ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.30(2020) No.2/3,pp.163-167

Scaling of Average Shortest Path on Weighted Crystal Network

Xiaoyan Li[1*] , Jun Li[2], Huixian Yang[1], Xiaobei Liu[1]
[1]School of Mathematical Sciences, Jiangsu University, Zhenjiang, 212013, P. R. China
[2]Zhenjiang No.1 High School, Jiangsu Province, Zhenjiang, 212013, P. R. China
(Received December 4 2020, accepted December 11 2020)

Abstract: In this paper, the weighted Crystal network is constructed by a new iterative method. Due to the particularity of network structure, we consider to calculate the average shortest path (ASP) of the network by dividing it into seven blocks to evaluate the influence of weight factor and network size on transmission efficiency. The results of ASP show that the scaled results are reasonable and the smaller weight, the higher efficiency.

Keywords: Crystal network; Average shortest path; Weight; Transmission efficiency.

1 Introduction

As an interdisciplinary subject, complex networks are widely concerned. A large number of scholars have studied its properties, such as network diameter, fractal dimension, clustering coefficient[1], coherence problem[2] and trapping problem[3]. Meanwhile, in addition to the fact that time is often used as a measure of travel efficiency, the average shortest path which is related to a lot of other topological properties of the network is also a way of judging, for example, centrality[4], degree distribution[5], fractal[6–8]. We usually think that the shorter distance between two nodes, the more efficient information is received.

In this paper, we construct a Crystal network to calculate its average shortest path (ASP). In section 2, we introduce the construction process of the network in detail. In section 3, the calculation formula and results of average shortest path are given. In section 4, we analyze the obtained results.

2 The weighted Crystal network

Let G_t denote the weighted Crystal network of generation $t (t \geq 0)$. At the initial state $t = 0$, the odd points (1, 3, 5) of the hexagon are regarded as the primary nodes and connected in pairs to form an uncompletely connected graph with unit edge weight. When $t \geq 1$, G_t can be obtained as follows: G_{t-1} is duplicated six times and the edge weights are scaled by a factor r, $(0 < r \leq 1)$, then their primary nodes are combined with the six vertices of G_0 respectively (see Fig.1). The numbers of nodes in G_t are $N_t = 6^{t+1}$.

3 Average weighted shortest path

In this part we will calculate the shortest path of Crystal network according to its definition[9, 10]:

$$\lambda_t = \frac{2}{N_t(N_t-1)} D_{tot}(t)$$

where $D_{tot}(t) = \sum_{\substack{i\neq j\\ i,j\in G_t}} d_{ij}(t)$. $d_{ij}(t)$ is the total shortest distance between nodes i and j in G_t.

Due to the structure of the Crystal network, G_t can be divided into seven branches which are labeled as G_t^x, $x =$

*Corresponding author. *E-mail address*: 2796590288@qq.com

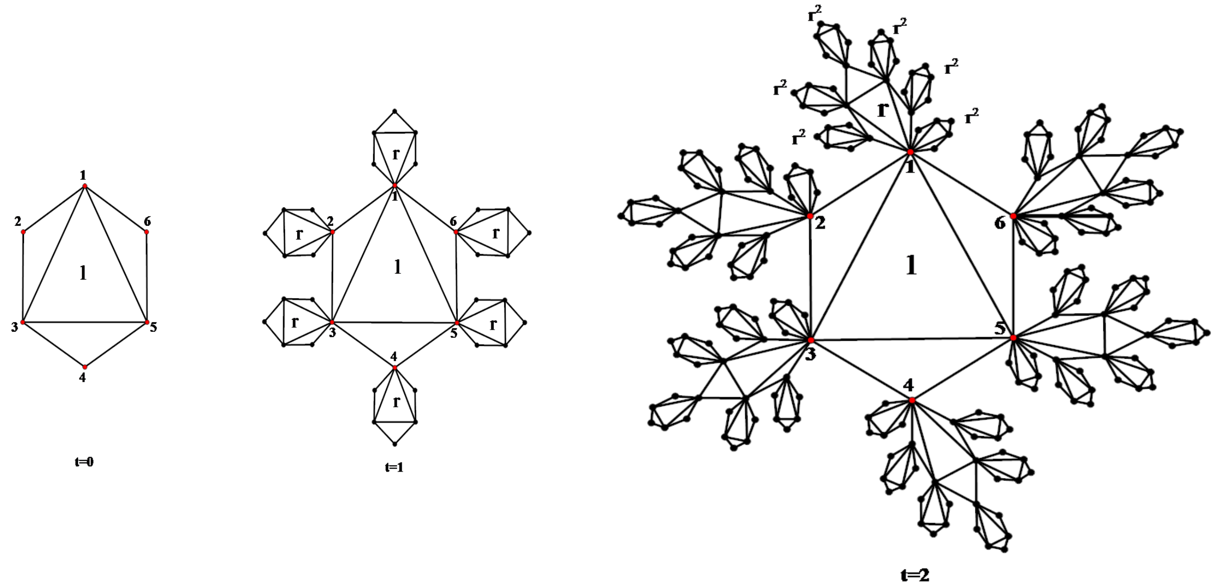

Figure 1: Iterative construction method for weighted Crystal network from $t = 0$ to $t = 2$.

A, B, C, D, E, F, H (see Fig.2). The middle G_t^H is a copy of G_0, and the rest of branches are copies of G_{t-1} which edge weights have been scaled by r. Therefore, we can write the recursive formula of $D_{tot}(t)$ in G_t:

$$D_{tot}(t) = 6rD_{tot}(t-1) + D_{tot}(0) + \Omega_t \tag{1}$$

In the above formula, $6rD_{tot}(t-1)$ is the shortest distance between the node pairs in G_t^x, $x = A, B, C, D, E, F$, $D_{tot}(0)$ is the shortest path linking nodes i and j in G_t^H and $D_{tot}(0) = 21$, the last item Ω_t is the sum of the distance between the node pairs belonging to different modules. Next, we will calculate Ω_t.
Let G_t^{xy} be the sum of all shortest paths with nodes in G_t^x and G_t^y. Because of symmetry:

$$\begin{aligned}
&\Omega_t^{AH} = \Omega_t^{CH} = \Omega_t^{EH} \\
&\Omega_t^{BH} = \Omega_t^{DH} = \Omega_t^{FH} \\
&\Omega_t^{AD} = \Omega_t^{BD} = \Omega_t^{BE} = \Omega_t^{BF} = \Omega_t^{CF} = \Omega_t^{DF} \\
&\Omega_t^{AB} = \Omega_t^{AC} = \Omega_t^{AE} = \Omega_t^{AF} = \Omega_t^{BC} = \Omega_t^{CD} = \Omega_t^{CE} = \Omega_t^{DE} = \Omega_t^{EF}
\end{aligned}$$

Thus, we can get the following equation:

$$\Omega_t = 9\Omega_t^{AB} + 6\Omega_t^{AD} + 3\Omega_t^{AH} + 3\Omega_t^{BH}. \tag{2}$$

In order to get Ω_t^{AB}, Ω_t^{AD}, Ω_t^{AH} and Ω_t^{BH}, we define

$$\triangle_t = \sum_{\substack{i \in G_t \\ i \neq 1}} d_{i1}$$

Where d_{i1} is the distance from any node in G_t to node 1 and $\triangle_0 = 6$.
According to the structure and symmetry of crystal network, We can derive the formula for $\triangle_t$

$$\begin{aligned}
\triangle_t &= r\triangle_{t-1} + 4(r\triangle_{t-1} + N_{t-1} - 1) + [r\triangle_{t-1} + 2(N_{t-1} - 1)] + \triangle_0 \\
&= (6r)^t \triangle_0 + 6^{t+1} \times [1 + r + r^2 + \cdots + r^{t-1}]
\end{aligned}$$

Substituting $\triangle_0 = 6$, we can obtain

$$\triangle_t = \begin{cases} 6^{t+1} \times \frac{1-r^{t+1}}{1-r} & (0 < r < 1) \\ (t+1) \times 6^{t+1} & (r = 1) \end{cases}$$

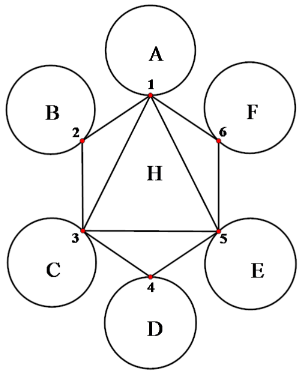

Figure 2: The division of G_t which is split into 7 branches.

After getting $\triangle_t$, we have

$$\begin{aligned}\Omega_t^{AB} &= \sum_{\substack{i\in G_t^A, j\in G_t^B\\ i\neq 1, j\neq 2}} d_{ij} \\ &= \sum_{\substack{i\in G_t^A, j\in G_t^B\\ i\neq 1, j\neq 2}} (d_{i1}+d_{12}+d_{j2}) \\ &= (N_{t-1}-1)\sum_{\substack{i\in G_t^A\\ i\neq 1}} d_{i1} + (N_{t-1}-1)^2 + (N_{t-1}-1)\sum_{\substack{j\in G_t^B\\ j\neq 2}} d_{j2} \\ &= 2r(N_{t-1}-1)\triangle_{t-1} + (N_{t-1}-1)^2\end{aligned} \tag{3}$$

Where $d_{12}=1$ has been used. Similarly, we could write the rest of Ω_t^{xy}.

$$\begin{aligned}\Omega_t^{AD} &= \sum_{\substack{i\in G_t^A, j\in G_t^D\\ i\neq 1, j\neq 4}} (d_{i1}+d_{13}+d_{34}+d_{j4}) \\ &= 2r(N_{t-1}-1)\triangle_{t-1} + 2(N_{t-1}-1)^2 \\ \Omega_t^{AH} &= \sum_{\substack{i\in G_t^A, j\in G_t^H\\ i,j\neq 1}} (d_{i1}+d_{j1}) \\ &= 5r\triangle_{t-1} + (N_{t-1}-1)\triangle_0 \\ \Omega_t^{BH} &= \sum_{\substack{i\in G_t^B, j\in G_t^H\\ i,j\neq 2}} (d_{i2}+d_{j2}) \\ &= 5r\triangle_{t-1} + (N_{t-1}-1)\xi_0\end{aligned} \tag{4}$$

Where $\xi_0 = \sum_{\substack{i\in G_0\\ i\neq 2}} d_{i2} = 8$.
Plugging Eqs.(3) and (4) into Eq.(2) gives

$$\Omega_t = \begin{cases} (30r \times \frac{1-r^t}{1-r} + 21) \times 6^{2t} - 21 & (0<r<1) \\ (30t+21)\times 6^{2t} - 21 & (r=1)\end{cases}$$

The equation Eq.(1) can be deduced as

$$D_{tot}(t) = [1+6r+(6r)^2+\cdots+(6r)^t]D_{tot}(0) + (6r)^{t-1}\Omega_1 + (6r)^{t-2}\Omega_2 + \cdots + +6r\Omega_{t-1} + \Omega_t$$

Using $D_{tot}(0) = 21$ and Ω_t, Eq.(1) is solved to yield

$$D_{tot}(t) = \begin{cases} \frac{-15}{r-6} \times 6^t r^{t+1} - [\frac{21+9r}{(1-r)(r-6)} + \frac{6}{1-r} \times r^{t+1}] \times 6^{2t+1} & (0 < r < 1) \\ (6t+3) \times 6^{2t+1} + \frac{1}{2} \times 6^{t+1} & (r = 1) \end{cases}$$

Subsequently, average weighted shortest path λ_t can be calculated as

$$\lambda_t = \begin{cases} \frac{-5}{r-6} \times \frac{r^{t+1}}{6^{t+1}-1} - \frac{2\times 6^t}{6^{t+1}-1}[\frac{21+9r}{(1-r)(r-6)} + \frac{6}{1-r} \times r^{t+1}] & (0 < r < 1) \\ \frac{(2t+1)\times 6^{t+1}+1}{6^{t+1}-1} & (r = 1) \end{cases} \quad (5)$$

When $t \to \infty$, the results in Eq.(5) can be evaluated as

$$\lambda_t \to \frac{7+3r}{(r-1)(r-6)}, \qquad if\ 0 < r < 1\ .$$

From $N_t = 6^{t+1}$, we note that $t = \log_6 N_t - 1$. Hence we have

$$\lambda_t \to 2t+1 = 2\log_6 N_t - 1, \qquad if\ r = 1\ .$$

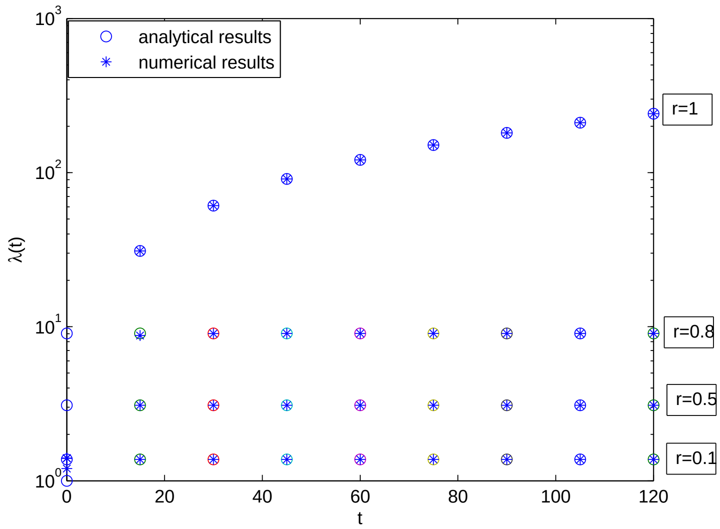

Figure 3: The asymptotic behavior of λ_t with t on a semilogarithmic scale for $r = 0.1, 0.5, 0.8$ and 1.

4 Conclusions

We calculate the average shortest path (ASP) of the constructed Crystal network and scale the results appropriately. As you can see (see Fig.3), the numerical results and the line represented by the analytical results are almost identical, which indicates that our scaling is reasonable. Furthermore, the ASP of Crystal network grows with a slope of 2 on t for $r = 1$, while when $0 < r < 1$, the size of ASP is independent of t. The smaller r is, the smaller result of ASP is, and the higher transmission efficiency.

5 Acknowledgements

Research is supported by Foundation for advanced talents of Jiangsu University (5503000039).

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