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# Option Valuation of Carbon Emission Allowance by a Recursive Algorithm

Xiling Zhang*, Yusheng Kong, Yue Liu
School of Finance and Economics, Jiangsu University, Zhenjiang, Jiangsu 212013, People's Republic of China
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**Abstract:** Regime switching model can efficiently reflect the changes of the external environment to the adjustment of model parameters. Therefore, we use Markov chain to describe the external regime and establish a regime switching model. Based on this, option valuation of carbon emission allowance is proceeded. In order to realize the numerical calculation of the value function, a set of backward recursive algorithm is designed and proved. The algorithm implements recursion according to the division of Markov chain jumps, thus overcoming the high computational complexity brought by Markov chain.

**Keywords:** regime switching model, carbon emission allowance, option pricing.

## 1 Introduction

In order to synergistically control global warming and reduce greenhouse gas emission, the Paris Agreement came into force in November 2016. Since the Kyoto Protocol entered into force in 2005, greenhouse gas emission allowances have been uniformly distributed, and traded freely, therefore it is necessary to establish a carbon emission allowance trading market [1–3] and model develop the carbon emission allowance trading mechanism [4–7].

Due to the demands for risk control and value investment, the carbon emission allowance options trading market has gradually emerged. The current research on carbon emission options mainly follows the model study of European options for commodities or stocks. Also, traditional pricing of carbon emission allowance options still relies on the Black-Scholes option pricing model [8, 9].

However, the Black-Scholes option pricing model that simulates the price process with geometric Brownian motion has some well-known shortcomings, especially when describing the carbon emission price. The continuous low random white noise of geometric Brownian motion does not easily reflect sudden effects from the external market. For example, sudden release of industrial policies and environmental policies, sudden fluctuations in exchange rate or commodity markets may make the geometrical Brownian motion model with fixed drift coefficients and volatility invalid or inefficient.

Thus, in this study we introduce a regime switching model that randomizes the drift coefficient and the diffusion coefficient (volatility) of the geometric Brownian motion. The Markov process is used to describe the transition between different regimes(states), and these states reflect the temperature of the market. For example, the volatility corresponding to the overheated market is large, meanwhile the drift coefficient is also a large positive value; the volatility corresponding to the market downturn is much smaller. The regime switching model has been widely used in the research and application of financial derivatives pricing, hedging and trading strategies for more than a decade [10, 11]. For setting of the regime switching model parameters (such as Markov chain transition rates matrix), refer to [12] and [13].

Considering to the calculation of option pricing of the regime switching model, [14] designs a recursive algorithm based on Markov chain state transition. However, to lower the complexity of the algorithm, this paper designs a backward recursive algorithm that does not depend on the Markov chain state transition for the carbon emission allowance option pricing calculation. The regime switching model is not only introduced into the pricing of carbon emission allowance options, but also provides a more efficient algorithm for the calculation of the regime switching model.

The rest of the paper is organized as follows. Section 2 establishes a theoretical model and selects the probability measure of risk-neutral valuation. Section 3 proposes and proves the backward recursive algorithm.

**Corresponding author*. *E-mail address*:xiaozhxixi@163.com


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## 2 Theoretical model

Throughout this paper, the geometric Brownian motion $\{X_t\}_{t\in[0,T]}$ with regime switching model is used to describe the valuation of carbon emission allowance during the time period $[0,T]$. $X_t$ is characterized by the following stochastic differential equation:

$$dX_t = X_t\mu(\alpha_t)dt + X_t\sigma(\alpha_t)dw_t, \qquad t\in[0,T], \tag{1}$$

where $\{\alpha_t\}_{t\in[0,T]}$ is a time-continuous Markov process on the state space $\mathcal{M} := 1,2,...,m$ ($m$ is positive integer). Let its transition probability matrix be $Q = {q_{i,j}}_{m\times m}$ ; drift variable $\mu(\cdot)$ is a real-valued function defined on $\mathcal{M}$ , the volatility variable $\sigma(\cdot)$ is a non-negative function defined on $\mathcal{M}$; $\{w_t\}_{t\in[0,T]}$ is the standard Brownian motion at probability measure $P_0$.

Let $r$ denote the risk-free rate, and we will select the appropriate equivalent martingale measure, under which risk-neutral pricing will be obtained. According to the main results in [14], we define the equivalent measure $P$ of $P_0$ as follows:

$$dP := \exp\left(\int_0^T \frac{r-\mu(\alpha_t)}{\sigma(\alpha_t)}dw_t - \frac{1}{2}\int_0^T\left(\frac{r-\mu(\alpha_t)}{\sigma(\alpha_t)}\right)^2 dt\right)dP_0.$$

Thus, the following two properties hold.

a) $B_t := w_t - \int_0^t \frac{r-\mu(\alpha_s)}{\sigma(\alpha_s)}ds$ is the standard Brownian motion at measure $P$.

b) The discounted asset $e^{-rt}X_t$ is the martingale under measure $P$.

From a), we get

$$dX_t = X_t r dt + X_t\sigma(\alpha_t)dB_t, \qquad t\in[0,T]. \tag{2}$$

Therefore, the rest of the paper only considers the pricing of options under the measure $P$.

Set

$$V(t,x,i) := E\left[e^{-r(T-t)}\phi(X_T)\,|\,X_t = x, \alpha_t = i\right], \tag{3}$$

where $t\in[0,T]$, $x\in$, $i\in\mathcal{M}$, and

$$\phi(x) := (x-K)^+ = \max\{x-K, 0\},$$

with the exercise price $K > 0$ . The option is a call option, and the pricing problem for the put option can be treated similarly.

## 3 Backward recursive algorithm (BRA)

In this section we calculate the value function $V(t,x,i)$ (Equation (3)) based on the backward recursive algorithm (BRA). The following proposition gives the limit expression and recursive formula of the BRA method approximation calculation for the value of $V(t,x,i)$.

**Proposition 1** *The following limiting equality holds any $t\in[0,T]$, $x>0$, $i\in\mathcal{M}$,*

$$V(t,x,i) = \lim_{n\to\infty} V(\lceil t\rceil_n, x, i)$$

*where $V_n(t_k^n, x, i)$ is defined by Backward recursive definition,*

$$\begin{aligned} V_n(t_{k-1}^n,x,i) &:= e^{(q_{i,i}-r)\delta_n}\int_{-\infty}^{\infty} V_n(t_k^n, xe^{\delta_n(r-\frac{1}{2}\sigma(i))+\sigma(i)z}, i)\varphi_{\delta_n}(z)dz, \\ &+\sum_{\substack{j=1\\ j\neq i}}^{m} q_{i,j}\int_0^{\delta_n} e^{(q_{i,i}-r)u}\int_{-\infty}^{\infty} V_n(t_k^n, xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z}, j)\varphi_u(z)dzdu, \end{aligned} \tag{4}$$

where $k = 1,2,\ldots,n$, terminal condition $V_n(T,x,i) = x$ holds for any $i\in\mathcal{M}$, $x>0$.

To prove the proposition, firstly we introduce the following two lemmas.

**Lemma 2** *For any $k \in \{1, \ldots, n\}$, $j \in \mathcal{M}$ and $a \geq 1$, we have*

$$\begin{aligned} V(t_k^n, x, i) =& e^{(q_{i,i}-r)\delta_n} \int_{-\infty}^{\infty} V(t_{k+1}^n, xe^{\delta_n(r-\frac{1}{2}\sigma(i))+\sigma(i)z}, i)\varphi_{\delta_n}(z)dz \\ &+ \sum_{\substack{j=1 \\ j\neq i}}^{m} q_{i,j} \int_0^{\delta_n} e^{(q_{i,i}-r)u} \int_{-\infty}^{\infty} V(t_k^n + u, xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z}, j)\varphi_u(z)dzdu. \end{aligned} \tag{5}$$

**Proof.** Applying the Ito formula (refer to [15]) to the stochastic differential equation (2) yields

$$X_T = X_t \exp\left(\int_t^T r - \frac{1}{2}\sigma^2(\alpha_s)ds + \int_t^T \sigma(\alpha_s)dB_s\right), \qquad t \in [t, T]. \tag{6}$$

Substituting (6) into (3) gives

$$V(t, x, i) = e^{-r(T-t)} E\left[\phi(xe^{\int_t^T r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_t^T \sigma(\alpha_s)dB_s}) \,|\, \alpha_t = i\right], \tag{7}$$

where $t \in [0, T]$, $x > 0$, $i \in \mathcal{M}$. Next, we consider $V(t_k^n, x, i)$ for $k \in \{0, \ldots, n-1\}$, which can be written as

$$V(t_k^n, x, i) = \Phi_n(t_k^n, x, i) + \Upsilon_n(t_k^n, x, i), \tag{8}$$

where $x > 0$, $i \in \mathcal{M}$ and

$$\Phi_n(t_k^n, x, i) := e^{-r(T-t_k^n)} E\left[\phi(xe^{\int_{t_k^n}^T r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_{t_k^n}^T \sigma(\alpha_s)dB_s})\mathbf{1}_{\{T_1(t_k^n)>t_{k+1}^n\}} \,\Big|\, \alpha_{t_k^n} = j\right], \tag{9}$$

with $T_1(t) := \inf\{s \geq t \,:\, \alpha_s \neq \alpha_t\}$ for any $t \in \mathbb{R}_+$ and

$$\Upsilon_n(t_k^n, x, i) := e^{-r(T-t_k^n)} E\left[\phi(xe^{\int_{t_k^n}^T r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_{t_k^n}^T \sigma(\alpha_s)dB_s})\mathbf{1}_{\{T_1(t_k^n)\leq t_{k+1}^n\}} \,\Big|\, \alpha_{t_k^n} = j\right]. \tag{10}$$

For $k \in \{0, \ldots, n-1\}$, $j \in \mathcal{M}$ and $a \geq 1$, it follows from (9) that

$$\Phi_n(t_k^n, x, i) = e^{(q_{i,i}-r)\delta_n} \int_{-\infty}^{\infty} V(t_{k+1}^n, xe^{\delta_n(r-\frac{1}{2}\sigma(i))+\sigma(i)z}, i)\varphi_{\delta_n}(z)dz,$$

where we have used the following. Given $\alpha_t = j$, the first jump time of $(\alpha_s)_{s\in[t,\infty)}$ obeys the exponential distribution of the parameter $-q_{j,j} > 0$. Next, for any $k \in \{0, \ldots, n-1\}$, $j \in \mathcal{M}$ and $a \geq 1$, we examine $\Upsilon_n(t_k^n, x, i)$. From (10) we have

$$\Upsilon_n(t_k^n, x, i) = \sum_{\substack{j=1 \\ j\neq i}}^{m} q_{i,j} \int_0^{\delta_n} e^{(q_{i,i}-r)u} \int_{-\infty}^{\infty} V(t_k^n + u, xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z}, j)\varphi_u(z)dzdu. \tag{11}$$

The proof of this Lemma is finished. ■

**Lemma 3** *$t \mapsto V(t, x, i)$ uniformly continuous for any $i \in \mathcal{M}$, $x > 0$.*

**Proof.** Firstly, the following inequality holds for any real numbers $x$ and $y$.

$$|\,(x-K)^+ - (y-K)^+\,| \leq |x-y|, \qquad x \in \mathbb{R}, y \in \mathbb{R}. \tag{12}$$

From (3) for any real number $x < 1$, we have

$$e^{s-t}V(t, x, i) = e^{-r(T-s)} E\left[\phi(X_T) \,|\, X_t = x, \alpha_t = i\right], \qquad s < t. \tag{13}$$

Using (7) and Markov properties, we deduce

$$E\left[\phi(X_T) \,|\, X_t = x, \alpha_t = i\right] = E\left[\phi(xe^{\int_0^{T-t} r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_0^{T-t} \sigma(\alpha_s)dB_s}) \,|\, \alpha_t = i\right]. \tag{14}$$

Combining the equations (3), (12), (13) and (14), we get

$$\begin{aligned} & e^{s-t}V(t,x,i)-V(s,x,i) \\ & \le\; e^{-r(T-s)}E\left[x\left|e^{\int_0^{T-t} r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_0^{T-t}\sigma(\alpha_s)dB_s}-e^{\int_0^{T-s} r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_0^{T-t}\sigma(\alpha_s)dB_s}\right|\,\Big|\,\alpha_t=i\right]. \end{aligned} \tag{15}$$

Note that

$$\begin{aligned} & \left|e^{\int_0^{T-t} r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_0^{T-t}\sigma(\alpha_s)dB_s}-e^{\int_0^{T-s} r-\frac{1}{2}\sigma^2(\alpha_s)ds+\int_0^{T-t}\sigma(\alpha_s)dB_s}\right| \\ & \le 2e^{\max\limits_{0\le h\le T}\int_0^h\sigma(\alpha_s)dB_s+T\,|\,r-\frac{1}{2}\max\limits_{k\in\mathcal{M}}\sigma^2(k)\,|}. \end{aligned}$$

The probability measure $P$ is integrable, so the limit of the control convergence theorem can be applied to the equation (15). Thereby, the proof is completed. ■

Finally, we prove Proposition 1. For any $n=1,2,\ldots$, we define

$$\Delta_k^n:=\max_{i\in\mathcal{M}}\sup_{x>0}|\,V_n(\delta_k^n,x,i)-V(\delta_k^n,x,i)\,|,\quad k\in\{0,1,\ldots,n\}. \tag{16}$$

Thus we have $\Delta_n^n=0$. By (4) and (5), we obtain

$$\begin{aligned} \Delta_{k-1}^n \le & \max_{i\in\mathcal{M}}\sup_{x>0}e^{(q_{i,i}-r)\delta_n}\int_{-\infty}^{\infty}\big|\,V_n(t_k^n,xe^{\delta_n(r-\frac{1}{2}\sigma(i))+\sigma(i)z},i)-V(t_k^n,xe^{\delta_n(r-\frac{1}{2}\sigma(i))+\sigma(i)z},i)\,\big|\,\varphi_{\delta_n}(z)dz \\ & +\max_{i\in\mathcal{M}}\sup_{x>0}e^{(q_{i,i}-r)\delta_n}\sum_{\substack{j=1\\ j\neq i}}^{m}q_{i,j}\int_0^{\delta_n}e^{(q_{i,i}-r)u}\int_{-\infty}^{\infty}\big|\,V_n(t_k^n,xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z},j) \\ & -V(t_{k-1}^n+u,xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z},j)\varphi_u(z)dzdu. \end{aligned} \tag{17}$$

$$\begin{aligned} & \big|\,V_n(t_k^n,xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z},j)-V(t_{k-1}^n+u,xe^{u(r-\frac{1}{2}\sigma^2(i))+\sigma(i)z},j)\,\big| \\ & \le\Delta_k^n+\varepsilon_{k-1}^n,u\in(0,\delta_n] \end{aligned} \tag{18}$$

and

$$\varepsilon_k^n:=\max_{j\in\mathcal{M}}\sup_{\substack{x>0\\ \delta_k^n\le s<t\le\delta_{k+1}^n}}|\,V(t,x,j)-V(s,x,j)\,|,\quad k=0,1,\ldots,n-1.$$

From (17) and (18), we derive

$$\Delta_{k-1}^n\le c(\Delta_k^n+\varepsilon_{k-1}^n\delta_n),$$

where $c>0$ is a constant independent of $n\ge1$. Thus,

$$\Delta_k^n\le c\left(\delta_n\sum_{i=k}^{n-1}\varepsilon_i^n\right),\qquad k=0,1,\ldots,n.$$

For $k$ of all the above formula, taking the maximum value of $\Delta_k^n$, we have

$$\max_{k=0,1,\ldots,n}\Delta_k^n=\max_{\substack{i\in\mathcal{M}\\ k=0,1,\ldots,n}}\sup_{x>0}|\,V_n(\delta_k^n,x,i)-V(\delta_k^n,x,i)\,|\le c\left(\max_{k=0,1,\ldots,n}\varepsilon_k^n\right). \tag{19}$$

By Lemma 2, it follows that $\max\limits_{k=0,1,\ldots,n}\Delta_k^n\to0$ as $n\to\infty$ . We get

$$\lim_{n\to\infty}|\,V(\lceil t\rceil_n,x,j)-V_n(\lceil t\rceil_n,x,j)\,|=0$$

Finally, we obtain from Lemma 2 that

$$V(t,x,j)=\lim_{n\to\infty}V(\lceil t\rceil_n,x,j)=\lim_{n\to\infty}V_n(\lceil t\rceil_n,x,j),t\in[0,T],j\in\mathcal{M},a\ge1.$$

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