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Exact Traveling Wave Solutions and Conservation Laws of (2+1) Dimensional Konopelchenko-Dubrovsky System

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Abstract: In this paper, we establish exact traveling wave solutions for the $(2{+}1)$ -dimensional Konopelchenko-Dubrovsky system using the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method. In addition, we derive local conservation laws of the $(2+1)$-dimensional Konopelchenko-Dubraovsky system by the multiplier approach.

Keywords: traveling wave solutions; (2+1) dimensional Konopelchneko-Dubrovsky equation; (((G')/G),(1/G))-expansion method; conservation laws;multiplier approach

1 Introduction

As stated in [1], nonlinear evolution equations (NLEEs) are widely used as models to describe complex physical phenomena in various fields of the sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics. When a NLEE is analysed, one of the most important question is the construction of the exact solutions for the equation [2]. In the open literature, quite a few methods for obtaining explicit traveling and solitary wave solutions to NLEEs have suggested such as the inverse scattering method [3], the bilinear transformation method [4], the tanh–sech method [5, 6] , the extended tanh method [7, 8], the sine–cosine method [9, 11], the homogeneous balance method [12, 13] , the pseudo spectral method [14],the (G'/G) -expansion method [15, 17], exp-function method [18], variational iteration method [19], homotopy perturbation method [20], the first Jacobi elliptic function method [21], Lie group analysis method [22] and so on.

The pioneer work Li et al. [23] introduced the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method for a reliable treatment of the nonlinear wave equations. The useful $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method is widely used by many such as in [24], [25], [26] and by the reference therein.

On the other hand, conservation laws plays an important role in the analysis of basic properties of solutions. The existence of a large number of conservation laws of a partial differential equation (PDE) is a strong indication of its integrability [22]. They are used for analysis, in particular, existence, uniqueness and stability analysis and construction of numerical schemes [27]. In addition, in the numerical integration of PDEs [28, 29], for example, to control numerical errors, conservation laws are also used.

The first objective of this paper is to apply the two variable $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method to find the exact traveling wave solutions of the following nonlinear (2+1)-dimensional Konopelchenko–Dubrovsky system [30, 32]

$$\begin{aligned} u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv &= 0, \\ u_y &= v_x, \end{aligned} \tag{1}$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ the subscripts denote partial differentiation, a and b are the real parameters. For $u_y = 0$, equation (1) is the Gardner equation (combined KdV and modified equation). For $a = 0$, equation (1) is the

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well-known Kadomtsev–Petviashvili equation and modified Kadomtsev–Petviashvili equation for $b = 0$. In [11], the authors have obtained exact traveling wave solutions of (1) by using the sine-cosine method. Wang and Zhang performed the modified extended tanh function method to the system (1) and obtain abundant new exact solutions of the equation in [33] (see also, [17]).

The other purpose of the work is the construct the local conservation laws of the system (1). For this aim, we performed the multiplier approach ([22, 34]) to the Konopelchenko–Dubrovsky system.

The plan of the paper is organized as follows : In Sec. 2, we give the description of the two variable $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method. In Secs. 3, we apply this method to system (1). Section 4 is devoted to the conservation laws. In Sec. 4, some concluding remarks are given.

2 $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method

First, we need the following preparations (see [23], [24], [26]):

If we consider the second order linear ordinary differential equation (ODE):

$$G''(\xi) + \lambda G(\xi) = \mu \tag{2}$$

and set $\phi = \frac{G'}{G}$, $\psi = \frac{1}{G}$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{3}$$

Case 1. If $\lambda < 0$, then the general solutions of Eq. (2) has the form:

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{4}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{5}$$

where $\sigma = A_1^2 - A_2^2$.

Case 2. If $\lambda > 0$, then the general solutions of Eq. (2) has the form:

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda} \tag{6}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{7}$$

where $\sigma = A_1^2 + A_2^2$.

Case 3. If $\lambda = 0$, then the general solutions of Eq. (2) has the form:

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2 \tag{8}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right). \tag{9}$$

Suppose we have the following nonlinear evolution equation

$$F\left(u, u_t, u_x, u_y, u_{xx}, \ldots\right) = 0 \tag{10}$$

where F is a polynomial in $u\left(x, y, t\right)$ and its partial derivatives. In the following, we give tha main steps of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expancion method [23], [24]:

Step 1. Traveling wave transformation

$$u\left(x, y, t\right) = u\left(\xi\right), \quad \xi = x + y - \omega t, \tag{11}$$

where ω is a constant, reduces Eq.(10) to an ODE in the form:

$$P\left(u, u', u'', ...\right) = 0, \tag{12}$$

where P is apolynomial of $u\left(\xi\right)$ and its total derivatives with respect to ξ.

Step 2. Assuming that the solution of Eq.(12) can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u\left(\xi\right) = \sum_{i=0}^{N} a_i \phi^i + \sum_{j=1}^{N} b_j \phi^{j-1} \psi \tag{13}$$

where a_i $(i = 0, 1, ..., N)$ and b_j $(j = 1, 2, ..., N)$ are constants to be determined later.

Step 3. Determine the positive integer N in Eq.(13) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in Eq.(12).

Step 4. Substitute Eq.(13) into Eq.(12) along with (3) and (5), the left-hand side Eq.(12) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equating each coefficients of this polynomial to zero, yields a system of algebraic equations which can be solved by using the Maple to get the values of a_i, b_j, ω, μ, A_1, A_2 and λ where $\lambda < 0$.

Step 5. Similar to step 4, substitute Eq.(13) into Eq. (12) along with (3) and (7) for $\lambda > 0$, (or (3) and (9) for $\lambda = 0$), we obtain the exact solutions of Eq.(12) expressed by trigonometric functions (or by rational functions) respectively.

3 Traveling wave solutions of system (1)

In this section, we will apply the method described in Sec.2 to find the exact traveling wave solutions of the nonlinear $(2+1)$-dimensional Konopelchenko-Dubrovsky system (1). Applying the transformations $u\left(x, y, t\right) = u\left(\xi\right)$ and $v\left(x, y, t\right) = v\left(\xi\right)$, where the wave variable $\xi = x + y - \omega t$, converts Eq.(1) into a system of ODE as

$$-\omega u' - u''' - 6buu' + \frac{3}{2}a^2u^2u' - 3v' + 3au'v = 0, \tag{14}$$

$$u' = v'. \tag{15}$$

We can rewrite (15) in the form

$$v = u. \tag{16}$$

Now, inserting (16) into (14) and integrating Eq. (14), yields

$$\frac{a^2}{2}u^3 + \frac{3}{2}\left(a - 2b\right)u^2 - \left(\omega + 3\right)u - u'' = \beta \tag{17}$$

with β is a constant of integration. By balancing between u'' and u^3 in Eq.(17) we get $N = 1$(see also, [17]). Consequently, we get

$$u\left(\xi\right) = a_0 + a_1\phi + b_1\psi \tag{18}$$

where a_0, a_1 and b_1 are constants to be determined later.

There are three cases to be discussed as follows:

3.1 Hyperbolic function solutions $(\lambda < 0)$

If $\lambda < 0$, substituting (18) into (17) and using (3) and (5), the left-hand side of Eq.(17) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, yields a system of algebraic equations in a_0 , a_1, b_1, ω, λ and μ as follows:

$$
\begin{aligned}
\phi^3 &: \quad a^2a_1^2\mu^2 - 4\mu^2 - 3a^2b_1^2\lambda + a^2a_1^2\lambda^2\sigma - 4\lambda^2\sigma = 0,\\
\phi^2\psi &: \quad 3a^2a_1^2\mu^2 - 4\mu^2 - a^2b_1^2\lambda - 4\lambda^2\sigma + 3a^2a_1^2\lambda^2\sigma = 0,\\
\phi^2 &: \quad -2b_1\lambda^3\mu\sigma + 6aa_1^2\lambda^2\mu^2\sigma - 3a^2a_0b_1^2\lambda^3\sigma + 6bb_1^2\lambda\mu^2 + 3a^2a_0a_1^2\lambda^4\sigma^2 - 3ab_1^2\lambda^3\sigma - 3ab_1^2\lambda\mu^2\\
&\quad +6a^2a_0a_1^2\lambda^2\sigma\mu^2 - 6ba_1^2\lambda^4\sigma^2 - 2a^2b_1^3\lambda^2\mu - 6ba_1^2\mu^4 + 3a^2a_0a_1^2\mu^4 + 3aa_1^2\mu^4 - 3a^2a_0b_1^2\lambda\mu^2\\
&\quad +3aa_1^2\lambda^4\sigma^2 - 12ba_1^2\mu^2\lambda^2\sigma + 6bb_1^2\lambda^3\sigma - 2b_1\lambda\mu^3 = 0,\\
\phi\psi &: \quad \mu^3 + ab_1\mu^2 + a^2a_0b_1\mu^2 - 2bb_1\mu^2 + \lambda^2\sigma\mu + a^2b_1^2\lambda\mu - 2bb_1\lambda^2\sigma + a^2a_0b_1\lambda^2\sigma + ab_1\lambda^2\sigma = 0,\\
\phi &: \quad -2\mu^2\omega + 3a^2a_0^2\mu^2 - 4\lambda\mu^2 - 12ba_0\mu^2 + 6aa_0\mu^2 - 6\mu^2 - 12ba_0\lambda^2\sigma + 6aa_0\lambda^2\sigma - 6\lambda^2\sigma\\
&\quad +3a^2a_0^2\lambda^2\sigma - 4\lambda^3\sigma - 2\omega\lambda^2\sigma - 3a^2b_1^2\lambda^2 = 0,\\
\psi &: \quad 12aa_0\lambda^2\sigma\mu^2 + 6a^2a_0^2\lambda^2\sigma\mu^2 - 24ba_0\lambda^2\sigma\mu^2 + 2\lambda\mu^4 - 2\omega\mu^4 - 4\omega\lambda^2\sigma\mu^2 + 6aa_0\lambda^4\sigma^2\\
&\quad -12ba_0\lambda^4\sigma^2 + 3a^2b_1^2\lambda^2\mu^2 - a^2b_1^2\lambda^4\sigma + 3a^2a_0^2\lambda^4\sigma^2 - 12bb_1\lambda^3\sigma\mu + 6ab_1\lambda^3\sigma\mu\\
&\quad +6a^2b_1a_0\lambda\mu^3 - 2\omega\lambda^4\sigma^2 + 6aa_0\mu^4 + 3a^2a_0^2\mu^4 - 12ba_0\mu^4 - 12bb_1\lambda\mu^3 + 6ab_1\lambda\mu^3\\
&\quad +6a^2b_1a_0\lambda^3\sigma\mu - 2\lambda^5\sigma^2 - 6\lambda^4\sigma^2 - 12\lambda^2\sigma\mu^2 - 6\mu^4 = 0,\\
\phi^0 &: \quad -6a_0\lambda^4\sigma^2 - 2\beta\lambda^4\sigma^2 - 6a_0\mu^4 - 2\beta\mu^4 + 3aa_0^2\mu^4 - 2\omega a_0\mu^4 - 6ba_0^2\mu^4 + a^2a_0^2\mu^4 - 2b_1\lambda^2\mu^3\\
&\quad -3a^2a_0b_1^2\lambda^4\sigma - 3a^2a_0b_1^2\lambda^2\mu^2 + 6aa_0^2\lambda^2\sigma\mu^2 - 4\omega a_0\lambda^2\sigma\mu^2 - 12ba_0^2\lambda^2\sigma\mu^2 + 2a^2a_0^3\lambda^2\sigma\mu^2\\
&\quad -2a^2b_1^3\lambda^3\mu + 3aa_0^2\lambda^4\sigma^2 - 2\omega a_0\lambda^4\sigma^2 - 6ba_0^2\lambda^4\sigma^2 + a^2a_0^3\lambda^4\sigma^2 - 12a_0\lambda^2\sigma\mu^2 - 4\beta\lambda^2\sigma\mu^2\\
&\quad -2b_1\lambda^4\sigma\mu - 3ab_1^2\lambda^4\sigma - 3ab_1^2\lambda^2\mu^2 + 6bb_1^2\lambda^4\sigma + 6bb_1^2\lambda^2\mu^2 = 0.
\end{aligned}
$$

On solving the above algebraic equations using the Maple, we get the following results:

$$
\begin{aligned}
a_0 &= -\frac{a-2b}{a^2}, \quad a_1 = \mp\frac{1}{a}, \quad b_1 = \mp\frac{1}{a}\sqrt{\frac{\mu^2+\lambda^2\sigma}{-\lambda}},\\
\omega &= -\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}, \quad \beta = -\frac{a^3-6a^2b+12ab^2-8b^3+a^3\lambda-2a^2b\lambda}{2a^4}.
\end{aligned}
$$

In this case, the exact solution of Eq.(17) has the form:

$$
\begin{aligned}
u_1(\xi) &= -\frac{a-2b}{a^2} + \frac{1}{a}\sqrt{-\lambda}\frac{A_1\cosh\left(\xi\sqrt{-\lambda}\right) + A_2\sinh\left(\xi\sqrt{-\lambda}\right)}{A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}}\\
&\quad +\frac{1}{a}\sqrt{\frac{\mu^2+\lambda^2\sigma}{-\lambda}}\frac{1}{A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}},\\
u_2(\xi) &= -\frac{a-2b}{a^2} - \frac{1}{a}\sqrt{-\lambda}\frac{A_1\cosh\left(\xi\sqrt{-\lambda}\right) + A_2\sinh\left(\xi\sqrt{-\lambda}\right)}{A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}}\\
&\quad -\frac{1}{a}\sqrt{\frac{\mu^2+\lambda^2\sigma}{-\lambda}}\frac{1}{A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}}
\end{aligned}
\tag{19}
$$

where μ is an arbitrary constant, $\sigma = A_1^2 - A_2^2$ and $\xi = x + y + \left(\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}\right)t$.

If $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$, then we have the solitary wave solution

$$\begin{aligned} u_1(x,y,t) &= v_1(x,y,t) \\ &= -\frac{a-2b}{a^2} + \frac{1}{a}\sqrt{-\lambda}\left\{ \begin{array}{c} \tanh\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \\ +\sec h\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \end{array} \right\}, \\ u_2(x,y,t) &= v_2(x,y,t) \\ &= -\frac{a-2b}{a^2} - \frac{1}{a}\sqrt{-\lambda}\left\{ \begin{array}{c} \tanh\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \\ +\sec h\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \end{array} \right\} \end{aligned} \tag{20}$$

If $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$, then we have the solitary wave solution

$$\begin{aligned} u_1(x,y,t) &= v_1(x,y,t) \\ &= -\frac{a-2b}{a^2} + \frac{1}{a}\sqrt{-\lambda}\left\{ \begin{array}{c} \coth\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \\ +co\sec h\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \end{array} \right\} \\ u_2(x,y,t) &= v_2(x,y,t) \\ &= -\frac{a-2b}{a^2} - \frac{1}{a}\sqrt{-\lambda}\left\{ \begin{array}{c} \coth\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \\ +co\sec h\left[\left(x+y+\frac{9a^2+a^2\lambda-12ab+12b^2}{2a^2}t\right)\sqrt{-\lambda}\right] \end{array} \right\} \end{aligned} \tag{21}$$

3.2 Trigonometric function solutions $(\lambda > 0)$

If $\lambda > 0$, substituting (18) into (17) and using (3) and (7), the left-hand side of Eq.(17) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, yields a system of algebraic equations in a_0 , a_1, b_1, ω, λ and μ as follows:

$$\phi^3 : \quad a^2a_1^2\mu^2 - 4\mu^2 - 3a^2b_1^2\lambda + 4\lambda^2\sigma - a^2a_1^2\lambda^2\sigma = 0,$$

$$\phi^2\psi : \quad -4\mu^2 + 3a_1^2a^2\mu^2 - a^2b_1^2\lambda - 3a^2a_1^2\lambda^2\sigma + 4\lambda^2\sigma = 0,$$

$$\begin{aligned} \phi^2 : \quad & -6aa_1^2\lambda^2\sigma\mu^2 + 2b_1\lambda^3\sigma\mu - 3a^2a_0b_1^2\lambda\mu^2 + 3a^2a_0a_1^2\mu^4 + 6bb_1^2\lambda\mu^2 + 3ab_1^2\lambda^3\sigma \\ & -6bb_1^2\lambda^3\sigma + 3aa_1^2\mu^4 - 2a^2b_1^3\lambda^2\mu + 3a^2a_0b_1^2\lambda^3\sigma - 3ab_1^2\lambda\mu^2 + 3aa_1^2\lambda^4\sigma^2 - 6ba_1^2\mu^4 \\ & +12ba_1^2\lambda^2\sigma\mu^2 + 3a^2a_0a_1^2\lambda^4\sigma^2 - 6a^2a_0a_1^2\lambda^2\sigma\mu^2 - 2b_1\lambda\mu^3 = 0, \end{aligned}$$

$$\phi\psi : \quad \mu^3 + a^2a_0b_1\mu^2 + ab_1\mu^2 - 2bb_1\mu^2 - \lambda^2\sigma\mu + a^2b_1^2\lambda\mu - a^2a_0b_1\lambda^2\sigma - ab_1\lambda^2\sigma + 2bb_1\lambda^2\sigma = 0,$$

$$\begin{aligned} \phi : \quad & 3a^2a_0^2\mu^2 - 2\omega\mu^2 + 6aa_0\mu^2 - 12ba_0\mu^2 - 6\mu^4 - 4\lambda\mu^2 + 4\lambda^3\sigma - 3a^2b_1^2\lambda^2 - 3a^2a_0^2\lambda^2\sigma \\ & +2\omega\lambda^2\sigma + 12ba_0\lambda^2\sigma + 6\lambda^2\sigma - 6aa_0\lambda^2\sigma = 0, \end{aligned}$$

$$\begin{aligned} \psi : \quad & -6\lambda^4\sigma^2 + 12\lambda^2\sigma\mu^2 - 6\mu^4 + 24ba_0\lambda^2\sigma\mu^2 - 6a^2a_0^2\lambda^2\sigma\mu^2 - 12aa_0\lambda^2\sigma\mu^2 - 2\lambda^5\sigma^2 \\ & +2\lambda\mu^4 - 2\omega\mu^4 + 3a^2a_0^2\lambda^4\sigma^2 + 6aa_0\lambda^4\sigma^2 + 3a^2b_1^2\lambda^2\mu^2 - 12ba_0\lambda^4\sigma^2 + a^2b_1^2\lambda^4\sigma \\ & +4\omega\lambda^2\sigma\mu^2 - 6ab_1\lambda^3\sigma\mu + 12bb_1\lambda^3\sigma\mu + 6a^2a_0b_1\lambda\mu^3 + 3a^2a_0^2\mu^4 - 2\omega\lambda^4\sigma^2 + 6aa_0\mu^4 \\ & -12ba_0\mu^4 + 6ab_1\lambda\mu^3 - 12bb_1\lambda\mu^3 - 6a^2a_0b_1\lambda^3\sigma\mu = 0, \end{aligned}$$

$$\begin{aligned} \phi^0 : \quad & -6a_0\mu^4 - 2\beta\mu^4 - 6a_0\lambda^4\sigma^2 - 2\beta\lambda^4\sigma^2 - 2\omega a_0\mu^4 - 6ba_0^2\mu^4 + a^2a_0^3\mu^4 + 3aa_0^2\mu^4 - 2b_1\lambda^2\mu^3 \\ & +3a^2a_0b_1^2\lambda^4\sigma - 3a^2a_0b_1^2\lambda^2\mu^2 + 4\omega a_0\lambda^2\sigma\mu^2 + 12ba_0^2\lambda^2\sigma\mu^2 - 2a^2a_0^3\lambda^2\sigma\mu^2 - 6aa_0^2\lambda^2\sigma\mu^2 \\ & -2a^2b_1^3\lambda^3\mu - 2\omega a_0\lambda^4\sigma^2 - 6ba_0^2\lambda^4\sigma^2 + a^2a_0^3\lambda^4\sigma^2 + 3aa_0^2\lambda^4\sigma^2 + 12a_0\lambda^2\sigma\mu^2 + 4\beta\lambda^2\sigma\mu^2 \\ & +3ab_1^2\lambda^4\sigma - 3ab_1^2\lambda^2\mu^2 - 6bb_1^2\lambda^4\sigma + 6bb_1^2\lambda^2\mu^2 + 2b_1\lambda^4\sigma\mu = 0. \end{aligned}$$

On solving the above algebraic equations using the Maple, we get the following results:

$$\begin{aligned} a_0 &= -\frac{a-2b}{a^2},\ \ a_1 = \mp\frac{1}{a},\ \ b_1 = \mp\frac{1}{a}\sqrt{\frac{\mu^2-\lambda^2\sigma}{\lambda}}, \\ \omega &= -\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2},\ \ \beta = -\frac{a^3-6a^2b+12ab^2-8b^3+a^3\lambda-2a^2b\lambda}{2a^4}. \end{aligned}$$

In this case, the exact solution of Eq.(17) has the form:

$$\begin{aligned} u_1(\xi) &= -\frac{a-2b}{a^2}+\frac{1}{a}\sqrt{\lambda}\frac{A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)}{A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}} \\ &\quad +\frac{\frac{1}{a}\sqrt{\frac{(\lambda^2\sigma-\mu^2)}{\lambda}}}{A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}}, \\ u_2(\xi) &= -\frac{a-2b}{a^2}-\frac{1}{a}\sqrt{\lambda}\frac{A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)}{A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}} \\ &\quad -\frac{\frac{1}{a}\sqrt{\frac{(\lambda^2\sigma-\mu^2)}{\lambda}}}{A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}} \end{aligned} \tag{22}$$

where μ is an arbitrary constant, $\sigma = A_1^2 + A_2^2$ and $\xi = x + y + \left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t$.

If $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$, then we have the solitary wave solution

$$\begin{aligned} u_1(x,y,t) &= v_1(x,y,t) \\ &= -\frac{a-2b}{a^2}+\frac{1}{a}\sqrt{\lambda}\left\{\begin{array}{c} -\tan\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \\ +i\sec\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \end{array}\right\}, \\ u_2(x,y,t) &= v_2(x,y,t) \\ &= -\frac{a-2b}{a^2}+\frac{1}{a}\sqrt{\lambda}\left\{\begin{array}{c} \tan\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \\ +i\sec\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \end{array}\right\}. \end{aligned} \tag{23}$$

If $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$, then we have the solitary wave solution

$$\begin{aligned} u_1(x,y,t) &= v_1(x,y,t) \\ &= -\frac{a-2b}{a^2}+\frac{1}{a}\sqrt{\lambda}\left\{\begin{array}{c} \cot\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \\ +ico\sec\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \end{array}\right\} \\ u_2(x,y,t) &= v_2(x,y,t) \\ &= -\frac{a-2b}{a^2}-\frac{1}{a}\sqrt{\lambda}\left\{\begin{array}{c} \cot\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \\ +i\,co\sec\left[\left(x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right)\sqrt{\lambda}\right] \end{array}\right\}. \end{aligned} \tag{24}$$

3.3 Rational function solutions $(\lambda = 0)$

If $\lambda = 0$, substituting (18) into (17) and using (3) and (9), the left-hand side of Eq.(17) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero, yields a system of algebraic equations in a_0 , a_1, b_1, ω, λ and μ as follows:

$$
\begin{aligned}
\phi^3 &: \quad a^2a_1^2A_1^2-4A_1^2+8A_2\mu-2a^2a_1^2A_2\mu+3a^2b_1^2=0,\\
\phi^2\psi &: \quad 3a^2a_1^2A_1^2-4A_1^2-6a^2a_1^2A_2\mu+a^2b_1^2+8A_2\mu=0,\\
\phi^2 &: \quad 2b_1A_1^2\mu-4b_1A_2\mu^2+12aa_1^2A_2^2\mu^2-24ba_1^2A_2^2\mu^2-2a^2b_1^3\mu-6bb_1^2A_1^2+3a^2a_0b_1^2A_1^2\\
&\quad -12a^2a_0a_1^2A_1^2A_2\mu-6ab_1^2A_2\mu+12bb_1^2A_2\mu-6a^2a_0b_1^2A_2\mu+3ab_1^2A_1^2-12aa_1^2A_1^2A_2\mu\\
&\quad +12a^2a_0a_1^2A_2^2\mu+3aa_1^2A_1^4-6ba_1^2A_1^4+3a^2a_0a_1^2A_1^4+24ba_1^2A_1^2A_2\mu=0,\\
\phi\psi &: \quad a^2a_0b_1A_1^2+ab_1A_1^2-2bb_1A_1^2+A_1^2\mu-2ab_1A_2\mu-a^2b_1^2\mu+4bb_1A_2\mu-2a^2a_0b_1A_2\mu-2A_2\mu^2=0,\\
\phi &: \quad -2\omega+6aa_0-12ba_0+3a^2a_0^2-6=0,\\
\psi &: \quad 4a^2b_1^2\mu^2-24bb_1A_2\mu^2+12bb_1A_1^2\mu-6a^2a_0b_1A_1^2\mu-6ab_1A_1^2\mu+12ab_1A_2\mu^2\\
&\quad +12a^2a_0b_1A_2\mu^2+12a^2a_0^2A_2^2\mu^2-2\omega A_1^4-24A_2^2\mu^2-4A_1^2\mu^2+8A_2\mu^3-8\omega A_2^2\mu^2\\
&\quad +48ba_0A_1^2A_2\mu+6aa_0A_1^4-12ba_0A_1^4+3a^2a_0^2A_1^4-6A_1^4+24aa_0A_2^2\mu^2+24A_1^2A_2\mu\\
&\quad -48ba_0A_2^2\mu^2-24aa_0A_1^2A_2\mu-12a^2a_0^2A_1^2A_2\mu+8\omega A_1^2A_2\mu=0,\\
\phi^0 &: \quad -2\omega a_0-6a_0-6ba_0^2+a^2a_0^3+3aa_0^2-2\beta=0.
\end{aligned}
$$

On solving the above algebraic equations using the Maple, we get the following results:

$$
\begin{aligned}
a_0 &= -\frac{a-2b}{a^2},\quad a_1=\mp\frac{1}{a},\quad b_1=\mp\frac{\sqrt{A_1^2-2\mu A_2}}{a},\\
\omega &= -\frac{3\left(3a^2-4ab+4b^2\right)}{2a^2},\quad \beta=-\frac{(a-2b)^3}{2a^4}.
\end{aligned}
$$

In this case, the exact solution of Eq.(17) has the form:

$$
\begin{aligned}
u_1(\xi) &= -\frac{a-2b}{a^2}+\frac{1}{a}\frac{\mu\xi+A_1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}\\
&\quad +\frac{\sqrt{A_1^2-2\mu A_2}}{a}\frac{1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2},\\
u_2(\xi) &= -\frac{a-2b}{a^2}-\frac{1}{a}\frac{\mu\xi+A_1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}\\
&\quad -\frac{\sqrt{A_1^2-2\mu A_2}}{a}\frac{1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}
\end{aligned}
\tag{25}
$$

where μ is an arbitrary constant and $\xi = x + y + \left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t$. We have the solitary wave solution

$$\begin{aligned}
u_1(x,y,t) &= v_1(x,y,t) \\
&= -\frac{a-2b}{a^2} + \frac{1}{a}\frac{\mu\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]+A_1}{\left\{\begin{array}{c}\frac{\mu}{2}\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]^2 \\ +A_1\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]+A_2\end{array}\right\}} \\
&\quad + \frac{\sqrt{A_1^2-2\mu A_2}}{a}\frac{1}{\left\{\begin{array}{c}\frac{\mu}{2}\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]^2 \\ +A_1\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]+A_2\end{array}\right\}}, \\
u_2(x,y,t) &= v_2(x,y,t) \\
&= -\frac{a-2b}{a^2} + \frac{1}{a}\frac{\mu\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]+A_1}{\left\{\begin{array}{c}\frac{\mu}{2}\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]^2 \\ +A_1\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]+A_2\end{array}\right\}} \\
&\quad - \frac{\sqrt{A_1^2-2\mu A_2}}{a}\frac{1}{\left\{\begin{array}{c}\frac{\mu}{2}\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]^2 \\ +A_1\left[x+y+\left(\frac{9a^2-12ab+12b^2+a^2\lambda}{2a^2}\right)t\right]+A_2\end{array}\right\}}.
\end{aligned} \tag{26}$$

4 Conservation laws

In this section, we construct conservation laws for (1). For the details see e. g., [22, 27, 34] and [35].

Consider a kth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^n)$, namely

$$E_\alpha\left(x, u, u_{(1)}, ..., u_{(k)}\right) = 0, \quad \alpha = 1, ..., m, \tag{27}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second, ..., kth order partial derivatives, i. e., $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$, ..., respectively, with the total derivative operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ..., \quad j = 1, ..., n, \tag{28}$$

where the summation convertion is used whenever appropriate. The n-tuple vector

$$T = \left(T^1, T^2, ..., T^n\right), \quad T^j \in A, \quad j = 1, ..., n,$$

is a conserved vector of (27) if T^i satisfied

$$D_i T^i|_{(27)} = 0. \tag{29}$$

The equation (29) is called a local consevation law of system (27).

It can be shown that every admitted consevation laws arises from multipliers $Q^\alpha\left(x, u, u_{(1)}, ...\right)$ such that

$$Q^\alpha E_\alpha = D_i T^i, \tag{30}$$

holds identically. In the multiplier approach for conservation laws, one takes the variational derivative of (30) that is,

$$\frac{\delta}{\delta u^\beta}\left(Q^\alpha E_\alpha\right) = 0, \tag{31}$$

holds for arbitrary functions of $u\left(x^1, x^2, ..., x^n\right)$. Here we will consider multipliers of the second order $\Lambda_\alpha = \Lambda_\alpha\left(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}\right)$. Once the multipliers are obtained the conserved vectors are calculated via a homotopy formula [35] and [37].

4.1 Conservation laws of system (1)

For the Konopelchenko-Dubrovsky system (1), we see that the three multipliers (with the aid of GeM [27], see also [36]-[37]), namely $\Lambda_1\left(x,t,u,v\right),\ \Lambda_2\left(x,t,u,v\right)$ are given by

$$\Lambda_1 = \frac{1}{3}(-c_1 au - c_2),$$

$$\Lambda_2 = F(t) + \frac{1}{2}(2v + au^2)c_1 + c_2 u +$$

where C_i, i = 1, 2 are arbitrary constants and $F(t)$ is arbitrary function of t .

Corresponding to the above multipliers we have the following three conserved vectors of (1) as

$$\begin{aligned}
C_1^t &= -\frac{1}{6}u^2,\\
C_1^x &= -\frac{1}{8}a^2u^4 + \frac{2}{3}bu^3 - \frac{1}{2}au^2v + \frac{1}{3}uu_{xx} - \frac{1}{6}u_x^2 - \frac{1}{2}v^2,\\
C_1^y &= -\frac{1}{6}au^3 + uv,\\
C_2^t &= -\frac{1}{3a}u,\\
C_2^x &= -\frac{a^2u^3 - 6bu^2 + 6auv - 2u_{xx}}{6a},\\
C_2^y &= \frac{2v + au^2}{2a},\\
C_3^t &= 0,\\
C_3^x &= -vF(t),\\
C_3^y &= uF(t).
\end{aligned}$$

5 Concluding remarks

In the present work, we have succeeded in implementing and applying the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method for finding some traveling wave solutions of the (2 + 1)-dimensional Konopelchenko-Dubrovsky system. The obtained exact solutions can be used as benchmarks against the numerical simulations. Some of these results are in agreement with the results reported by others in the literature, and new results are formally developed in this work. From our results obtained in this paper, we conclude that the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method is powerful, effective and convenient for NLEEs. As one can see, this method has more general applications than the other methods.

Moreover, we constructed conservation laws for the system (1) via the multiplier approach. For the system (1), this method gave rise to three multipliers and thus three conserved vectors were obtained. The conserved vectors obtained here can be used in reductions and solutions of the underlying system [38].

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