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Critical Traveling Waves for a Four-component Diffusive Influenza Epidemic Model with Vaccination

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Abstract: In this paper, with the aid of a perturbed system and Schauder's fixed point theorem, we derived the existence of a critical traveling wave solution for a four-component diffusive influenza epidemic model with vaccination. This partially solves an open problem in [Z. Xu, C. Ai. Appl. Math. Model., 40, 7265-7280(2016)]. Moreover, our work improves some previous results.

Keywords: Influenza epidemic model; Critical traveling wave; Reaction-diffusion; Perturbed system

1 Introduction

The reaction-diffusion system

$$\begin{cases} \partial_t S = d_1\partial_{xx}S - \beta(\delta_A A + I)S, \\ \partial_t A = d_2\partial_{xx}A + \beta[(1-p)S + (1-p_V)\delta_V V](\delta_A A + I) - \mu_A A, \\ \partial_t I = d_3\partial_{xx}I + \beta(pS + p_V\delta_V V)(\delta_A A + I) - \mu I, \\ \partial_t V = d_4\partial_{xx}V - \delta_V\beta(\delta_A A + I)V, \end{cases} \tag{1}$$

was proposed by Xu and Ai [13] as a model to describe the spatial propagation of influenza disease with vaccination. In Eq. (1), $S(t,x)$, $A(t,x)$, $I(t,x)$ and $V(t,x)$ denote the densities of susceptible, asymptomatically infected, symptomatically infected and vaccinated individuals at time t and position x, respectively. For the biological interpretation of the parameters in Eq. (1), we refer to [13]. Xu and Ai [13] investigated the traveling wave solutions in the form of

$$(S(t,x), A(t,x), I(t,x), V(t,x)) = (S(\xi), A(\xi), I(\xi), V(\xi)), \quad \xi = x + ct,$$

which satisfies the wave system

$$\begin{cases} cS'(\xi) = d_1S''(\xi) - \beta[\delta_A A(\xi) + I(\xi)]S(\xi), \\ cA'(\xi) = d_2A''(\xi) + \beta[(1-p)S(\xi) + (1-p_V)\delta_V V(\xi)][\delta_A A(\xi) + I(\xi)] - \mu_A A(\xi), \\ cI'(\xi) = d_3I''(\xi) + \beta[pS(\xi) + p_V\delta_V V(\xi)][\delta_A A(\xi) + I(\xi)] - \mu I(\xi), \\ cV'(\xi) = d_4V''(\xi) - \delta_V\beta[\delta_A A(\xi) + I(\xi)]V(\xi) \end{cases} \tag{2}$$

and the asymptotic boundary conditions

$$\begin{aligned} (S(-\infty), A(-\infty), I(-\infty), V(-\infty)) &= (S^0, 0, 0, V^0), \\ (S(+\infty), A(+\infty), I(+\infty), V(+\infty)) &= (S^1, 0, 0, V^1), \end{aligned} \tag{3}$$

where c is the wave speed, S^0, V^0, S^1 and V^1 are constants. Their results showed that if the basic reproduction number

$$R_0 = \beta S^0\left[\frac{(1-p)\delta_A}{\mu_A} + \frac{p}{\mu}\right] + \beta\delta_V V^0\left[\frac{(1-p_V)\delta_A}{\mu_A} + \frac{p_V}{\mu}\right] > 1,$$

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then there exists a critical wave speed $c^* > 0$ such that for each $c > c^*$ system Eq.1 admits a non-trivial and non-negative traveling wave solution subjected to Eq. (3). When $R_0 < 1$ or $0 < c < c^*$, system Eq. (1) has no non-trivial and non-negative traveling wave solutions. When $R_0 > 1$ and $c = c^*$, the existence of traveling wave solutions for Eq. (1) is still an open problem.

It is known that traveling wave solutions, especially the ones with the critical speed, play an important role in the investigation of epidemic propagation. Until recently, there has been a lot of work on the existence of super-critical traveling wave solutions for diffusive epidemic models [1, 2, 4-7, 9, 10, 12-19]. There also has been some research on the existence of critical traveling wave solutions for three-component diffusive epidemic models [1, 7, 11, 20]. However, the study of the existence of critical traveling wave solutions for four-component diffusive epidemic models is quite little [3]. Ai and Albashaireh [1] applied dynamical system theory to derive the existence of the critical waves for a diffusive SIRS epidemic model. Wu [11] used a limiting argument to obtain the existence of critical waves for a discrete diffusive epidemic model. Later, Deng and Zhang [3] also applied a limiting argument to obtain the existence of critical travelling waves for a four-component influenza model with treatment. Fu [7] utilized Schauder's fixed point theorem to establish the existence of critical waves for a diffusive SIR epidemic model. Note that the method in [1] is applicable to the epidemic models which have two equilibria, while model Eq. (1) has infinitely many equilibria. The limiting argument in [3, 11] seems to be abstract and obscure.

Inspired by [7], in the present paper, we will deal with the existence of critical traveling wave solutions for Eq. (1) with $d_2 = d_3$. We will introduce a perturbed system of Eq. (2) and use Schauder's fixed point theorem together with a limiting argument to obtain the existence result. Moreover, we will derive some properties of the critical traveling wave solution. Here we should point out that the method adopted in this paper is different from that in [3]. Compared with the results in [13], we not only obtain the existence of a critical traveling wave solution, but also derive the positiveness of the solution and the strict monotonicity of S-component and V-component in this solution. Our arguments can be applied to improve the properties of super-critical traveling wave solution for Eq. (1). Additionally, we should mention that when $R_0 > 1$ and $d_2 \neq d_3$, the existence of critical waves is still open due to the technical difficulty.

The rest of this paper is organized as follows. In Section 2, we suggest a perturbed system of Eq. (2) and construct a pair of upper and lower solutions for this system. Then, we apply Schauder's fixed point theorem to derive the existence of a critical traveling wave solution for this system. In Section 3, we use a limiting argument to obtain the existence of a critical traveling wave solution for the original system. Moreover, we obtain some properties of the critical traveling wave solution.

2 Existence of a critical traveling wave solution for a perturbed system

For our purpose, we let $d_2 = d_3 := d$ and introduce a perturbed system of Eq. (2)

$$\begin{cases} cS'(\xi) = d_1 S''(\xi) - \beta[\delta_A A(\xi) + I(\xi)]S(\xi), \\ cA'(\xi) = dA''(\xi) + \beta[(1-p)S(\xi) + (1-p_V)\delta_V V(\xi)][\delta_A A(\xi) + I(\xi)] - \mu_A A(\xi) - \varepsilon A^2(\xi), \\ cI'(\xi) = dI''(\xi) + \beta[pS(\xi) + p_V \delta_V V(\xi)][\delta_A A(\xi) + I(\xi)] - \mu I(\xi) - \varepsilon I^2(\xi), \\ cV'(\xi) = d_4 V''(\xi) - \delta_V \beta[\delta_A A(\xi) + I(\xi)]V(\xi), \end{cases} \tag{4}$$

where $\varepsilon > 0$ is a small parameter. The aim of constructing this perturbed system is to get a pair of bounded upper and lower solutions and, therefore, to establish a bounded cone for utilizing Schauder's fixed point theorem.

2.1 Eigenvalue problem

Linearizing the second and third equations in Eq. (4) at $(S^0, 0, 0, V^0)$ yields

$$\begin{cases} dA''(\xi) - cA'(\xi) + (\nu_1\delta_A - \mu_A)A(\xi) + \nu_1 I(\xi) = 0, \\ dI''(\xi) - cI'(\xi) + (\nu_2 - \mu)I(\xi) + \nu_2\delta_A A(\xi) = 0, \end{cases} \tag{5}$$

where

$$\nu_1 = \beta[(1-p)S^0 + (1-p_V)\delta_V V^0] \quad \text{and} \quad \nu_2 = \beta(pS^0 + p_V\delta_V V^0).$$

Plugging $A(\xi) = L_1 e^{\rho\xi}$ and $I(\xi) = L_2 e^{\rho\xi}$ into Eq. (5) gives

$$B(\rho)\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$B(\rho) = \begin{pmatrix} h_1(\rho) & \nu_1 \\ \nu_2\delta_A & h_2(\rho) \end{pmatrix}$$

with

$$h_1(\rho) = d\rho^2 - c\rho + \nu_1\delta_A - \mu_A \quad \text{and} \quad h_2(\rho) = d\rho^2 - c\rho + \nu_2 - \mu.$$

Lemma 1 *Assume that $R_0 > 1$, then there exists a positive real number c^* such that for $c = c^*$, the equation $\det B(\rho) = 0$ admits a positive double root ρ^*, a positive simple root ρ_1 and a negative simple root ρ_2.*

Proof. The proof can be carried out as that in Lemma 2.1 of [8]. So we omit it here for simplicity. ■

In the rest of the paper, we always suppose that $R_0 > 1$ and $c = c^*$. Let ρ^* be the smallest positive eigenvalue defined as in Lemma 1 and $(L_1, L_2) > 0$ is the corresponding eigenvector with

$$h_1(\rho^*)L_1 + \nu_1 L_2 = 0 \quad \text{and} \quad \nu_2\delta_A L_1 + h_2(\rho^*)L_2 = 0. \tag{6}$$

From Lemma 1, we have that

$$h_1(\rho^*)h_2(\rho^*) - \nu_1\nu_2\delta_A = 0 \quad \text{and} \quad h_1'(\rho^*)h_2(\rho^*) + h_1(\rho^*)h_2'(\rho^*) = 0. \tag{7}$$

By Eq. (6) one can get that

$$h_1(\rho^*) < 0 \quad \text{and} \quad h_2(\rho^*) < 0. \tag{8}$$

Note that $h_1'(\rho^*) = h_2'(\rho^*) = 2d\rho^* - c^*$, then it follows from Eq. (8) and the second equation in Eq. (7) that

$$2d\rho^* - c^* = 0. \tag{9}$$

2.2 Upper and lower solutions

In this subsection, we restrict our attention to the upper and lower solutions of Eq. (4). For $\xi \in \mathbb{R}$, we define the following non-negative continuous functions

$$\begin{aligned}
&S_+(\xi) := S^0,\\
&A_+(\xi) := \begin{cases} -L_1\xi e^{\rho^*\xi}, & \xi \le \xi_1,\\ L_1 M, & \xi > \xi_1,\end{cases}\\
&I_+(\xi) := \begin{cases} -L_2\xi e^{\rho^*\xi}, & \xi \le \xi_1,\\ L_2 M, & \xi > \xi_1,\end{cases}\\
&V_+(\xi) := V^0,\\
&S_-(\xi) := \begin{cases} S^0 - \sigma^{-1}e^{\sigma\xi}, & \xi \le \xi_2,\\ 0, & \xi > \xi_2,\end{cases}\\
&A_-(\xi) := \begin{cases} -L_1\xi e^{\rho^*\xi} - L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}, & \xi \le \xi_3,\\ 0, & \xi > \xi_3,\end{cases}\\
&I_-(\xi) := \begin{cases} -L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}, & \xi \le \xi_4,\\ 0, & \xi > \xi_4,\end{cases}\\
&V_-(\xi) := \begin{cases} V^0 - \sigma^{-1}e^{\sigma\xi}, & \xi \le \xi_5,\\ 0, & \xi > \xi_5,\end{cases}
\end{aligned}$$

where L_1, L_2, ρ^* satisfy Eq. (6),

$$\xi_1 = -\frac{1}{\rho^*},\ \ \xi_2 = \frac{\ln(\sigma S^0)}{\sigma},\ \ \xi_3 = -\frac{L_3^2}{L_1^2},\ \ \xi_4 = -\frac{L_4^2}{L_2^2},\ \ \xi_5 = \frac{\ln(\sigma V^0)}{\sigma},$$

L_3, L_4, M and σ are positive constants to be determined in the following lemma.

Lemma 2 *For given sufficiently large constants* $L_3 > 0$ *and* $L_4 > 0$*, the functions* $S_\pm(\xi)$*,* $A_\pm(\xi)$*,* $I_\pm(\xi)$ *and* $V_\pm(\xi)$ *satisfy*

$$d_1 S_+''(\xi) - c^* S_+'(\xi) - \beta[\delta_A A_-(\xi) + I_-(\xi)]S_+(\xi) \le 0, \ \ \xi \in \mathbb{R}, \tag{10}$$

$$dA_+''(\xi) - c^* A_+'(\xi) + \beta[(1-p)S_+(\xi) + (1-p_V)\delta_V V_+(\xi)][\delta_A A_+(\xi) + I_+(\xi)] - \mu_A A_+(\xi) - \varepsilon A_+^2(\xi) \le 0, \ \ \xi \ne \xi_1, \tag{11}$$

$$dI_+''(\xi) - c^* I_+'(\xi) + \beta[pS_+(\xi) + p_V\delta_V V_+(\xi)][\delta_A A_+(\xi) + I_+(\xi)] - \mu I_+(\xi) - \varepsilon I_+^2(\xi) \le 0, \ \ \xi \ne \xi_1, \tag{12}$$

$$d_4 V_+''(\xi) - c^* V_+'(\xi) - \delta_V\beta[\delta_A A_-(\xi) + I_-(\xi)]V_+(\xi) \le 0, \ \ \xi \in \mathbb{R}, \tag{13}$$

$$d_1 S_-''(\xi) - c^* S_-'(\xi) - \beta[\delta_A A_+(\xi) + I_+(\xi)]S_-(\xi) \ge 0, \ \ \xi \ne \xi_2, \tag{14}$$

$$dA_-''(\xi) - c^* A_-'(\xi) + \beta[(1-p)S_-(\xi) + (1-p_V)\delta_V V_-(\xi)][\delta_A A_-(\xi) + I_-(\xi)] - \mu_A A_-(\xi) - \varepsilon A_-^2(\xi) \ge 0, \ \ \xi \ne \xi_3, \tag{15}$$

$$dI_-''(\xi) - c^* I_-'(\xi) + \beta[pS_-(\xi) + p_V\delta_V V_-(\xi)][\delta_A A_-(\xi) + I_-(\xi)] - \mu I_-(\xi) - \varepsilon I_-^2(\xi) \ge 0, \ \ \xi \ne \xi_4, \tag{16}$$

$$d_4 V_-''(\xi) - c^* V_-'(\xi) - \delta_V\beta[\delta_A A_+(\xi) + I_+(\xi)]V_-(\xi) \ge 0, \ \ \xi \ne \xi_5 \tag{17}$$

for

$$M > \max\left\{0, \frac{\nu_1\delta_A L_1 + \nu_1 L_2 - \mu_A L_1}{\varepsilon L_1^2}, \frac{\nu_2\delta_A L_1 + \nu_2 L_2 - \mu L_2}{\varepsilon L_2^2}\right\} \quad \textit{and} \quad 0 < \sigma \le \min\left\{\rho^*, \frac{c^*}{d_1}, \frac{c^*}{d_4}\right\}.$$

Proof. Proof of Eq. (10) and Eq. (13). Since $S_+(\xi) = S^0$, $V_+(\xi) = V^0$, $A_-(\xi) \ge 0$ and $I_-(\xi) \ge 0$ for $\xi \in \mathbb{R}$, inequalities Eq. (10) and Eq. (13) hold obviously.

Proof of Eq. (11). When $\xi < \xi_1$, $A_+(\xi) = -L_1\xi e^{\rho^*\xi}$, $I_+(\xi) = -L_2\xi e^{\rho^*\xi}$, $S_+(\xi) = S^0$ and $V_+(\xi) = V^0$. We then infer from Eq. (6) and Eq. (9) that

$$\begin{aligned}
&dA_+''(\xi) - c^* A_+'(\xi) + \beta[(1-p)S_+(\xi) + (1-p_V)\delta_V V_+(\xi)][\delta_A A_+(\xi) + I_+(\xi)] - \mu_A A_+(\xi) - \varepsilon A_+^2(\xi) \\
&= -dL_1[2\rho^* e^{\rho^*\xi} + (\rho^*)^2\xi e^{\rho^*\xi}] + c^* L_1(e^{\rho^*\xi} + \rho^*\xi e^{\rho^*\xi}) + \beta[(1-p)S^0 + (1-p_V)\delta_V V^0](-\delta_A L_1\xi e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi}) \\
&\quad + u_A L_1\xi e^{\rho^*\xi} - \varepsilon A_+^2(\xi) \\
&= -L_1\xi e^{\rho^*\xi}[d(\rho^*)^2 - c^*\rho^* + \nu_1\delta_A - \mu_A] - 2dL_1\rho^* e^{\rho^*\xi} + c^* L_1 e^{\rho^*\xi} - \nu_1 L_2\xi e^{\rho^*\xi} - \varepsilon A_+^2(\xi) \\
&= -\xi e^{\rho^*\xi}[h_1(\rho^*)L_1 + \nu_1 L_2] - L_1 e^{\rho^*\xi}(2d\rho^* - c^*) - \varepsilon A_+^2(\xi) \\
&= -\varepsilon A_+^2(\xi) \\
&\le 0.
\end{aligned}$$

When $\xi > \xi_1$, $A_+(\xi) = L_1 M$, $I_+(\xi) = L_2 M$, $S_+(\xi) = S^0$ and $V_+(\xi) = V^0$. Noting that $M > \frac{\nu_1\delta_A L_1 + \nu_1 L_2 - \mu_A L_1}{\varepsilon L_1^2}$, we have

$$\begin{aligned}
&dA_+''(\xi) - c^* A_+'(\xi) + \beta[(1-p)S_+(\xi) + (1-p_V)\delta_V V_+(\xi)][\delta_A A_+(\xi) + I_+(\xi)] - \mu_A A_+(\xi) - \varepsilon A_+^2(\xi) \\
&= \nu_1(\delta_A L_1 M + L_2 M) - \mu_A L_1 M - \varepsilon L_1^2 M^2 \\
&= M(\nu_1\delta_A L_1 + \nu_1 L_2 - \mu_A L_1 - \varepsilon L_1^2 M) \\
&\le 0.
\end{aligned}$$

Proof of Eq. (12). When $\xi < \xi_1$, $I_+(\xi) = -L_2\xi e^{\rho^*\xi}$, $A_+(\xi) = -L_1\xi e^{\rho^*\xi}$, $S_+(\xi) = S^0$ and $V_+(\xi) = V^0$. In view of Eq. (6) and Eq. (9), we obtain

$$\begin{aligned}
&dI_+''(\xi) - c^* I_+'(\xi) + \beta[pS_+(\xi) + p_V\delta_V V_+(\xi)][\delta_A A_+(\xi) + I_+(\xi)] - \mu I_+(\xi) - \varepsilon I_+^2(\xi) \\
&= -dL_2[2\rho^* e^{\rho*\xi} + \xi(\rho^*)^2 e^{\rho^*\xi}] + c^* L_2(e^{\rho^*\xi} + \xi\rho^* e^{\rho^*\xi}) + \beta(pS^0 + p_V\delta_V V^0)(-\delta_A L_1\xi e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi}) + \mu L_2\xi e^{\rho^*\xi} - \varepsilon I_+^2(\xi) \\
&= -L_2\xi e^{\rho^*\xi}[d(\rho^*)^2 - c^*\rho^* + \nu_2 - \mu] - \nu_2\delta_A L_1\xi e^{\rho^*\xi} - 2dL_2\rho^* e^{\rho^*\xi} + c^* L_2 e^{\rho^*\xi} - \varepsilon I_+^2(\xi) \\
&= -\xi e^{\rho^*\xi}[h_2(\rho^*)L_2 + \nu_2\delta_A L_1] - L_2 e^{\rho^*\xi}(2d\rho^* - c^*) - \varepsilon I_+^2(\xi) \\
&= -\varepsilon I_+^2(\xi) \\
&\le 0.
\end{aligned}$$

When $\xi > \xi_1$, $I_+(\xi) = L_2M$, $A_+(\xi) = L_1M$, $S_+(\xi) = S^0$ and $V_+(\xi) = V^0$. Using $M > \frac{\nu_2\delta_A L_1+\nu_2 L_2-\mu L_2}{\varepsilon L_2^2}$, we deduce

$$\begin{aligned}&dI''_+(\xi) - c^*I'_+(\xi) + \beta[pS_+(\xi) + p_V\delta_V V_+(\xi)][\delta_A A_+(\xi) + I_+(\xi)] - \mu I_+(\xi) - \varepsilon I_+^2(\xi)\\&= \nu_2(\delta_A L_1 M + L_2 M) - \mu L_2 M - \varepsilon L_2^2 M^2\\&= M(\nu_2\delta_A L_1 + \nu_2 L_2 - \mu L_2 - \varepsilon L_2^2 M)\\&\le 0.\end{aligned}$$

Proof of Eq. (14). Choose sufficiently small $\sigma \in (0, \min\{\rho^*, c^*/d_1\})$ such that $\xi_2 < \xi_1$ and $c^* - d_1\sigma + \beta(\delta_A L_1 + L_2)S^0\xi e^{(\rho^*-\sigma)\xi} \ge 0$ for $\xi < \xi_2$. When $\xi < \xi_2$, $S_-(\xi) = S^0 - \sigma^{-1}e^{\sigma\xi}$, $A_+(\xi) = -L_1\xi e^{\rho^*\xi}$ and $I_+(\xi) = -L_2\xi e^{\rho^*\xi}$. Then we get

$$\begin{aligned}&d_1S''_-(\xi) - c^*S'_-(\xi) - \beta[\delta_A A_+(\xi) + I_+(\xi)]S_-(\xi)\\&= -d_1\sigma e^{\sigma\xi} + c^*e^{\sigma\xi} + \beta(\delta_A L_1\xi e^{\rho^*\xi} + L_2\xi e^{\rho^*\xi})(S^0 - \sigma^{-1}e^{\sigma\xi})\\&\ge -d_1\sigma e^{\sigma\xi} + c^*e^{\sigma\xi} + \beta(\delta_A L_1 + L_2)S^0\xi e^{\rho^*\xi}\\&= e^{\sigma\xi}[c^* - d_1\sigma + \beta(\delta_A L_1 + L_2)S^0\xi e^{(\rho^*-\sigma)\xi}]\\&\ge 0.\end{aligned}$$

When $\xi > \xi_2$, $S_-(\xi) = 0$ and inequality Eq. (14) holds.

Proof of Eq. (15). Recall that $h_1(\rho^*)h_2(\rho^*) - \nu_1\nu_2\delta_A = 0$ (see Eq. (7)), we can choose sufficiently large constants $L_3 > 0$ and $L_4 > 0$ such that

$$-h_1(\rho^*)L_3 - \nu_1 L_4 = -h_2(\rho^*)L_4 - \nu_2\delta_A L_3 = 0, \tag{18}$$

$$\frac{1}{4}dL_3(-\xi)^{-2} - \varepsilon L_1^2(-\xi)^{\frac{3}{2}}e^{\rho^*\xi} + \sigma^{-1}e^{\sigma\xi}[\beta(1-p)+\beta(1-p_V)\delta_V][-\delta_A L_1(-\xi)^{\frac{1}{2}} + \delta_A L_3 - L_2(-\xi)^{\frac{1}{2}} + L_4] > 0 \quad \text{for} \quad \xi < \xi_3 \tag{19}$$

and

$$\frac{1}{4}dL_4(-\xi)^{-2} - \varepsilon L_2^2(-\xi)^{\frac{3}{2}}e^{\rho^*\xi} + \sigma^{-1}e^{\sigma\xi}[\beta p + \beta p_V\delta_V][-\delta_A L_1(-\xi)^{\frac{1}{2}} + \delta_A L_3 - L_2(-\xi)^{\frac{1}{2}} + L_4] > 0 \quad \text{for} \quad \xi < \xi_4. \tag{20}$$

When $\xi < \xi_3$, $A_-(\xi) = -L_1\xi e^{\rho^*\xi} - L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}$, $S_-(\xi) \ge S^0 - \sigma^{-1}e^{\sigma\xi}$, $V_-(\xi) \ge V^0 - \sigma^{-1}e^{\sigma\xi}$ and $I_-(\xi) \ge -L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}$. Utilizing Eq. (6), Eq. (9), Eq. (18) and Eq. (19), we derive that

$$\begin{aligned}&dA''_-(\xi) - c^*A'_-(\xi) + \beta[(1-p)S_-(\xi) + (1-p_V)\delta_V V_-(\xi)][\delta_A A_-(\xi) + I_-(\xi)] - \mu_A A_-(\xi) - \varepsilon A_-^2(\xi)\\
&\ge -2dL_1\rho^*e^{\rho^*\xi} - dL_1\xi(\rho^*)^2e^{\rho^*\xi} + \frac{1}{4}dL_3(-\xi)^{-\frac{3}{2}}e^{\rho^*\xi} + dL_3(-\xi)^{-\frac{1}{2}}\rho^*e^{\rho^*\xi} - dL_3(-\xi)^{\frac{1}{2}}(\rho^*)^2e^{\rho^*\xi}\\
&\quad + c^*L_1e^{\rho^*\xi} + c^*L_1\xi\rho^*e^{\rho^*\xi} - \frac{1}{2}c^*L_3(-\xi)^{-\frac{1}{2}}e^{\rho^*\xi} + c^*L_3(-\xi)^{\frac{1}{2}}\rho^*e^{\rho^*\xi}\\
&\quad + [\nu_1 - \beta(1-p)\sigma^{-1}e^{\sigma\xi} - \beta(1-p_V)\delta_V\sigma^{-1}e^{\sigma\xi}][-\delta_A L_1\xi e^{\rho^*\xi} - \delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}]\\
&\quad - \mu_A[-L_1\xi e^{\rho^*\xi} - L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] - \varepsilon[-L_1\xi e^{\rho^*\xi} - L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}]^2\\
&\ge -L_1\xi e^{\rho^*\xi}h_1(\rho^*) - (-\xi)^{\frac{1}{2}}e^{\rho^*\xi}L_3h_1(\rho^*) - L_1e^{\rho^*\xi}(2d\rho^* - c^*) + \frac{1}{4}dL_3(-\xi)^{-\frac{3}{2}}e^{\rho^*\xi} + (-\xi)^{-\frac{1}{2}}e^{\rho^*\xi}L_3(d\rho^* - \frac{1}{2}c^*)\\
&\quad - \nu_1L_2\xi e^{\rho^*\xi} - \nu_1L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} + [-\beta(1-p)\sigma^{-1}e^{\sigma\xi} - \beta(1-p_V)\delta_V\sigma^{-1}e^{\sigma\xi}]\\
&\quad \times [-\delta_A L_1\xi e^{\rho^*\xi} - \delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] - \varepsilon(-L_1\xi e^{\rho^*\xi})^2\\
&= -\xi e^{\rho^*\xi}[h_1(\rho^*)L_1 + \nu_1L_2] - (-\xi)^{\frac{1}{2}}e^{\rho^*\xi}[h_1(\rho^*)L_3 + \nu_1L_4] + \frac{1}{4}dL_3(-\xi)^{-\frac{3}{2}}e^{\rho^*\xi} + [-\beta(1-p)\sigma^{-1}e^{\sigma\xi} - \beta(1-p_V)\delta_V\sigma^{-1}e^{\sigma\xi}]\\
&\quad \times [-\delta_A L_1\xi e^{\rho^*\xi} - \delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] - \varepsilon(-L_1\xi e^{\rho^*\xi})^2\\
&\ge (-\xi)^{\frac{1}{2}}e^{\rho^*\xi}\Big\{ - h_1(\rho^*)L_3 - \nu_1L_4 + \frac{1}{4}dL_3(-\xi)^{-2} - \varepsilon L_1^2(-\xi)^{\frac{3}{2}}e^{\rho^*\xi} + \sigma^{-1}e^{\sigma\xi}[\beta(1-p) + \beta(1-p_V)\delta_V]\\
&\quad \times [-\delta_A L_1(-\xi)^{\frac{1}{2}} + \delta_A L_3 - L_2(-\xi)^{\frac{1}{2}} + L_4]\Big\}\\
&\ge 0.\end{aligned}$$

When $\xi > \xi_3$, $A_-(\xi) = 0$ and inequality Eq. (15) holds.

Proof of Eq. (16). When $\xi < \xi_4$, $I_-(\xi) = -L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}$, $S_-(\xi) \geq S^0 - \sigma^{-1}e^{\sigma\xi}$, $V_-(\xi) \geq V^0 - \sigma^{-1}e^{\sigma\xi}$ and $A_-(\xi) \geq -L_1\xi e^{\rho^*\xi} - L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}$. Then by Eq. (6), Eq. (9), Eq. (18) and Eq. (20), we obtain

$$\begin{aligned}
& dI_-''(\xi) - c^*I_-'(\xi) + \beta[pS_-(\xi) + p_V\delta_V V_-(\xi)][\delta_A A_-(\xi) + I_-(\xi)] - \mu I_-(\xi) - \varepsilon I_-^2(\xi) \\
& \geq -2dL_2\rho^*e^{\rho^*\xi} - dL_2\xi(\rho^*)^2e^{\rho^*\xi} + \frac{1}{4}dL_4(-\xi)^{-\frac{3}{2}}e^{\rho^*\xi} + dL_4(-\xi)^{-\frac{1}{2}}\rho^*e^{\rho^*\xi} \\
& \quad - dL_4(-\xi)^{\frac{1}{2}}(\rho^*)^2e^{\rho^*\xi} + c^*L_2e^{\rho^*\xi} + c^*L_2\xi\rho^*e^{\rho^*\xi} - \frac{1}{2}c^*L_4(-\xi)^{-\frac{1}{2}}e^{\rho^*\xi} + c^*L_4(-\xi)^{\frac{1}{2}}\rho^*e^{\rho^*\xi} \\
& \quad + [\nu_2 - \beta p\sigma^{-1}e^{\sigma\xi} - \beta p_V\delta_V\sigma^{-1}e^{\sigma\xi}][-\delta_A L_1\xi e^{\rho^*\xi} - \delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] \\
& \quad - \mu[-L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] - \varepsilon[-L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}]^2 \\
& \geq -L_2\xi e^{\rho^*\xi}h_2(\rho^*) - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}h_2(\rho^*) - L_2e^{\rho^*\xi}(2d\rho^* - c^*) + \frac{1}{4}dL_4(-\xi)^{-\frac{3}{2}}e^{\rho^*\xi} + (-\xi)^{-\frac{1}{2}}e^{\rho^*\xi}L_4(d\rho^* - \frac{1}{2}c^*) \\
& \quad - \nu_2\delta_A L_1\xi e^{\rho^*\xi} - \nu_2\delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} + [-\beta p\sigma^{-1}e^{\sigma\xi} - \beta p_V\delta_V\sigma^{-1}e^{\sigma\xi}] \\
& \quad \times [-\delta_A L_1\xi e^{\rho^*\xi} - \delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] - \varepsilon(-L_2\xi e^{\rho^*\xi})^2 \\
& = -\xi e^{\rho^*\xi}[h_2(\rho^*)L_2 + \nu_2\delta_A L_1] - (-\xi)^{\frac{1}{2}}e^{\rho^*\xi}[h_2(\rho^*)L_4 + \nu_2\delta_A L_3] + \frac{1}{4}dL_4(-\xi)^{-\frac{3}{2}}e^{\rho^*\xi} \\
& \quad + [-\beta p\sigma^{-1}e^{\sigma\xi} - \beta p_V\delta_V\sigma^{-1}e^{\sigma\xi}][-\delta_A L_1\xi e^{\rho^*\xi} - \delta_A L_3(-\xi)^{\frac{1}{2}}e^{\rho^*\xi} - L_2\xi e^{\rho^*\xi} - L_4(-\xi)^{\frac{1}{2}}e^{\rho^*\xi}] - \varepsilon(-L_2\xi e^{\rho^*\xi})^2 \\
& \geq (-\xi)^{\frac{1}{2}}e^{\rho^*\xi}\Big\{ - h_2(\rho^*)L_4 - \nu_2\delta_A L_3 + \frac{1}{4}dL_4(-\xi)^{-2} - \varepsilon L_2^2(-\xi)^{\frac{3}{2}}e^{\rho^*\xi} \\
& \quad + \sigma^{-1}e^{\sigma\xi}[\beta p + \beta p_V\delta_V][-\delta_A L_1(-\xi)^{\frac{1}{2}} + \delta_A L_3 - L_2(-\xi)^{\frac{1}{2}} + L_4]\Big\} \\
& \geq 0.
\end{aligned}$$

When $\xi > \xi_4$, $I_-(\xi) = 0$ and inequality Eq. (16) holds.

Proof of Eq. (17). Choose sufficiently small $\sigma \in (0, \min\{\rho^*, c^*/d_4\})$ such that $\xi_5 < \xi_1$ and $c^* - d_4\sigma + \delta_V\beta(\delta_A L_1 + L_2)V^0\xi e^{(\rho^*-\sigma)\xi} \geq 0$ for $\xi < \xi_5$. When $\xi < \xi_5$, $V_-(\xi) = V^0 - \sigma^{-1}e^{\sigma\xi}$, $A_+(\xi) = -L_1\xi e^{\rho^*\xi}$ and $I_+(\xi) = -L_2\xi e^{\rho^*\xi}$. Then we have

$$\begin{aligned}
& d_4V_-''(\xi) - c^*V_-'(\xi) - \delta_V\beta[\delta_A A_+(\xi) + I_+(\xi)]V_-(\xi) \\
& \geq -d_4\sigma e^{\sigma\xi} + c^*e^{\sigma\xi} + \delta_V\beta(\delta_A L_1\xi e^{\rho^*\xi} + L_2\xi e^{\rho^*\xi})(V^0 - \sigma^{-1}e^{\sigma\xi}) \\
& \geq (-d_4\sigma + c^*)e^{\sigma\xi} + \delta_V\beta(\delta_A L_1 + L_2)V^0\xi e^{\rho^*\xi} \\
& = e^{\sigma\xi}[c^* - d_4\sigma + \delta_V\beta(\delta_A L_1 + L_2)V^0\xi e^{(\rho^*-\sigma)\xi}] \\
& \geq 0.
\end{aligned}$$

When $\xi > \xi_5$, $V_-(\xi) = 0$ and inequality Eq. (17) holds. The proof of this lemma is completed. ■

2.3 Application of Schauder's fixed point theorem

Let $M^* = M\max\{L_1, L_2\}$ and choose

$$\beta_1 > \beta(\delta_A + 1)M^*, \quad \beta_2 > \mu_A + 2\varepsilon M^*, \quad \beta_3 > \mu + 2\varepsilon M^*, \quad \beta_4 > \delta_V\beta(\delta_A + 1)M^*, \tag{21}$$

such that for $(S, A, I, V) \in C(\mathbb{R}, \Omega)$ with $\Omega = [0, S^0] \times [0, M^*] \times [0, M^*] \times [0, V^0]$,

$$H_1[S, A, I, V](\xi) := \beta_1S(\xi) - \beta[\delta_A A(\xi) + I(\xi)]S(\xi)$$

is monotonically increasing in S and monotonically decreasing in both A and I,

$$H_2[S, A, I, V](\xi) := \beta_2A(\xi) + \beta[(1-p)S(\xi) + (1-p_V)\delta_V V(\xi)][\delta_A A(\xi) + I(\xi)] - \mu_A A(\xi) - \varepsilon A^2(\xi)$$

and

$$H_3[S,A,I,V](\xi) := \beta_3 I(\xi) + \beta[pS(\xi) + p_V\delta_V V(\xi)][\delta_A A(\xi) + I(\xi)] - \mu I(\xi) - \varepsilon I^2(\xi)$$

are monotonically increasing in S, A, I, V,

$$H_4[S,A,I,V](\xi) := \beta_4 V(\xi) - \delta_V \beta[\delta_A A(\xi) + I(\xi)]V(\xi)$$

is monotonically increasing in V, and monotonically decreasing in both A and I. Then Eq. (4) with $c = c^*$ is equivalent to

$$\begin{cases} d_1 S''(\xi) - c^* S'(\xi) - \beta_1 S(\xi) + H_1[S,A,I,V](\xi) = 0, \\ dA''(\xi) - c^* A'(\xi) - \beta_2 A(\xi) + H_2[S,A,I,V](\xi) = 0, \\ dI''(\xi) - c^* I'(\xi) - \beta_3 I(\xi) + H_3[S,A,I,V](\xi) = 0, \\ d_4 V''(\xi) - c^* V'(\xi) - \beta_4 V(\xi) + H_4[S,A,I,V](\xi) = 0. \end{cases} \tag{22}$$

Define a set

$$\begin{aligned} T := \big\{ &(S,A,I,V) \in C(\mathbb{R},\Omega) : S_-(\xi) \le S(\xi) \le S_+(\xi), A_-(\xi) \le A(\xi) \le A_+(\xi), \\ &I_-(\xi) \le I(\xi) \le I_+(\xi) \text{ and } V_-(\xi) \le V(\xi) \le V_+(\xi) \big\}. \end{aligned}$$

Clearly, T is bounded, nonempty, closed and convex in $C(\mathbb{R},\Omega)$. Moreover, we define an operator: $G = (G_1, G_2, G_3, G_4)$: $T \mapsto C(\mathbb{R},\Omega)$

$$G_i[S,A,I,V](\xi) = \frac{1}{\Lambda_i}\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)} H_i[S,A,I,V](s)ds + \frac{1}{\Lambda_i}\int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)} H_i[S,A,I,V](s)ds, \tag{23}$$

where

$$\rho_{i1} = \frac{c^* - \sqrt{(c^*)^2 + 4d_i\beta_i}}{2d_i}, \quad \rho_{i2} = \frac{c^* + \sqrt{(c^*)^2 + 4d_i\beta_i}}{2d_i} \quad \text{and} \quad \Lambda_i = d_i(\rho_{i2} - \rho_{i1}), \quad i = 1,2,3,4.$$

Lemma 3 $G = (G_1, G_2, G_3, G_4) : T \mapsto T$.

Proof. By the monotonicity of $H_i[S,A,I,V](\xi)$ with respect to S, A, I and V, we only need to show that

$$S_-(\xi) \le G_1[S_-, A_+, I_+, V](\xi) \le G_1[S,A,I,V](\xi) \le G_1[S_+, A_-, I_-, V](\xi) \le S_+(\xi), \tag{24}$$
$$A_-(\xi) \le G_2[S_-, A_-, I_-, V_-](\xi) \le G_2[S,A,I,V](\xi) \le G_2[S_+, A_+, I_+, V_+](\xi) \le A_+(\xi), \tag{25}$$
$$I_-(\xi) \le G_3[S_-, A_-, I_-, V_-](\xi) \le G_3[S,A,I,V](\xi) \le G_3[S_+, A_+, I_+, V_+](\xi) \le I_+(\xi), \tag{26}$$
$$V_-(\xi) \le G_4[S, A_+, I_+, V_-](\xi) \le G_4[S,A,I,V](\xi) \le G_4[S, A_-, I_-, V_+](\xi) \le V_+(\xi) \tag{27}$$

for any $(S,A,I,V) \in T$.

Proof of Eq. (24). It follows from Eq. (14) that

$$\begin{aligned} G_1[S_-, A_+, I_+, V](\xi) &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)} H_1[S_-, A_+, I_+, V](s)ds + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)} H_1[S_-, A_+, I_+, V](s)ds \\ &\ge \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}[-d_1 S_-''(s) + c^* S_-'(s) + \beta_1 S_-(s)]ds \\ &\quad + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}[-d_1 S_-''(s) + c^* S_-'(s) + \beta_1 S_-(s)]ds \\ &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}[-d_1 S_-''(s) + c^* S_-'(s) + \beta_1 S_-(s)]ds \\ &\quad + \frac{1}{\Lambda_1}\int_{\xi}^{\xi_2} e^{\rho_{12}(\xi-s)}[-d_1 S_-''(s) + c^* S_-'(s) + \beta_1 S_-(s)]ds \\ &\quad + \frac{1}{\Lambda_1}\int_{\xi_2}^{+\infty} e^{\rho_{12}(\xi-s)}[-d_1 S_-''(s) + c^* S_-'(s) + \beta_1 S_-(s)]ds \\ &= S_-(\xi) + \frac{d_1}{\Lambda_1} e^{\rho_{12}(\xi-\xi_2)}[S_-'(\xi_2 + 0) - S_-'(\xi_2 - 0)] \\ &\ge S_-(\xi) \quad \text{for } \xi < \xi_2. \end{aligned}$$

When $\xi > \xi_2$, one can get

$$G_1[S_-, A_+, I_+, V](\xi) \geq S_-(\xi) = 0.$$

Then by the continuity of $G_1[S_-, A_+, I_+, V](\xi)$ and $S_-(\xi)$ at the point ξ_2, we have

$$G_1[S_-, A_+, I_+, V](\xi) \geq S_-(\xi) \quad \text{for} \quad \xi \in \mathbb{R}.$$

On the other hand, we infer from Eq. (10) that for all $\xi \in \mathbb{R}$,

$$H_1[S_+, A_-, I_-, V](\xi) \leq -d_1 S''_+(\xi) + c^* S'_+(\xi) + \beta_1 S_+(\xi) = \beta_1 S^0,$$

which leads to

$$\begin{aligned} G_1[S_+, A_-, I_-, V](\xi) &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)} H_1[S_+, A_-, I_-, V](s)ds + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)} H_1[S_+, A_-, I_-, V](s)ds \\ &\leq \frac{\beta_1 S^0}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}ds + \frac{\beta_1 S^0}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}ds \\ &= S^0. \end{aligned}$$

The proof of Eqs. (25)-(27) are similar to that of Eq. (24), so we omit it here. This ends the proof. ■

Now we introduce a functional space

$$B_\omega(\mathbb{R}, \Omega) = \left\{ \Psi = (\Psi_1(\xi), \Psi_2(\xi), \Psi_3(\xi), \Psi_4(\xi)) \in C(\mathbb{R}, \Omega) : \|\Psi\|_\omega := \max_{1\leq i\leq 4}\{\sup_{\xi\in\mathbb{R}} |\Psi_i(\xi)| e^{-\omega|\xi|}\} < +\infty \right\},$$

where $\omega \in (0, \min_{1\leq i\leq 4}\{-\rho_{i1}, \rho_{i2}\})$.

Lemma 4 *The operator* $G = \big(G_1, G_2, G_3, G_4\big) : T \mapsto T$ *is continuous with respect to the norm* $||\cdot||_\omega$ *in* $B_\omega(\mathbb{R}, \Omega)$.

Proof. For any (S, A, I, V) and $(\tilde{S}, \tilde{A}, \tilde{I}, \tilde{V}) \in T$, it is not difficult to verify that there exists constants $K_i > 0$ $(i = 1, 2, 3, 4)$ such that

$$|H_i[S, A, I, V](\xi) - H_i[\tilde{S}, \tilde{A}, \tilde{I}, \tilde{V}](\xi)|e^{-\omega|\xi|} \leq K_i[||S - \tilde{S}||_\omega + ||A - \tilde{A}||_\omega + ||I - \tilde{I}||_\omega + ||V - \tilde{V}||_\omega]. \tag{28}$$

Then using Eq. (28), we have

$$\begin{aligned} &|G_i[S, A, I, V](\xi) - G_i[\tilde{S}, \tilde{A}, \tilde{I}, \tilde{V}](\xi)|e^{-\omega|\xi|} \\ &\leq \frac{K_i}{\Lambda_i}\left[\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)} e^{\omega|s|-\omega|\xi|}ds + \int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)} e^{\omega|s|-\omega|\xi|}ds\right][||S - \tilde{S}||_\omega + ||A - \tilde{A}||_\omega + ||I - \tilde{I}||_\omega + ||V - \tilde{V}||_\omega] \\ &\leq \frac{K_i}{\Lambda_i}\left[\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)} e^{\omega|s-\xi|}ds + \int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)} e^{\omega|s-\xi|}ds\right][||S - \tilde{S}||_\omega + ||A - \tilde{A}||_\omega + ||I - \tilde{I}||_\omega + ||V - \tilde{V}||_\omega] \\ &= \frac{K_i}{\Lambda_i}\left(\frac{1}{\rho_{i2} - \omega} - \frac{1}{\rho_{i1} + \omega}\right)[||S - \tilde{S}||_\omega + ||A - \tilde{A}||_\omega + ||I - \tilde{I}||_\omega + ||V - \tilde{V}||_\omega], \end{aligned}$$

which implies that the operator G_i $(i = 1, 2, 3, 4)$ are continuous with respect to the norm $||\cdot||_\omega$. This finishes the proof. ■

Lemma 5 *The operator* $G = \big(G_1, G_2, G_3, G_4\big) : T \mapsto T$ *is compact with respect to the norm* $||\cdot||_\omega$ *in* $B_\omega(\mathbb{R}, \Omega)$.

Proof. By the definition of $H_i[S, A, I, V](\xi)$, we have that

$$|H_i[S, A, I, V](\xi)| \leq L_0, \quad i = 1, 2, 3, 4,$$

where $L_0 := \max\{\beta_1 S^0,\ \beta_2 L_1 M,\ \beta_3 L_2 M,\ \beta_4 V^0\}$. Then it follows that

$$\begin{aligned} \left|\frac{d}{d\xi} G_i[S, A, I, V](\xi)\right| &= \frac{1}{\Lambda_i}\left|\rho_{i1}\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)} H_i[S, A, I, V](s)ds + \rho_{i2}\int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)} H_i[S, A, I, V](s)ds\right| \\ &\leq \frac{L_0}{\Lambda_i}\left(|\rho_{i1}|\int_{-\infty}^{\xi} e^{\rho_{i1}(\xi-s)}ds + |\rho_{i2}|\int_{\xi}^{+\infty} e^{\rho_{i2}(\xi-s)}ds\right) \\ &\leq \frac{2L_0}{\Lambda_i}, \quad i = 1, 2, 3, 4. \end{aligned} \tag{29}$$

Using Lemma 3, we have

$$\sum_{i=1}^{4}|G_i[S,A,I,V](\xi)| \leq (S^0 + L_1M + L_2M + V^0). \tag{30}$$

Hence, for any $\epsilon > 0$, there exists a constant $N > 0$ such that

$$\left(\sum_{i=1}^{4}|G_i[S,A,I,V](\xi)|\right)e^{-\omega|\xi|} < (S^0 + L_1M + L_2M + V^0)e^{-\omega N} < \epsilon \quad \text{for } |\xi| > N. \tag{31}$$

Utilizing Eq. (29), Eq. (30) and Arzelà-Ascoli's theorem, we can choose finite elements in $G(T)$ such that there are a finite ϵ-net of $G(T)(\xi)$ with the supremum norm if we restrict them on $\xi \in [-N, N]$, which is also a finite ϵ-net of $G(T)(\xi)$ for $\xi \in \mathbb{R}$ with the norm $||\cdot||_\omega$ (by Eq. (31)). Therefore, the operator G is compact with respect to the norm $||\cdot||_\omega$ in $B_\omega(\mathbb{R}, \Omega)$. The proof is completed. ■

In the following, we will derive the existence and some properties of critical traveling wave solutions for system Eq. (4) with $c = c^*$.

Proposition 6 *Assume that $R_0 > 1$ and $c = c^*$, then system Eq. (4) has a solution $(S, A, I, V) \in T$ satisfying*

(i) $S(\xi) > 0$, $A(\xi) > 0$, $I(\xi) > 0$ *and* $V(\xi) > 0$ *for* $\xi \in \mathbb{R}$*;*

(ii) $S(\xi) \to S^0$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V^0$, $A(\xi) = O(-\xi e^{\rho^*\xi})$ *and* $I(\xi) = O(-\xi e^{\rho^*\xi})$ *as* $\xi \to -\infty$*;*

(iii) $S(\xi)$ *and* $V(\xi)$ *are strictly decreasing for* $\xi \in \mathbb{R}$*;*

(iv) $S(\xi) \to S^1$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V^1$ *as* $\xi \to +\infty$*;*

(v) $S'(\xi)$, $A'(\xi)$, $I'(\xi)$, $V'(\xi)$, $S''(\xi)$, $A''(\xi)$, $I''(\xi)$, $V''(\xi) \to 0$ *as* $\xi \to \pm\infty$*;*

(vi)

$$\int_{-\infty}^{+\infty}[\mu_A A(\eta) + \varepsilon A^2(\eta)]d\eta = c^*(1-p)(S^0 - S^1) + c^*(1-p_V)(V^0 - V^1) \tag{32}$$

and

$$\int_{-\infty}^{+\infty}[\mu I(\eta) + \varepsilon I^2(\eta)]d\eta = c^*p(S^0 - S^1) + c^*p_V(V^0 - V^1); \tag{33}$$

(vii) $S(\xi) < S^0$, $A(\xi) < (1-p)(S^0 - S^1) + (1-p_V)(V^0 - V^1)$, $I(\xi) < p(S^0 - S^1) + p_V(V^0 - V^1)$ *and* $V(\xi) < V^0$ *for* $\xi \in \mathbb{R}$.

Proof. According to Lemma 3-Lemma 5 and Schauder's fixed point theorem, we conclude that the operator G admits a fixed point $(S(\xi), A(\xi), I(\xi), V(\xi)) \in T$, which is a solution of system Eq. (4) with $c = c^*$ satisfying

$$\begin{aligned} &S_-(\xi) \leq S(\xi) \leq S_+(\xi), \quad A_-(\xi) \leq A(\xi) \leq A_+(\xi), \\ &I_-(\xi) \leq I(\xi) \leq I_+(\xi), \quad V_-(\xi) \leq V(\xi) \leq V_+(\xi), \quad \xi \in \mathbb{R}. \end{aligned} \tag{34}$$

(i) Noting that $H_1[S, A, I, V](\xi)$ is monotonically increasing with respect to S and monotonically decreasing in both A and I, we have from Eq. (21) that

$$\begin{aligned} S(\xi) &= \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}H_1[S,A,I,V](s)ds + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}H_1[S,A,I,V](s)ds \\ &\geq \frac{1}{\Lambda_1}\int_{-\infty}^{\xi} e^{\rho_{11}(\xi-s)}[\beta_1 - \beta(\delta_A+1)M^*]S_-(s)ds + \frac{1}{\Lambda_1}\int_{\xi}^{+\infty} e^{\rho_{12}(\xi-s)}H_1[S,A,I,V](s)ds \\ &> 0. \end{aligned} \tag{35}$$

Similarly, one can obtain that the positiveness of $A(\xi)$, $I(\xi)$ and $V(\xi)$ for $\xi \in \mathbb{R}$.

(ii) Applying squeeze rule to Eq. (34), we get

$$\begin{aligned}&S(\xi)\to S^0,\ \ A(\xi)\to 0,\ \ I(\xi)\to 0,\ \ V(\xi)\to V^0,\\ &A(\xi)=O(-\xi e^{\rho^*\xi})\ \text{ and }\ I(\xi)=O(-\xi e^{\rho^*\xi})\quad\text{as}\quad \xi\to-\infty.\end{aligned} \tag{36}$$

(iii) In view of Eq. (23), we have

$$S'(\xi)=\frac{\rho_{11}}{\Lambda_1}\int_{-\infty}^{\xi}e^{\rho_{11}(\xi-s)}H_1[S,A,I,V](s)ds+\frac{\rho_{12}}{\Lambda_1}\int_{\xi}^{+\infty}e^{\rho_{12}(\xi-s)}H_1[S,A,I,V](s)ds, \tag{37}$$

which together with L'Hospital rule yields

$$\lim_{\xi\to\pm\infty}S'(\xi)=0. \tag{38}$$

Similarly, one can get

$$\lim_{\xi\to\pm\infty}A'(\xi)=\lim_{\xi\to\pm\infty}I'(\xi)=\lim_{\xi\to\pm\infty}V'(\xi)=0. \tag{39}$$

Rewrite the first equation in system Eq. (4) with $c=c^*$ as

$$-[e^{-\frac{c^*}{d_1}\xi}S'(\xi)]'=-\frac{\beta}{d_1}[\delta_A A(\xi)+I(\xi)]S(\xi)e^{-\frac{c^*}{d_1}\xi}. \tag{40}$$

Then using Eq. (38) together with the positiveness of A, I, S on $\mathbb{R}$ and integrating Eq. (40) from ξ to $+\infty$, we obtain

$$S'(\xi)=-\frac{\beta}{d_1}e^{\frac{c^*}{d_1}\xi}\int_{\xi}^{+\infty}[\delta_A A(\eta)+I(\eta)]S(\eta)e^{-\frac{c^*}{d_1}\eta}d\eta<0\quad\text{for }\ \xi\in\mathbb{R}, \tag{41}$$

which implies that $S(\xi)$ is strictly decreasing on $\mathbb{R}$. Analogously, one can deduce that $V(\xi)$ is strictly decreasing on $\mathbb{R}$.

(iv) Using Eq. (35) and Eq. (41), we get that $\lim_{\xi\to+\infty}S(\xi)$ exists, which is denoted by S^1. Then we can infer that $S^0>S^1\geq 0$. Similarly, we can obtain that $\lim_{\xi\to+\infty}V(\xi):=V^1$ exists and $V^0>V^1\geq 0$.

Next we show that $\lim_{\xi\to+\infty}A(\xi)=\lim_{\xi\to+\infty}I(\xi)=0$. From Eq. (29), we know that $S'(\xi)$, $A'(\xi)$, $I'(\xi)$ and $V'(\xi)$ are uniformly bounded on $\mathbb{R}$. Integrating the first equation in system Eq. (4) with $c=c^*$ over $(-\infty,\xi)$ gives that

$$\int_{-\infty}^{\xi}\beta[\delta_A A(\eta)+I(\eta)]S(\eta)d\eta=d_1S'(\xi)-c^*[S(\xi)-S^0]<+\infty \tag{42}$$

for any $\xi\in\mathbb{R}$, which together with the positiveness of the integrand implies that

$$\int_{-\infty}^{+\infty}[\delta_A A(\eta)+I(\eta)]S(\eta)d\eta<+\infty. \tag{43}$$

By the similar argument as above, one can obtain that

$$\int_{-\infty}^{+\infty}[\delta_A A(\eta)+I(\eta)]V(\eta)d\eta<+\infty. \tag{44}$$

Integrating the second equation in system Eq. (4) with $c=c^*$ from $-\infty$ to ξ, we have

$$\begin{aligned}\mu_A\int_{-\infty}^{\xi}A(\eta)d\eta+\varepsilon\int_{-\infty}^{\xi}A^2(\eta)d\eta=&dA'(\xi)-c^*A(\xi)+\beta(1-p)\int_{-\infty}^{\xi}[\delta_A A(\eta)+I(\eta)]S(\eta)d\eta\\ &+\beta(1-p_V)\delta_V\int_{-\infty}^{\xi}[\delta_A A(\eta)+I(\eta)]V(\eta)d\eta\\ &<+\infty\quad\text{for }\ \xi\in\mathbb{R},\end{aligned} \tag{45}$$

where we have used Eq. (43), Eq. (44) and the uniform boundedness of $A(\xi)$ and $A'(\xi)$ on $\mathbb{R}$. Then we derive that $\int_{\mathbb{R}}A(\eta)d\eta<+\infty$. Consequently, we have $\lim_{\xi\to+\infty}A(\xi)=0$. Similarly, one can prove that $\lim_{\xi\to+\infty}I(\xi)=0$.

(v) In the proof of (iii), we have derived that $S'(\xi),\ A'(\xi),\ I'(\xi),\ V'(\xi) \to 0$ as $\xi \to \pm\infty$. Then we obtain from system Eq. (4) with $c = c^*$ and the asymptotic boundary of S, A, I, V at infinity that

$$S''(\xi),\ A''(\xi),\ I''(\xi),\ V''(\xi) \to 0 \ \text{ as } \xi \to \pm\infty. \tag{46}$$

(vi) Using Eq.42 and the asymptotic boundary of S, A, I, V at plus infinity, we have

$$\beta \int_{-\infty}^{+\infty} [\delta_A A(\eta) + I(\eta)] S(\eta) d\eta = c^*(S^0 - S^1). \tag{47}$$

Similarly, we get

$$\delta_V \beta \int_{-\infty}^{+\infty} [\delta_A A(\eta) + I(\eta)] V(\eta) d\eta = c^*(V^0 - V^1). \tag{48}$$

Using $A(+\infty) = A'(+\infty) = 0$, Eq. (45), Eq. (47) and Eq. (48), we obtain

$$\int_{-\infty}^{+\infty} [\mu_A A(\eta) + \varepsilon A^2(\eta)] d\eta = c^*(1-p)(S^0 - S^1) + c^*(1-p_V)(V^0 - V^1). \tag{49}$$

Analogously, one can prove

$$\int_{-\infty}^{+\infty} [\mu I(\eta) + \varepsilon I^2(\eta)] d\eta = c^* p(S^0 - S^1) + c^* p_V (V^0 - V^1). \tag{50}$$

(vii) By Lemma 3, we have that $S(\xi) \le S^0$ for $\xi \in \mathbb{R}$. Suppose that there exists some $\acute{\xi} \in \mathbb{R}$ such that $S(\acute{\xi}) = S^0$, then $S'(\acute{\xi}) = 0$ and $S''(\acute{\xi}) \le 0$. Using the positiveness of $S(\xi)$, $A(\xi)$ and $I(\xi)$ on $\mathbb{R}$, we deduce from system Eq. (4) with $c = c^*$ that

$$\begin{aligned} 0 &= d_1 S''(\acute{\xi}) - c^* S'(\acute{\xi}) - \beta[\delta_A A(\acute{\xi}) + I(\acute{\xi})] S(\acute{\xi}) \\ &\le -\beta[\delta_A A(\acute{\xi}) + I(\acute{\xi})] S(\acute{\xi}) \\ &< 0, \end{aligned}$$

which leads to a contradiction. Hence $S(\xi) < S^0$ for $\xi \in \mathbb{R}$. In a similar way, one can get that $V(\xi) < V^0$ for $\xi \in \mathbb{R}$.

Construct a function

$$F(\xi) := \frac{1}{c^*} \int_{-\infty}^{\xi} [\mu I(\eta) + \varepsilon I^2(\eta)] d\eta + \frac{1}{c^*} \int_{\xi}^{+\infty} e^{\frac{c^*}{d}(\xi - \eta)} [\mu I(\eta) + \varepsilon I^2(\eta)] d\eta. \tag{51}$$

It is easy to check that $F(-\infty) = 0$, $F(+\infty) = p(S^0 - S^1) + p_V(V^0 - V^1)$ and $F(\xi)$ satisfies the following equation

$$c^* F'(\xi) - dF''(\xi) = \mu I(\xi) + \varepsilon I^2(\xi). \tag{52}$$

Define $\tau(\xi) := F(\xi) + I(\xi)$. By Eq. (52) and the third equation in system Eq. (4) with $c = c^*$, we have

$$c^* \tau'(\xi) - d\tau''(\xi) = \beta[pS(\xi) + p_V \delta_V V(\xi)][\delta_A A(\xi) + I(\xi)], \tag{53}$$

i. e.,

$$[-de^{-\frac{c^*}{d}\xi} \tau'(\xi)]' = e^{-\frac{c^*}{d}\xi} \beta[pS(\xi) + p_V \delta_V V(\xi)][\delta_A A(\xi) + I(\xi)]. \tag{54}$$

An elementary computation gives that $\lim_{\xi \to +\infty} F'(\xi) = 0$ and $\lim_{\xi \to +\infty} \tau'(\xi) = 0$. Integrating Eq. (54) from ξ to $+\infty$ yields

$$\tau'(\xi) = \frac{1}{d} \int_{\xi}^{+\infty} e^{\frac{c^*}{d}(\xi - \eta)} \beta[pS(\eta) + p_V \delta_V V(\eta)][\delta_A A(\eta) + I(\eta)] d\eta > 0 \quad \text{for} \quad \xi \in \mathbb{R}. \tag{55}$$

Hence, $\tau(\xi)$ is strictly increasing on $\mathbb{R}$. Noting that

$$\tau(+\infty) = F(+\infty) = p(S^0 - S^1) + p_V(V^0 - V^1),$$

we have

$$I(\xi) < p(S^0 - S^1) + p_V(V^0 - V^1) \quad \text{for} \quad \xi \in \mathbb{R}. \tag{56}$$

Similarly, we have that $A(\xi) < (1-p)(S^0 - S^1) + (1-p_V)(V^0 - V^1)$ for $\xi \in \mathbb{R}$. The proof is finished. ■

3 Existence of critical waves for original system

Theorem 7 *Suppose that $\mathcal{R}_0 > 1$ and $c = c^*$, then system Eq. (1) admits a traveling wave solution satisfying*

(i) $S(\xi) > 0$, $A(\xi) > 0$, $I(\xi) > 0$ *and* $V(\xi) > 0$ *for* $\xi \in \mathbb{R}$*;*

(ii) $S(\xi) \to S^0$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V^0$, $A(\xi) = O(-\xi e^{\rho^* \xi})$ *and* $I(\xi) = O(-\xi e^{\rho^* \xi})$ *as* $\xi \to -\infty$*;*

(iii) $S(\xi)$ *and* $V(\xi)$ *are strictly decreasing for* $\xi \in \mathbb{R}$*;*

(iv) $S(\xi) \to S^1$, $A(\xi) \to 0$, $I(\xi) \to 0$, $V(\xi) \to V^1$ *as* $\xi \to +\infty$*;*

(v) $S'(\xi)$, $A'(\xi)$, $I'(\xi)$, $V'(\xi)$, $S''(\xi)$, $A''(\xi)$, $I''(\xi)$, $V''(\xi) \to 0$ *as* $\xi \to \pm\infty$*;*

(vi)

$$\int_{-\infty}^{+\infty} \mu_A A(\eta) d\eta = c^*(1-p)(S^0 - S^1) + c^*(1-p_V)(V^0 - V^1)$$

and

$$\int_{-\infty}^{+\infty} \mu I(\eta) d\eta = c^* p(S^0 - S^1) + c^* p_V (V^0 - V^1);$$

(vii) $S(\xi) < S^0$, $A(\xi) < (1-p)(S^0 - S^1) + (1-p_V)(V^0 - V^1)$, $I(\xi) < p(S^0 - S^1) + p_V(V^0 - V^1)$ *and* $V(\xi) < V^0$ *for* $\xi \in \mathbb{R}$.

Proof. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence satisfying $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. By Proposition 6, we have that for each $n \in \mathbb{N}$, there exists a solution $(S_n(\xi), A_n(\xi), I_n(\xi), V_n(\xi))$ for the system

$$\begin{cases} c^* S_n'(\xi) = d_1 S_n''(\xi) - \beta[\delta_A A_n(\xi) + I_n(\xi)] S_n(\xi), \\ c^* A_n'(\xi) = d A_n''(\xi) + \beta[(1-p) S_n(\xi) + (1-p_V)\delta_V V_n(\xi)][\delta_A A_n(\xi) + I_n(\xi)] - \mu_A A_n(\xi) - \varepsilon_n A_n^2(\xi), \\ c^* I_n'(\xi) = d I_n''(\xi) + \beta[p S_n(\xi) + p_V \delta_V V_n(\xi)][\delta_A A_n(\xi) + I_n(\xi)] - \mu I_n(\xi) - \varepsilon_n I_n^2(\xi), \\ c^* V_n'(\xi) = d_4 V_n''(\xi) - \delta_V \beta[\delta_A A_n(\xi) + I_n(\xi)] V_n(\xi), \end{cases} \tag{57}$$

which satisfies the asymptotic boundary conditions

$$\begin{aligned} (S_n(-\infty), A_n(-\infty), I_n(-\infty), V_n(-\infty)) &= (S^0, 0, 0, V^0), \\ (S_n(+\infty), A_n(+\infty), I_n(+\infty), V_n(+\infty)) &= (S^1, 0, 0, V^1). \end{aligned} \tag{58}$$

Using (vii) of Proposition 6, we obtain that there exists a constant $M_1 > 0$ independent of ε_n such that

$$\big|[\delta_A A_n(\xi) + I_n(\xi)] S_n(\xi)\big| \le M_1, \quad \xi \in \mathbb{R}.$$

Then it follows from the first equation in Eq. (57) that

$$\begin{aligned} |S_n'(\xi)| &= \frac{\beta}{d_1} e^{\frac{c^*}{d_1}\xi} \int_\xi^{+\infty} [\delta_A A_n(\eta) + I_n(\eta)] S_n(\eta) e^{-\frac{c^*}{d_1}\eta} d\eta \\ &\le \frac{\beta}{d_1} e^{\frac{c^*}{d_1}\xi} M_1 \int_\xi^{+\infty} e^{-\frac{c^*}{d_1}\eta} d\eta \\ &\le \frac{\beta}{c^*} M_1, \quad \xi \in \mathbb{R}. \end{aligned} \tag{59}$$

Similarly, it can be shown that there exist constants $M_i > 0$ $(i = 2, 3, 4)$ independent of ε_n such that

$$|A'(\xi)| \le M_2, \quad |I'(\xi)| \le M_3, \quad |V'(\xi)| \le M_4, \quad \xi \in \mathbb{R}. \tag{60}$$

By Eq. (57), Eq. (59) and Eq. (60), we can deduce that $S_n''(\xi)$, $A_n''(\xi)$, $I_n''(\xi)$ and $V_n''(\xi)$ are uniformly bounded on $\mathbb{R}$. Differentiating Eq. (57) with respect to ξ, we can conclude that $S_n'''(\xi)$, $A_n'''(\xi)$, $I_n'''(\xi)$ and $V_n'''(\xi)$ are uniformly bounded on $\mathbb{R}$. Then we infer from Arzelà-Ascoli's theorem that there exists a subsequence $\{\varepsilon_n\}$ such that

$$\begin{aligned} &S_n(\xi) \to S(\xi), \quad A_n(\xi) \to A(\xi), \quad I_n(\xi) \to I(\xi), \quad V_n(\xi) \to V(\xi), \quad S_n'(\xi) \to S'(\xi), \quad A_n'(\xi) \to A'(\xi), \\ &I_n'(\xi) \to I'(\xi), \quad V_n'(\xi) \to V'(\xi), \quad S_n''(\xi) \to S''(\xi), \quad A_n''(\xi) \to A''(\xi), \quad I_n''(\xi) \to I''(\xi), \quad V_n''(\xi) \to V''(\xi) \end{aligned}$$

uniformly in any bounded closed interval of $\mathbb{R}$ as $n \to +\infty$. Passing to the limit in Eq. (57) as $n \to \infty$ yields that $(S(\xi), A(\xi), I(\xi), V(\xi))$ is the solution of Eq. (2). It is easy to see that this solution satisfies the asymptotic boundary conditions Eq. (3). Similar to the proof of Proposition (6) with $c = c^*$, one can derive the rest of the assertions in this theorem. The proof is completed. ■

Remark 8 *Using a similar argument as in the proof of Theorem 7, one can obtain the positiveness of the super-critical traveling wave solution for Eq. (1) and the strict monotonicity of S-component and V-component in the super-critical traveling wave solution.*

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