rightarrow F_j u \;\text{ in }\; L^{r'}(\Omega, \nu_1). \tag{4}$$

Step 2. We define the operator $G_j : X \to L^{s'}(\Omega, \nu_2)$ $(j = 1, ..., n)$ by

$$(G_j u)(x) = \mathcal{B}_j(x, u(x), \nabla u(x)).$$

This operator is continuous and bounded. In fact,
(i) Using (H8), Theorem 3 (since $\nu_1 \in A_r$) and Remark 5(b) we obtain

$$\begin{aligned}
&\|G_j u\|^{s'}_{L^{s'}(\Omega,\nu_2)} = \int_\Omega |G_j u(x)|^{s'} \nu_2\, dx = \int_\Omega |\mathcal{B}_j(x, u, \nabla u)|^{s'} \nu_2\, dx \\
&\leq \int_\Omega \left(K_2 + g_1 |u|^{s/s'} + g_2 |\nabla u|^{s/s'} \right)^{s'} \nu_2\, dx \\
&\leq C_s \int_\Omega \left[(K_2^{s'} + g_1^{s'} |u|^s + g_2^{s'} |\nabla u|^s) \nu_2 \right] dx \\
&= C_s \left[\int_\Omega K_2^{s'} \nu_2\, dx + \int_\Omega g_1^{s'} |u|^s \nu_2\, dx + \int_\Omega g_2^{s'} |\nabla u|^s \nu_2\, dx \right] \\
&\leq C_s \left(\|K_2\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|g_1\|^{s'}_{L^\infty(\Omega)} \|u\|^s_{L^s(\Omega,\nu_2)} + \|g_2\|^{s'}_{L^\infty(\Omega)} \|\, |\nabla u|\, \|^s_{L^s(\Omega,\nu_2)} \right) \\
&\leq C_s \left(\|K_2\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|g_1\|^{s'}_{L^\infty(\Omega)} M_2^s \|u\|^s_{L^r(\Omega,\nu_1)} + M_2^s \|g_2\|^{s'}_{L^\infty(\Omega)} \|\, |\nabla u|\, \|^s_{L^r(\Omega,\nu_1)} \right) \\
&\leq C_s \Big(\|K_2\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \|g_1\|^{s'}_{L^\infty(\Omega)} C_\Omega M_2^s \|\, |\nabla u|\, \|^s_{L^r(\Omega,\nu_1)} \\
&+ M_2^s \|g_2\|^{s'}_{L^\infty(\Omega)} \|\, |\nabla u|\, \|^s_{L^r(\Omega,\nu_1)} \Big) \\
&\leq C_s \left(\|K_2\|^{s'}_{L^{s'}(\Omega,\nu_2)} + C_\Omega M_2^s \|g_1\|^{s'}_{L^\infty(\Omega)} \|u\|^s_X + M_2^s \|g_2\|^{s'}_{L^\infty(\Omega)} \|u\|^s_X \right),
\end{aligned} \tag{5}$$

where the C_s depends only on s. Therefore, in (5), we obtain

$$\begin{aligned}
&\|G_j u\|_{L^{s'}(\Omega,\nu_2)} \\
&\leq C_s^{1/s'} \left(\|K_2\|_{L^{s'}(\Omega,\nu_2)} + M_2^{s-1} (C_\Omega^{1/s'} \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)})\, \|u\|_X^{s-1} \right).
\end{aligned} \tag{6}$$

(ii) Let $u_m \rightarrow u$ in X as $m \to \infty$. We need to show that $G_j u_m \rightarrow G_j u$ in $L^{s'}(\Omega, \nu_2)$. We will apply the Lebesgue Dominated Theorem. If $u_m \rightarrow u$ in X, then $|\nabla u_m| \rightarrow |\nabla u|$ in $L^r(\Omega, \nu_1)$. Using Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_1 \in L^r(\Omega, \nu_1)$ such that

$$\begin{aligned}
&u_{m_k}(x) \rightarrow u(x) \text{ a.e. in } \Omega, \\
&D_j u_{m_k}(x) \rightarrow D_j u(x) \text{ a.e. in } \Omega, \\
&|\nabla u_{m_k}(x)| \leq \Phi_1(x), \text{ a.e. in } \Omega.
\end{aligned}$$

By Theorem 3 (since $\nu_1 \in A_r$, with $k = 1$) we have

$$\|u_{m_k}\|_{L^r(\Omega,\nu_1)} \leq C_\Omega \, \| \, |\nabla u_{m_k}| \, \|_{L^r(\Omega,\nu_1)} \leq C_\Omega \, \|\Phi_1\|_{L^r(\Omega,\nu_1)}$$
$$\|u\|_{L^r(\Omega,\nu_1)} \leq C_\Omega \, \| \, |\nabla u| \, \|_{L^r(\Omega,\nu_1)} \leq C_\Omega \, \|\Phi_1\|_{L^r(\Omega,\nu_1)}.$$

Next, applying (H8), Theorem 3 and Remark 5(b) we obtain

$$\begin{aligned}
&\|G_j u_{m_k} - G_j u\|^{s'}_{L^{s'}(\Omega,\nu_2)} = \int_\Omega |G_j u_{m_k}(x) - G_j u(x)|^{s'} \nu_2 \, dx \\
&= \int_\Omega |\mathcal{B}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{B}_j(x, u, \nabla u)|^{s'} \nu_2 \, dx \\
&\leq C_s \int_\Omega \left(|\mathcal{B}_j(x, u_{m_k}, \nabla u_{m_k})|^{s'} + |\mathcal{B}_j(x, u, \nabla u)|^{s'} \right) \nu_2 \, dx \\
&\leq C_s \left[\int_\Omega \left(K_2 + g_1 |u_{m_k}|^{s/s'} + g_2 |\nabla u_{m_k}|^{s/s'} \right)^{s'} \nu_2 \, dx \right. \\
&\left. + \int_\Omega \left(K_2 + g_1 |u|^{s/s'} + g_2 |\nabla u|^{s/s'} \right)^{s'} \nu_2 \, dx \right] \\
&\leq C_s \left[\int_\Omega K_2^{s'} \nu_2 \, dx + \|g_1\|^{s'}_{L^\infty(\Omega)} \int_\Omega |u_{m_k}|^s \, \nu_2 \, dx + \|g_2\|^{s'}_{L^\infty(\Omega)} \int_\Omega |\nabla u_{m_k}|^s \, \nu_2 \, dx \right. \\
&\left. + \int_\Omega K_2^{s'} \, \nu_2 \, dx + \|g_1\|^{s'}_{L^\infty(\Omega)} \int_\Omega |u|^s \, \nu_2 \, dx + \|g_2|^{s'}_{L^\infty(\Omega)} \int_\Omega |\nabla u|^s \, \nu_2 \, dx \right] \\
&\leq 2 \, C_s \left(\|K_2\|^{s'}_{L^{s'}(\Omega,\nu_2)} + C_\Omega^s \, \|g_1\|^{s'}_{L^\infty(\Omega)} \int_\Omega |\Phi_1|^s \, \nu_2 \, dx + \|g_2\|^{s'}_{L^\infty(\Omega)} \int_\Omega |\Phi_1|^s \, \nu_2 \, dx \right) \\
&\leq 2 \, C_s \Big(\|K_2\|^{s'}_{L^{s'}(\Omega,\nu_2)} + \left(C_\Omega \, \|g_1\|^{s'}_{L^\infty(\Omega)} + \|g_2\|^{s'}_{L^\infty(\Omega)} \right) M_2^s \|\Phi_1\|^s_{L^r(\Omega,\nu_1)}
\end{aligned}$$

By condition (H5), we have

$$G_j u_{m_k}(x) = \mathcal{B}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{B}_j(x, u(x), \nabla u(x)) = G_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|G_j u_{m_k} - G_j u\|_{L^{s'}(\Omega,\nu_2)} \to 0,$$

that is,

$$G_j u_{m_k} \to G_j u \ \text{ in } \ L^{s'}(\Omega, \nu_2).$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$G_j u_m \to G_j u \ \text{ in } \ L^{s'}(\Omega, \nu_2). \tag{7}$$

Step 3. We define the operator $F : X \to L^{p'}(\Omega, \omega_1)$ by

$$(Fu)(x) = |\Delta u(x)|^{p-2} \, \Delta u(x).$$

We now show that operator F is bounded and continuous.
(i) We have

$$\begin{aligned}
\|Fu\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_1 u(x)|^{p'} \, \omega_1 \, dx \\
&= \int_\Omega \left| |\Delta u|^{p-2} \, \Delta u \right|^{p'} \omega_1 \, dx \\
&= \int_\Omega |\Delta u|^p \, \omega_1 \, dx \\
&= \|\Delta u\|^p_{L^p(\Omega,\omega_1)} \\
&\leq \|u\|^p_X.
\end{aligned} \tag{8}$$

Therefore, in (8), we obtain

$$\|F\,u\|_{L^{p\,\prime}(\Omega,\omega_1)} \le \|u\|_X^{p-1}, \tag{9}$$

and hence the boundedness.
(ii) Let $u_m \to u$ in X as $m \to 0$. We need to show that $Fu_m \to Fu$ in $L^{p\,\prime}(\Omega,\omega_1)$. If $u_m \to u$ in X then $\Delta u_m \to \Delta u$ in $L^p(\Omega,\omega_1)$. Using Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_2 \in L^p(\Omega,\omega_1)$ such that

$$\Delta u_{m_k}(x) \to \Delta u(x) \text{ a.e. in } \Omega, \tag{10}$$
$$|\Delta u_{m_k}(x)| \le \Phi_2(x) \text{ a.e. in } \Omega. \tag{11}$$

Now, since $p > 2$, using (10), (11), $a = p/p' = p-1$ and $a' = (p-1)/(p-2)$, there exists a constant $\alpha_p > 0$ (by Lemma 4(a)) such that

$$\begin{aligned}
&\|Fu_{m_k} - Fu\|^{p\,\prime}_{L^{p\,\prime}(\Omega,\omega_1)} = \int_\Omega |Fu_{m_k} - Fu|^{p\,\prime}\,\omega_1\,dx\\
&= \int_\Omega \left| |\Delta u_{m_k}|^{p-2}\Delta u_{m_k} - |\Delta u|^{p-2}\Delta u \right|^{p\,\prime}\omega_1\,dx\\
&\le \int_\Omega \left[\alpha_p\,|\Delta u_{m_k} - \Delta u|\,(\,|\Delta u_{m_k}| + |\Delta u|\,)^{p-2}\right]^{p\,\prime}\omega_1\,dx\\
&\le \alpha_p^{p\,\prime}\int_\Omega |\Delta u_{m_k} - \Delta u|^{p\,\prime}(2\Phi_2)^{(p-2)p\,\prime}\,\omega_1\,dx\\
&= 2^{(p-2)p\,\prime}\alpha_p^{p\,\prime}\int_\Omega |\Delta u_{m_k} - \Delta u|^{p\,\prime}\,\Phi_2^{(p-2)p\,\prime}\,\omega_1\,dx\\
&\le 2^{(p-2)p\,\prime}\alpha_p^{p\,\prime}\left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p\,\prime a}\omega_1\,dx\right)^{1/a}\left(\int_\Omega \Phi_2^{(p-2)p\,\prime a\,\prime}\,\omega_1\,dx\right)^{1/a\,\prime}\\
&= 2^{(p-2)p\,\prime}\alpha_p^{p\,\prime}\left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{p}\,\omega_1\,dx\right)^{p\,\prime/p}\left(\int_\Omega \Phi_2^{p}\,\omega_1\,dx\right)^{(p-2)/(p-1)}\\
&= 2^{(p-2)p\,\prime}\alpha_p^{p\,\prime}\,\|\Delta u_{m_k} - \Delta u\|^{p\,\prime}_{L^p(\Omega,\omega_1)}\|\Phi_2\|^{p\,\prime(p-2)}_{L^p(\Omega,\omega_1)}\\
&\le 2^{(p-2)p\,\prime}\alpha_p^{p\,\prime}\,\|u_{m_k} - u\|^{p\,\prime}_X\,\|\Phi_2\|^{p\,\prime(p-2)}_{L^p(\Omega,\omega_1)}.
\end{aligned}$$

Hence,

$$\|Fu_{m_k} - Fu\|_{L^{p\,\prime}(\Omega,\omega_1)} \le 2^{p-2}\,\alpha_p\,\|u_{m_k} - u\|_X\,\|\Phi_2\|^{p-2}_{L^p(\Omega,\omega_1)}.$$

Therefore (since $2 < p < \infty$), we obtain $\|Fu_{m_k} - Fu\|_{L^{p\,\prime}(\Omega,\omega_1)} \to 0$, that is,

$$Fu_{m_k} \to Fu \text{ in } L^{p\,\prime}(\Omega,\omega_1).$$

By the Convergence Principle in Banach spaces (see Proposition 10.13 in [19], we have

$$Fu_m \to Fu \text{ in } L^{p\,\prime}(\Omega,\omega_1). \tag{12}$$

Step 4. Define the operator $G : X \to L^{q\,\prime}(\Omega,\omega_2)$, $(Gu)(x) = |\Delta u(x)|^{q-2}\Delta u(x)$. We also have that the operator G is continuous and bounded. In fact:
(i) If $q > 2$, we have by Remark 5(a)

$$\begin{aligned}
\|Gu\|^{q\,\prime}_{L^{q\,\prime}(\Omega,\omega_2)} &= \int_\Omega \big|\,|\Delta u|^{q-2}\,\Delta u\big|^{q\,\prime}\,\omega_2\,dx = \int_\Omega |\Delta u|^q\,\omega_2\,dx\\
&= \|\Delta u\|^q_{L^q(\Omega,\omega_2)}\\
&\le M_1^q\|\Delta u\|^q_{L^p(\Omega,\omega_1)}\\
&\le M_1^q\,\|u\|^q_X.
\end{aligned}$$

Hence, $\|Gu\|_{L^{q\,\prime}(\Omega,\omega_2)} \le M_1^{q-1}\|u\|_X^{q-1}$.

(ii) Now using (10), (11), Remark 5(a), $b = q/q' = q-1$ and $b' = (q-1)/(q-2)$ (if $q > 2$), there exists a constant $\alpha_q > 0$ (by Lemma 4(a)) such that

$$\begin{aligned}
&\|Gu_{m_k} - Gu\|^{q'}_{L^{q'}(\Omega,\omega_2)} = \int_\Omega |\, Gu_{m_k} - Gu|^{q'}\,\omega_2\,dx \\
&= \int_\Omega \left| \, |\Delta u_{m_k}|^{q-2}\,\Delta u_{m_k} - |\,\Delta u|^{q-2}\,\Delta u \right|^{q'} \omega_2\,dx \\
&\le \int_\Omega \left[\alpha_q\,|\Delta u_{m_k} - \Delta u|\,(\,|\Delta u_{m_k}| + |\Delta u|)^{(q-2)}\right]^{q'} \omega_2\,dx \\
&\le \alpha_q^{q'} \int_\Omega |\,\Delta u_{m_k} - \Delta u|^{q'}\,(2\,\Phi_2)^{(q-2)\,q'}\,\omega_2\,dx \\
&\le 2^{(q-2)q'}\alpha_q^{q'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^{q'b}\omega_2\,dx\right)^{1/b} \left(\int_\Omega \Phi_2^{(q-2)q'b'}\,\omega_2\,dx\right)^{1/b'} \\
&= \alpha_q^{q'}\,2^{(q-2)q'} \left(\int_\Omega |\Delta u_{m_k} - \Delta u|^q\,\omega_2\,dx\right)^{q'/q} \left(\int_\Omega \Phi_2^q\,\omega_2\,dx\right)^{(q-2)/(q-1)} \\
&= \alpha_q^{q'}\,2^{(q-2)q'}\|\Delta u_{m_k} - \Delta u\|^{q'}_{L^q(\Omega,\omega_2)}\|\Phi_2\|^{q'(q-2)}_{L^q(\Omega,\omega_2)} \\
&\le \alpha_q^{q'}\,2^{(q-2)q'}M_1^{q'}\|\Delta u_{m_k} - \Delta u\|^{q'}_{L^p(\Omega,\omega_1)} M_1^{q'(q-2)}\|\Phi_2\|^{q'(q-2)}_{L^p(\Omega,\omega_1)} \\
&\le \alpha_q^{q'}\,2^{(q-2)q'}M_1^q\|u_{m_k} - u\|^{q'}_X\|\Phi_2\|^{q'(q-2)}_{L^p(\Omega,\omega_1)}.
\end{aligned}$$

Hence, $\|Gu_{m_k} - Gu\|_{L^{q'}(\Omega,\omega_2)} \le 2^{q-2}\alpha_q\,M_1^{q-1}\|\Phi_2\|^{q-2}_{L^p(\Omega,\omega_1)}\,\|u_{m_k} - u\|_X$.
In the case $q = 2$ we have $(Gu)(x) = \Delta u(x)$. Hence,

$$\begin{aligned}
&\|Gu\|_{L^2(\Omega,\omega_2)} = \|\Delta u\|_{L^2(\Omega,\omega_2)} \le M_1\|\Delta u\|_{L^p(\Omega,\omega_1)} \le M_1\|u\|_X, \\
&\|Gu_{m_k} - Gu\|_{L^2(\Omega,\omega_2)} \le M_1\|\Delta u_{m_k} - \Delta u\|_{L^p(\Omega,\omega_1)} \le M_1\|u_{m_k} - u\|_X.
\end{aligned}$$

Therefore (for $2 \le q < \infty$), we obtain $\|Gu_{m_k} - Gu\|_{L^{q'}(\Omega,\omega_2)} \to 0$, that is, $Gu_{m_k} \to Gu$ in $L^{q'}(\Omega,\omega_2)$. By the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]), we have

$$Gu_m \to Gu \text{ in } L^{q'}(\Omega,\omega_2). \tag{13}$$

Step 5. Since $\dfrac{f_j}{\nu_1} \in L^{r'}(\Omega,\nu_1)$ $(j = 0,1,...,n)$ then $\mathbf{T} \in X^*$. Moreover, by Theorem 3 (with $k = 1$, since $\nu_1 \in A_r$) we have

$$\begin{aligned}
|\mathbf{T}(\varphi)| &\le \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\nu_1}|\varphi|\nu_1\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\nu_1}|D_j\varphi|\,\nu_1\,dx \\
&\le \|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\|\varphi\|_{L^r(\Omega,\nu_1)} + \left(\sum_{j=1}^n \|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\right)\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\
&\le C_\Omega\|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\
&+ \left(\sum_{j=1}^n \|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\right)\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\
&\le \left(C_\Omega\,\|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\right)\|\varphi\|_X.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned} |\mathbf{B}(u,\varphi)| &\leq |\mathbf{B_1}(u,\varphi)|+|\mathbf{B_2}(u,\varphi)|+|\mathbf{B_3}(u,\varphi)|+|\mathbf{B_4}(u,\varphi)| \\ &\leq \int_\Omega |\mathcal{A}(x,\nabla u)||\nabla\varphi|\,\nu_1\,dx+\int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\nu_2\,dx \\ &+ \int_\Omega |\Delta u|^{p-1}|\Delta\varphi|\,\omega_1\,dx+\int_\Omega |\Delta u|^{q-1}|\Delta\varphi|\,\omega_2\,dx. \end{aligned} \tag{14}$$

In (14) we have:
(i) By (H4),

$$\begin{aligned} &\int_\Omega |\mathcal{A}(x,\nabla u)|\,|\nabla\varphi|\,\nu_1\,dx \leq \int_\Omega \left(K_1+h_1|\nabla u|^{r/r'}\right)|\nabla\varphi|\,\nu_1\,dx \\ &\leq \|K_1\|_{L^{r'}(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)}+\|h_1\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{r/r'}_{L^r(\Omega,\nu_1)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\ &\leq \left(\|K_1\|_{L^{r'}(\Omega,\nu_1)}+\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{r/r'}\right)\|\varphi\|_X; \end{aligned}$$

(ii) By (H8), Theorem 3 (with $k=1$, since $\nu_2\in A_s$) and Remark 5(b)

$$\begin{aligned} &\int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\nu_2\,dx \leq \int_\Omega \left(K_2+g_1|u|^{s/s'}+g_2|\nabla u|^{s/s'}\right)|\nabla\varphi|\,\nu_2\,dx \\ &\leq \|K_2\|_{L^{s'}(\Omega,\nu_2)}\|\,|\nabla\varphi|\,\|_{L^s(\Omega,\nu_2)}+\|g_1\|_{L^\infty(\Omega)}\|u\|^{s/s'}_{L^s(\Omega,\nu_2)}\|\,|\nabla\varphi|\,\|_{L^s(\Omega,\nu_2)} \\ &+\|g_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{s/s'}_{L^s(\Omega,\nu_2)}\|\,|\nabla\varphi|\,\|_{L^s(\Omega,\nu_2)} \\ &\leq M_2\|K_2\|_{L^{s'}(\Omega,\nu_2)}\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\ &+M_2^{s-1}\,C_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{s-1}_{L^r(\Omega,\nu_1)}\,M_2\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\ &+M_2^{s-1}\,\|g_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{s-1}_{L^r(\Omega,\nu_1)}\,M_2\|\,|\nabla\varphi|\,\|_{L^r(\Omega,\nu_1)} \\ &\leq\left[M_2\|K_2\|_{L^{s'}(\Omega,\nu_2)}+\left(M_2^s\,C_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)}+M_2^s\|g_2\|_{L^\infty(\Omega)}\right)\|u\|_X^{s-1}\right]\|\varphi\|_X; \end{aligned}$$

(iii) We have

$$\begin{aligned} \int_\Omega |\Delta u|^{p-1}|\Delta\varphi|\,\omega_1\,dx &\leq \left(\int_\Omega |\Delta u|^{(p-1)\,p'}\omega_1\,dx\right)^{1/p'}\left(\int_\Omega |\Delta\varphi|^p\,\omega_1\,dx\right)^{1/p} \\ &= \|\Delta u\|^{p-1}_{L^p(\Omega,\omega_1)}\,\|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\ &\leq \|u\|_X^{p-1}\,\|\varphi\|_X; \end{aligned}$$

(iv) By Remark 5(a),

$$\begin{aligned} \int_\Omega |\Delta u|^{q-1}|\Delta\varphi|\,\omega_2\,dx &\leq \left(\int_\Omega |\Delta u|^{(q-1)\,q'}\omega_2\,dx\right)^{1/q'}\left(\int_\Omega |\Delta\varphi|^q\,\omega_2\,dx\right)^{1/q} \\ &= \|\Delta u\|^{q-1}_{L^q(\Omega,\omega_2)}\,\|\Delta\varphi\|_{L^q(\Omega,\omega_2)} \\ &\leq M_1^{q-1}\|\Delta u\|^{q-1}_{L^p(\Omega,\omega_1)}\,M_1\,\|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\ &\leq M_1^q\,\|u\|_X^{q-1}\,\|\varphi\|_X. \end{aligned}$$

Hence, in (14) we obtain, for all $u,\varphi\in X$

$$\begin{aligned} |\mathbf{B}(u,\varphi)| &\leq \Big[\|K_1\|_{L^{r'}(\Omega,\nu_1)}+\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{r-1}+M_2\|K_2\|_{L^{s'}(\Omega,\nu_2)} \\ &+ \left(M_2^sC_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)}+M_2^s\|g_2\|_{L^\infty(\Omega)}\right)\|u\|_X^{s-1} \\ &+ \|u\|_X^{p-1}+M_1^q\|u\|_X^{q-1}\Big]\|\varphi\|_X. \end{aligned}$$

Since $\mathbf{B}(u,.)$ is linear, for each $u\in X$, there exists a linear and continuous functional on X denoted by Au such that $(Au|\varphi)=\mathbf{B}(u,\varphi)$, for all $u,\ \varphi\in X$ (here $(f|x)$ denotes the value of the linear functional f at the point x). Moreover

$$\begin{aligned}\|Au\|_* \quad \le \quad & \Big[\|K_1\|_{L^{r\,'}(\Omega,\nu_1)}+\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{r-1}+M_2\|K_2\|_{L^{s\,'}(\Omega,\nu_2)}\\ & +\left(M_2^sC_\Omega^{s-1}\|g_1\|_{L^\infty(\Omega)}+M_2^s\|g_2\|_{L^\infty(\Omega)}\right)\|u\|_X^{s-1}+\|u\|_X^{p-1}+M_1^q\|u\|_X^{q-1}\Big],\end{aligned}$$

where $\|Au\|_*=\sup\{|(Au|\varphi)|=|\mathbf{B}(u,\varphi)|:\varphi\in X,\|\varphi\|_X=1\}$ is the norm of the operator Au. Hence, we obtain the operator $A:X\to X^*$, $u\mapsto Au$. Consequently, problem (P) is equivalent to the operator equation $Au=\mathbf{T},\ u\in X$.

Step 6. Using (H2), (H6) and Lemma 4(b), we obtain (for $u_1,u_2\in X$, $u_1\neq u_2$)

$$\begin{aligned}&(Au_1-Au_2|u_1-u_2)=B(u_1,u_1-u_2)-B(u_2,u_1-u_2)\\ &=\int_\Omega\Big(|\Delta u_1|^{p-2}\Delta u_1-|\Delta u_2|^{p-2}\Delta u_2\Big)\,\Delta(u_1-u_2)\,\omega_1\,dx\\ &+\int_\Omega\Big(|\Delta u_1|^{q-2}\Delta u_1-|\Delta u_2|^{q-2}\Delta u_2\Big)\,\Delta(u_1-u_2)\,\omega_2\,dx\\ &+\int_\Omega\langle\mathcal{A}(x,\nabla u_1)-\mathcal{A}(x,\nabla u_2),\nabla(u_1-u_2)\rangle\,\nu_1\,dx\\ &+\int_\Omega\langle\mathcal{B}(x,u_1,\nabla u_1)-\mathcal{B}(x,u_2,\nabla u_2),\nabla(u_1-u_2)\rangle\,\nu_2\,dx\\ &\ge\beta_p\int_\Omega\Big(|\Delta u_1|+|\Delta u_2|\Big)^{p-2}|\Delta u_1-\Delta u_2|^2\,\omega_1\,dx\\ &+\beta_q\int_\Omega\Big(|\Delta u_1|+|\Delta u_2|\Big)^{q-2}|\Delta u_1-\Delta u_2|^2\,\omega_2\,dx\\ &+\theta_1\int_\Omega|\nabla(u_1-u_2)|^r\,\nu_1\,dx+\theta_2\int_\Omega|\nabla(u_1-u_2)|^s\,\nu_2\,dx\\ &\ge\beta_p\int_\Omega\Big(|\Delta u_1|+|\Delta u_2|\Big)^{p-2}|\Delta u_1-\Delta u_2|^2\,\omega_1\,dx\\ &+\theta_1\int_\Omega|\nabla(u_1-u_2)|^r\,\nu_1\,dx\\ &\ge\beta_p\int_\Omega|\Delta u_1-\Delta u_2|^{p-2}|\Delta u_1-\Delta u_2|^2\,\omega_1+\theta_1\int_\Omega|\nabla(u_1-u_2)|^r\,\nu_1\,dx\\ &=\int_\Omega|\Delta u_1-\Delta u_2|^p\,\omega_1\,dx+\theta_1\int_\Omega|\nabla(u_1-u_2)|^r\,\nu_1\,dx>\ 0.\end{aligned}$$

Therefore, the operator A is strictly monotone. Moreover, we have by (H3) and (H7),

$$\begin{aligned}(Au|u)\quad&=\quad\mathbf{B}(u,u)=\mathbf{B}_1(u,u)+\mathbf{B}_2(u,u)+\mathbf{B}_3(u,u)+\mathbf{B}_4(u,u)\\ &=\quad\int_\Omega|\Delta u|^p\omega_1\,dx+\int_\Omega|\Delta u|^q\omega_2\,dx\\ &+\quad\int_\Omega\langle\mathcal{A}(x,\nabla u),\nabla u\rangle\,\nu_1\,dx+\int_\Omega\langle\mathcal{B}(x,u,\nabla u),\nabla u\rangle\,\nu_2\,dx\\ &\ge\quad\int_\Omega|\Delta u|^p\omega_1\,dx+\int_\Omega|\nabla u|^q\,\omega_2\,dx\\ &+\quad\lambda_1\int_\Omega|\nabla u|^r\,\nu_1\,dx+\lambda_2\int_\Omega|\nabla u|^s\,\nu_2\,dx+\Lambda_2\int_\Omega|u|^s\,\nu_2\,dx\\ &\ge\quad\int_\Omega|\Delta u|^p\omega_1\,dx+\lambda_1\int_\Omega|\nabla u|^r\,\nu_1\,dx\\ &\ge\quad\gamma\left(\|\Delta u\|_{L^p(\Omega,\omega_1)}^p+\|\,|\nabla u|\,\|_{L^r(\Omega,\nu_1)}^r\right),\end{aligned}$$

where $\gamma = \min\{\lambda_1, 1\}$. Hence, since $2 < r, p < \infty$, we have

$$\frac{(Au|u)}{\|u\|_X} \to +\infty, \text{ as } \|u\|_X \to +\infty,$$

that is, A is coercive (using that $\lim\limits_{t+a\to\infty} \dfrac{t^p + a^r}{t+a} = \infty$, with $t > 0$ and $a > 0$).

Step 7. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}
|\mathbf{B_1}(u_m, \varphi) - \mathbf{B_1}(u, \varphi)| &\leq \sum_{j=1}^{n} \int_\Omega |\mathcal{A}_j(x, \nabla u_m) - \mathcal{A}_j(x, \nabla u)||D_j\varphi|\, \nu_1\, dx \\
&= \sum_{j=1}^{n} \int_\Omega |F_j u_m - F_j u||D_j\varphi|\, \nu_1\, dx \\
&\leq \left(\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{r'}(\Omega,\nu_1)}\right) \|\, |\nabla\varphi|\, \|_{L^r(\Omega,\nu_1)} \\
&\leq \left(\sum_{j=1}^{n} \|F_j u_m - F_j u\|_{L^{r'}(\Omega,\nu_1)}\right) \|\varphi\|_X,
\end{aligned}$$

and, by Remark 5(b),

$$\begin{aligned}
|\mathbf{B_2}(u_m, \varphi) - \mathbf{B_2}(u, \varphi)| &\leq \sum_{j=1}^{n} \int_\Omega |\mathcal{B}_j(x, u_m, \nabla u_m) - \mathcal{B}_j(x, u, \nabla u)||D_j\varphi|\, \nu_2\, dx \\
&= \sum_{j=1}^{n} \int_\Omega |G_j u_m - G_j u||D_j\varphi|\, \nu_2\, dx \\
&\leq \left(\sum_{j=1}^{n} \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)}\right) \|\, |\nabla\varphi|\, \|_{L^s(\Omega,\nu_2)} \\
&\leq M_2 \left(\sum_{j=1}^{n} \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)}\right) \|\, |\nabla\varphi|\, \|_{L^r(\Omega,\nu_1)} \\
&\leq M_2 \left(\sum_{j=1}^{n} \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)}\right) \|\varphi\|_X,
\end{aligned}$$

and we also have

$$\begin{aligned}
|\mathbf{B}_3(u_m, \varphi) - \mathbf{B}_3(u, \varphi)| &\leq \int_\Omega \left| |\Delta u_m|^{p-2}\Delta u_m - |\Delta u|^{p-2}\Delta u \right| |\Delta\varphi|\, \omega_1\, dx \\
&= \int_\Omega |F u_m - F u||\Delta\varphi|\, \omega_1\, dx \\
&\leq \|F u_m - F u\|_{L^{p'}(\Omega,\omega_1)}\, \|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq \|F u_m - F u\|_{L^{p'}(\Omega,\omega_1)}\, \|\varphi\|_X,
\end{aligned}$$

and by Remark 5(a)

$$\begin{aligned}
|\mathbf{B}_4(u_m, \varphi) - \mathbf{B}_4(u, \varphi)| &\leq \int_\Omega \left| |\Delta u_m|^{q-2}\Delta u_m - |\Delta u|^{q-2}\Delta u \right| |\Delta\varphi|\, \omega_2\, dx \\
&= \int_\Omega |G u_m - G u||\Delta\varphi|\, \omega_2\, dx \\
&\leq \|G u_m - G u\|_{L^{q'}(\Omega,\omega_2)}\, \|\Delta\varphi\|_{L^q(\Omega,\omega_2)} \\
&\leq M_1\, \|G u_m - G u\|_{L^{q'}(\Omega,\omega_1)}\, \|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq M_1\, \|G u_m - G u\|_{L^{q'}(\Omega,\omega_2)}\, \|\varphi\|_X,
\end{aligned}$$

for all $\varphi \in X$. Hence,

$$\begin{aligned}
&|\mathbf{B}(u_m,\varphi)-\mathbf{B}(u,\varphi)| \\
&\le |\mathbf{B_1}(u_m,\varphi)-\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)| \\
&+ |\mathbf{B_3}(u_m,\varphi)-\mathbf{B_3}(u,\varphi)| + |\mathbf{B_4}(u_m,\varphi)-\mathbf{B_4}(u,\varphi)| \\
&\le \Bigg[\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{r'}(\Omega,\nu_1)} + M_2 \sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)} \\
&+ \|Fu_m - Fu\|_{L^{p'}(\Omega,\omega_1)} + M_1 \|Gu_m - Gu\|_{L^{q'}(\Omega,\omega_2)}\Bigg] \|\varphi\|_X.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|Au_m - Au\|_* &\le \sum_{j=1}^n \left(\|F_j u_m - F_j u\|_{L^{r'}(\Omega,\nu_1)} + M_2 \|G_j u_m - G_j u\|_{L^{s'}(\Omega,\nu_2)}\right) \\
&+ \|Fu_m - Fu\|_{L^{p'}(\Omega,\omega_1)} + M_1 \|Gu_m - Gu\|_{L^{q'}(\Omega,\omega_2)}.
\end{aligned}$$

Therefore, using (4), (7), (12) and (13) we have $\|Au_m - Au\|_* \to 0$ as $m \to +\infty$, that is, A is continuous and this implies that A is hemicontinuous.
Therefore, by Theorem 6, the operator equation $Au = T$ has a unique solution $u \in X$ and it is the unique solution for problem (P).
Step 8. Estimates for $\|u\|_X$. In particular, by setting $\varphi = u$ in Definition 4, we have

$$\mathbf{B}(u,u) = \mathbf{B_1}(u,u) + \mathbf{B_2}(u,u) + \mathbf{B_3}(u,u) + \mathbf{B_4}(u,u) = T(u). \tag{15}$$

Hence, using (H3) and (H7) we obtain

$$\begin{aligned}
\mathbf{B}(u,u) &= \mathbf{B_1}(u,u) + \mathbf{B_2}(u,u) + \mathbf{B_3}(u,u) + \mathbf{B_4}(u,u) \\
&= \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla u\rangle\, \nu_1\, dx + \int_\Omega \langle \mathbf{B}(x,u,\nabla u), \nabla u\rangle\, \nu_2\, dx \\
&+ \int_\Omega |\Delta u|^p\, \omega_1\, dx + \int_\Omega |\Delta u|^q\, \omega_2\, dx \\
&\ge \lambda_1 \int_\Omega |\nabla u|^r\, \nu_1\, dx + \int_\Omega |\Delta u|^p\, \omega_1\, dx,
\end{aligned} \tag{16}$$

and, since $\nu_1 \in A_r$,

$$\begin{aligned}
T(u) &= \int_\Omega f_0\, u\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j u\, dx \\
&\le \|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)} \|u\|_{L^r(\Omega,\nu_1)} + \left(\sum_{j=1}^n \|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\right) \|\,|\nabla u|\,\|_{L^r(\Omega,\nu_1)} \\
&\le \left(C_\Omega\, \|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}\right) \|u\|_X \\
&= M\, \|u\|_X,
\end{aligned} \tag{17}$$

where $M = C_\Omega\, \|f_0/\nu_1\|_{L^{r'}(\Omega,\nu_1)} + \sum_{j=1}^n \|f_j/\nu_1\|_{L^{r'}(\Omega,\nu_1)}$. Hence in (15), using (16) and (17), we obtain

$$\lambda_1 \int_\Omega |\nabla u|^r\, \nu_1\, dx + \int_\Omega |\Delta u|^p\, \omega_1\, dx \le M\, \|u\|_X.$$

Therefore,

$$\|\Delta u\|^p_{L^p(\Omega,\omega_1)} \le M\, \|u\|_X \ \text{ and } \ \|\,|\nabla u|\,\|^r_{L^r(\Omega,\nu_1)} \le \frac{M}{\lambda_1}\, \|u\|_X.$$

By Young's inequality, we obtain

$$\begin{aligned}\|u\|_X &= \|\Delta u\|_{L^p(\Omega,\omega_1)} + \|\,|\nabla u|\,\|_{L^r(\Omega,\nu_1)} \\ &\leq M^{1/p}\,\|u\|_X^{1/p} + \left(\frac{M}{\lambda_1}\right)^{1/r}\|u\|_X^{1/r} \\ &\leq \frac{1}{p'}M^{p'/p} + \frac{1}{p}\,\|u\|_X + \frac{1}{r'}\left(\frac{M}{\lambda_1}\right)^{r'/r} + \frac{1}{r}\|u\|_X.\end{aligned}$$

Since $2 < r, p < \infty$, then $1/r + 1/p < 1$. Therefore, we obtain

$$\|u\|_X \leq \gamma_{p,r}\left(\frac{1}{p'}M^{p'-1} + \frac{1}{r'}\big(M/\lambda_1\big)^{r'-1}\right),$$

where $\gamma_{p,r} = p\,r/(p\,r - p - r)$.

Example 7 *Let $\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 < 1\}$, the weight functions $\omega_1(x,y) = (x^2+y^2)^{5/2}$, $\omega_2(x,y) = (x^2+y^2)^{3/2}$, $\nu_1(x,y) = (x^2+y^2)^{-1/2}$ and $\nu_2(x,y) = (x^2+y^2)^{-1/3}$ ($\nu\in A_4$, $\nu_2\in A_3$, $\omega_1\in A_5$, $\omega_2\in A_3$, $p = 5$, $q = 3$, $r = 4$ and $s = 3$), and the function*

$$\begin{aligned}&\mathcal{A} : \Omega\times\mathbb{R}^2\to\mathbb{R}^2 \\ &\mathcal{A}((x,y),\xi) = h_1(x,y)\,|\xi|^2\,\xi,\end{aligned}$$

where $h_1(x,y) = 2\,\mathrm{e}^{(x^2+y^2)}$, and

$$\begin{aligned}&\mathcal{B} : \Omega\times\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2 \\ &\mathcal{B}((x,y),\eta,\xi) = g_2(x,y)\,|\xi|\,\xi,\end{aligned}$$

where $g_2(x,y) = 2 + \cos(x^2+y^2)$. Let us consider the partial differential operator

$$Lu(x,y) = \Delta\big[|\Delta u|^3\Delta u\,\omega_1 + |\Delta u|\Delta u\,\omega_2\big] - \operatorname{div}\big(\mathcal{A}((x,y),\nabla u) + \mathcal{B}((x,y),u,\nabla u)\big).$$

Therefore, by Theorem 1, the problem

$$(P)\begin{cases} Lu(x) = \dfrac{\cos(xy)}{\sqrt{x^2+y^2}} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) \text{ in } \Omega, \\ u(x) = \Delta u(x) = 0 \text{ on } \partial\Omega \end{cases}$$

has a unique solution $u\in X = W_0^{1,4}(\Omega,\nu_1)\cap W^{2,5}(\Omega,\omega_1)$.

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