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A Matrix Inverse Pair and Elliptic Hypergeometric Summations

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Abstract: In this paper, we use matrix inversion technique to derive three summation formulas for elliptic hypergeometric series. To my knowledge, two of them are new and the other formula was first discovered by Warnaar.

Keywords: elliptic hypergeometric series; basic hypergeometric series; matrix inverse pair.

1 Introduction

We will follow the standard notations on basic hypergeometric series (q-series) and elliptic hypergeometric series [5], and we always assume $|q| < 1$ and $|p| < 1$. Define the q-shifted factorial for all integers n by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

and a theta function $\theta(a;p)$ by

$$\theta(a;p) = (a;p)_\infty (p/a;p)_\infty,$$

where $a \neq 0$. $(a;p,q)_n$ which is an elliptic shifted factorial analogue of the q-shifted factorial is defined by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1}\theta(aq^k;p), & n = 1,2,\ldots, \\ 1, & n = 0, \\ \left(\prod_{k=0}^{-n-1}\theta(aq^{n+k};p)\right)^{-1}, & n = -1,-2,\ldots. \end{cases}$$

We call q and p in $(a;q,p)_n$ the *base* and *nome*, respectively.

As usual,

$$\begin{aligned}(a_1,\ldots,a_k;q)_n =&(a_1;q)_n\cdots(a_k;q)_n,\\ \theta(a_1,\ldots,a_k;p) =&\theta(a_1;p)\cdots\theta(a_k;p),\\ (a_1,\ldots,a_k;q,p)_n =&(a_1;q,p)_n\cdots(a_k;q,p)_n.\end{aligned}$$

We use the abbreviation

$$_{r+1}V_r(a_1;a_6,\ldots,a_{r+1};q,p) = \sum_{i=0}^{\infty}\frac{\theta(a_1q^{2i};p)}{\theta(a_1;p)}\frac{(a_1,a_6,\ldots,a_{r+1};q,p)_i}{(q,a_1q/a_6,\ldots,a_1q/a_{r+1};q,p)_i}q^i. \tag{1.1}$$

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One must impose that one of the parameters a_k is of the form q^{-n} such that the series (1.1) converges.

The following identities appeared in [5] are often used in this paper

$$\begin{aligned}\theta(a;p) &= -a\theta(1/a;p) = -a\theta(ap;p),\\ \theta(a^2;p^2) &= \theta(a,-a;p),\\ (a^2;q^2,p^2)_n &= (a,-a;q,p)_n,\\ (a;q,p)_{2n} &= (a,aq;q^2,p)_n,\\ (a;q,p)_{n-k} &= \frac{(a;q,p)_n}{(q^{1-n/a};q,p)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}.\end{aligned} \tag{1.2}$$

From a century and a half ago, till the present day, there has been much interest in finding many q-analogues of classical results in the theory of hypergeometric series. It is natural to find elliptic analogues of those classical results. This active research field has aroused much interest in the world of mathematics and theoretical physics, see [2–4, 10–16]. Frenkel and Turaev [4] found the following elliptic analogue of Jackson's ${}_8\phi_7$ summation formula in their study of elliptic $6j$-symbols

$${}_{10}V_9(a;b,c,d,a^2q^{n+1}/bcd,q^{-n};q,p) = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n}. \tag{1.3}$$

Schlosser [12] gave a combinatorial proof of (1.3). By Abel's method, Chu and Jia [2] proved and discovered some formulas for elliptic hypergeometric series. Warnaar [14] used matrix inversion and determinant evaluation techniques to prove several summation and transformation formulas for terminating, balanced, very-well-poised, elliptic hypergeometric series. Later, he [16] also proved the following summation formulas

$${}_{12}V_{11}(ab;b,bq,b/p,bpq,aq^2/b,a^2q^{2n},q^{-2n};q^2,p^2) = \frac{\theta(a;p)(-q,aq/b;q,p)_n(abq^2;q^2,p^2)_n}{\theta(aq^{2n};p)(a,-b;q,p)_n(a/b;q^2,p^2)_n}q^{-n}, \tag{1.4}$$

$${}_{12}V_{11}(ab;b,-b,bp,-b/p,aq/b,a^2q^{n+1},q^{-n};q,p^2)d = \chi(n\text{ is even})\frac{(q,a^2q^2/b^2;q^2,p^2)_{n/2}(abq;q,p^2)_n}{(a^2q^2,b^2q;q^2,p^2)_{n/2}(aq/b;q,p^2)_n}, \tag{1.5}$$

where the function χ is defined by

$$\chi(x) = \begin{cases} 1, & \text{if } x \text{ is true},\\ 0, & \text{otherwise}.\end{cases} \tag{1.6}$$

and

$${}_{12}V_{11}(b;-b,bp,-b/p,c/b,bq/c,q^{n+1},q^{-n};q,p^2) = \frac{(bq,c/b^2;q,p^2)_n(cq^{-n};q^2,p^2)_n}{(q/b,c;q,p^2)_n(cq^{-n}/b^2;q^2,p^2)_n}\left(-\frac{1}{b}\right)^n. \tag{1.7}$$

2 An inverse pair

Matrix inversion is a powerful technique for derivation identities [1, 6–9]. We say that f and f^{-1} is an inverse pair if f and f^{-1} are two infinite-dimensional, lower-triangular matrices and satisfying

$$\sum_{k=l}^{n} f^{-1}_{n,k} f_{k,l} = \delta_{n,l}. \tag{2.1}$$

Matrix inverse technique states that if (2.1) holds, then the following two statements are equivalent

$$\sum_{k=0}^{n} f_{n,k} a_k = b_n \tag{2.2}$$

and

$$\sum_{k=0}^{n} f_{n,k}^{-1} b_k = a_n. \tag{2.3}$$

Obviously, in order to prove an identity, we should select suitable explicit inverse pair of infinite-dimensional, lower-triangular matrices $f_{n,k}$, $f_{n,k}^{-1}$ satisfying (2.1) and another known identity. Warnaar [14] used this technique to obtain some summation and transformation formulas for elliptic hypergeometric series. He also proved the following elliptic analogue of a results due to Krattenthaler [7]

Lemma 2.1 (Warnaar [14], Lemma 3.2) *Let a and b_i, c_i $(i \in \mathbb{Z})$ be indeterminates (such that $c_i \neq c_j$ for $i \neq j$ and $ac_ic_j \neq 1$ for $i, j \in \mathbb{Z}$). Then*

$$\sum_{k=l}^{n} f_{n,k}^{-1} f_{k,l} = \delta_{n,l},$$

where

$$f_{n,k} = \frac{\prod_{j=k}^{n-1} \theta(c_k b_j; p)\theta(ac_k/b_j; p)}{\prod_{j=k+1}^{n} c_j \theta(ac_k c_j; p)\theta(c_k/c_j; p)} \tag{2.4}$$

and

$$f_{n,k}^{-1} = \frac{\theta(c_k b_k, ac_k/b_k; p)}{\theta(c_n b_n, ac_n/b_n; p)} \frac{\prod_{j=k+1}^{n} \theta(c_n b_j, ac_n/b_j; p)}{\prod_{j=k}^{n-1} c_j \theta(ac_n c_j, c_n/c_j; p)}. \tag{2.5}$$

In this paper, we also use the matrix inversion technique to prove some summation formulas for elliptic hypergeometric series. For (2.4) and (2.5), we set $b_i = bq^i$ and $c_i = cq^i$ for all i. Now we compute the products

$$\prod_{j=k}^{n-1} \theta(c_k b_j; p) = \prod_{j=k}^{n-1} \theta(bcq^{k+j}; p) = \frac{(bc; q, p)_{n+k}}{(bc; q, p)_{2k}}$$

and

$$\frac{\prod_{j=k}^{n-1} \theta(ac_k/b_j; p)}{\prod_{j=k+1}^{n} \theta(c_k/c_j; p)} = \frac{\prod_{j=k}^{n-1} \theta(acq^{k-j}/b; p)}{\prod_{j=k+1}^{n} \theta(q^{k-j}; p)} = \frac{\prod_{j=k}^{n-1}(-acq^{k-j}/b)\theta(bq^{j-k}/ac; p)}{\prod_{j=k+1}^{n}(-q^{k-j})\theta(q^{j-k}; p)}$$

$$= \frac{(b/ac; q, p)_{n-k}}{(q; q, p)_{n-k}} \left(\frac{ac}{b}\right)^{n-k} = \frac{(b/ac; q, p)_n (q^{-n}; q, p)_k}{(q; q, p)_n (acq^{1-n}/b; q, p)_k} \left(\frac{ac}{b}\right)^{n} q^k.$$

Similarly, we can compute other products appearing in (2.4) and (2.5), after simplification, we obtain the following inverse pair which is different from that of Warnaar derived from Lemma 2.1 in [14].

Lemma 2.2 *Let*

$$f_{n,k} = \frac{(bc, b/ac; q, p)_n}{(q, ac^2q; q, p)_n} \left(\frac{a}{b}\right)^n q^{-\binom{n}{2}} \frac{(ac^2q; q, p)_{2k}(bcq^n, q^{-n}; q, p)_k}{(bc; q, p)_{2k}(acq^{1-n}/b, ac^2q^{n+1}; q, p)_k} c^k q^{\binom{k+1}{2}} \tag{2.6}$$

and

$$f_{n,k}^{-1} = c^{-n} q^{-\binom{n}{2}} \frac{\theta(bcq^{2k}; p)(bcq^{n+1}, ac/b; q, p)_n}{\theta(bcq^{2n}; p)(q, ac^2q^n; q, p)_n} \frac{(ac^2q^n, q^{-n}; q, p)_k}{(bcq^{n+1}, bq^{1-n}/ac; q, p)_k} \left(\frac{bq}{a}\right)^k q^{\binom{k}{2}}. \tag{2.7}$$

Then $f_{n,k}$ and $f_{n,k}^{-1}$ are an inverse pair.

Remark. For (2.4) and (2.5), substituting $a \to ab$, $b_i \to aq^i$ and $c_i \to q^{ri}$, Warnaar [14] derived the following inverse pair

$$f_{n,k} = \frac{\theta(abq^{2rk};p)}{\theta(ab;p)} \frac{(aq^n;q,p)_{rk}}{(bq^{1-n};q,p)_{rk}} \frac{(ab,q^{-rn};q^r,p)_k}{(q^r,abq^{rn+r};q^r,p)_k} q^{rk}$$

and

$$f^{-1}_{n,k} = \frac{(b;q,p)_{rn}}{(aq;q,p)_{rn}} \frac{\theta\left(aq^{(r+1)k},bq^{(r-1)k};p\right)}{\theta(a,b;p)} \frac{(a,1/b;q,p)_k}{(q^r,abq^r;q^r,p)_k} \frac{(abq^{rn},q^{-rn};q^r,p)_k}{(q^{1-rn}/b,aq^{rn+1};q,p)_k} q^k.$$

3 Some summation formulas

In this section, the above inverse pair (2.6) and (2.7) will be repeatedly used to derive some summation formulas for elliptic hypergeometric series. The first formula derived by us can be stated as follows.

Theorem 3.1 *Suppose none of the denominators vanish, then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\theta(aq^{4k};p)(a,b^2;q^2,p)_k(aq^n/b,q^{-n};q,p)_{2k}}{\theta(a;p)(q^2,aq^2/b^2;q^2,p)_k(bq^{1-n},aq^{n+1};q,p)_{2k}} q^{2k}$$
$$= \frac{(aq/b^2,apq/b^2;q^2,p^2)_n(aq,b;q,p)_n}{(aq,apq;q^2,p^2)_n(aq/b^2,1/b;q,p)_n} \left(-\frac{1}{b}\right)^n. \quad (3.1)$$

Proof. Let the inverse pair $f_{n,k}$ and $f^{-1}_{n,k}$ be defined by (2.6) and (2.7), respectively. It follows from (1.5) that (2.2) holds for

$$a_n = \frac{(ac\sqrt{cq}/\sqrt{b},-ac\sqrt{cq}/\sqrt{b},ac\sqrt{cpq}/\sqrt{b};q,p)_n}{(q,\sqrt{bcq},-\sqrt{bcq},\sqrt{bcq}/\sqrt{p};q,p)_n}$$
$$\times \frac{(-ac\sqrt{cq}/\sqrt{bp},b/ac;q,p)_n(bc;q,p)_{2n}}{(-\sqrt{bcpq},a^2c^3q/b;q,p)_n(ac^2q^n;q,p)_n} c^{-n} q^{-\binom{n}{2}}$$

and

$$b_n = \chi(n \text{ is enen}) \frac{(q,b^2/a^2c^2;q^2,p)_{n/2}(bc;q,p)_n}{(bcq,a^2c^3q^2/b;q^2,p)_{n/2}(q;q,p)_n} \left(\frac{a}{b}\right)^n q^{-\binom{n}{2}},$$

where the function χ is defined by (1.6). This implies that the identity (2.3) holds for the above values a_n and b_n, i.e.,

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\theta(bcq^{4k};p)(bc,b^2/a^2c^2;q^2,p)_k(ac^2q^n,q^{-n};q,p)_{2k}}{\theta(bc;p)(q^2,a^2q^2c^3/b;q^2,p)_k(bcq^{n+1},bq^{1-n}/ac;q,p)_{2k}} q^{2k}$$
$$= \frac{(a^2c^3q/b,a^2c^3pq/b;q^2,p^2)_n(bcq,b/ac;q,p)_n}{(bcq,bcpq;q^2,p^2)_n(a^2c^3q/b,ac/b;q,p)_n} \left(-\frac{ac}{b}\right)^n.$$

Here, the identities (1.2) are used to simplify. Making the simultaneous changes $bc \to a$ and $b^2/a^2c^2 \to b^2$ yields (3.1). ∎

In theorem 3.1, let $p \to 0$, we obtain the following corollary

Corollary 3.1 *Suppose none of the denominators vanish, then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(1-aq^{4k})(a,b^2;q^2)_k(aq^n/b,q^{-n};q)_{2k}}{(1-a)(q^2,aq^2/b^2;q^2)_k(bq^{1-n},aq^{n+1};q,p)_k} q^{2k} = \frac{(aq/b^2;q^2)_n(aq,b;q)_n}{(aq;q^2)_n(aq/b^2,1/b;q)_n} \left(\frac{1}{b}\right)^n. \quad (3.2)$$

We also discover the following elliptic hypergeometric summation formula.

Theorem 3.2 *Suppose none of the denominators vanish, then*

$$\sum_{k=0}^{n}\frac{\theta(q^{2k+1};p)(a^2,bq^n/a,q^{-n};q,p)_k(b^2q^{-k};q^2,p)_k}{\theta(q;p)(b^2,aq^{2-n}/b,q^{n+2};q,p)_k(a^2q^{-k};q^2,p)_k}\left(-\frac{aq}{b}\right)^k$$

$$=\frac{(-b/a,b\sqrt{p}/a,-b/a\sqrt{p},ab,q/ab,q^2;q,p)_n}{(-q,q/\sqrt{p},-q\sqrt{p},q/a^2,b^2,b/aq;q,p)_n}. \quad (3.3)$$

Proof. We consider a special case of the inverse pair $f_{n,k}$ and $f_{n,k}^{-1}$ which are defined by (2.6) and (2.7), respectively. Let $c=q/b$ in (2.6) and (2.7), then

$$f_{n,k}=\frac{(b^2/aq;q,p)_n}{(aq^3/b^2;q,p)_n}\left(\frac{a}{b}\right)^n q^{-\binom{n}{2}}\frac{(aq^3/b^2;q,p)_{2k}(q^{n+1},q^{-n};q,p)_k}{(q;q,p)_{2k}(aq^{2-n}/b^2,aq^{n+3}/b^2;q,p)_k}\left(\frac{q}{b}\right)^k q^{\binom{k+1}{2}} \quad (3.4)$$

and

$$f_{n,k}^{-1}=\left(\frac{b}{q}\right)^n q^{-\binom{n}{2}}\frac{\theta(q^{2k+1};p)(q^{n+2},aq/b^2;q,p)_n(aq^{n+2}/b^2,q^{-n};q,p)_k}{\theta(q^{2n+1};p)(q,aq^{n+2}/b^2;q,p)_n(q^{n+2},b^2q^{-n}/a;q,p)_k}\left(\frac{bq}{a}\right)^k q^{\binom{k}{2}}. \quad (3.5)$$

In view of (1.7), we see that (2.2) holds for

$$a_n=\frac{(-aq^2/b^2,aq^2\sqrt{p}/b^2,-aq^2/b^2\sqrt{p},b^2c/aq^2,aq^3/b^2c,q^{n+1};q,p)_n}{(-q,q/\sqrt{p},-\sqrt{p}q,a^2q^5/b^4c,c,aq^{n+2}/b^2;q,p)_n}b^nq^{-\binom{n+1}{2}}$$

and

$$b_n=\frac{(cb^4/a^2q^4;q,p)_n(cq^{-n};q^2,p)_n}{(c;q,p)_n(cb^4q^{-4-n}/a^2;q^2,p)_n}\left(-\frac{b}{q^2}\right)^n q^{-\binom{n}{2}}.$$

This implies that the identity (2.3) holds for the above value a_n and b_n, where $f_{n,k}$ and $f_{n,k}^{-1}$ are defined by (3.4) and (3.5), respectively, that is,

$$\sum_{k=0}^{n}\frac{\theta(q^{2k+1};p)(cb^4/a^2q^4,aq^{n+2}/b^2,q^{-n};q,p)_k(cq^{-k};q^2,p)_k}{\theta(q;p)(c,q^{n+2},b^2q^{-n}/a;q,p)_k(cb^4q^{-4-k}/a^2;q^2,p)_k}\left(-\frac{b^2}{aq}\right)^k$$

$$=\frac{(-aq^2/b^2,aq^2\sqrt{p}/b^2,-aq^2/b^2\sqrt{p},b^2c/aq^2,aq^3/b^2c,q^2;q,p)_n}{(-q,q/\sqrt{p},-\sqrt{p}q,a^2q^5/b^4c,c,aq/b^2;q,p)_n}.$$

Making the simultaneous changes $cb^4/a^2q^4\to a^2$ and $c\to b^2$ yields (3.3). ∎

For the theorem 3.2, let $p\to 0$, we obtain the following corollary

Corollary 3.2 *Suppose none of the denominators vanish, then*

$$\sum_{k=0}^{n}\frac{(1-q^{2k+1})(a^2,bq^n/a,q^{-n};q)_k(b^2q^{-k};q^2)_k}{(1-q)(b^2,aq^{2-n}/b,q^{n+2};q)_k(a^2q^{-k};q^2)_k}\left(-\frac{aq}{b}\right)^k=\frac{(-b/a,ab,q/ab,q^2;q)_n}{(-q,q/a^2,b^2,b/aq;q)_n}\left(-\frac{b}{aq}\right)^n. \quad (3.6)$$

The following theorem was first discovered by Warnaar in [16], here we give another proof.

Theorem 3.3 *Suppose none of the denominators vanish, then the identity* (1.4) *holds.*

Proof. By the elliptic analogue of Jackson's ${}_8\phi_7$ sum (1.3), we have

$$\sum_{k=0}^{n}\frac{\theta(-bcq^{2k};p)(-bc,b^2q/ac^2;q,p)_k(ac^4q^{2n},q^{-2n};q^2,p^2)_k}{\theta(-bc;p)(q,-ac^3/b;q,p)_k(b^2c^2q^{2n+2},b^2q^{2-2n}/ac^2;q^2,p^2)_k}q^k$$

$$=\frac{(-bcq,\sqrt{a}cq^{-n}/b,-\sqrt{a}cq^{-n}/b,bq^{1-2n}/ac^3;q,p)_n}{(-ac^3/b,bq^{1-n}/\sqrt{a}c,-bq^{1-n}/\sqrt{a}c,q^{-2n};q,p)_n}. \quad (3.7)$$

It follows from (1.2) that

$$\frac{(ac^3/b, ac^3q/b, ac^3/bp, ac^3pq/b, b^2q^2/ac^2, b^2c^2q^2; q^2, p^2)_n}{(bcq^2, bcq, bcpq^2, bcq/p, a^2c^6/b^2, ac^2/b^2; q^2, p^2)_n}$$
$$= \frac{(-bcq, \sqrt{a}cq^{-n}/b, -\sqrt{a}cq^{-n}/b, bq^{1-2n}/ac^3; q, p)_n}{(-ac^3/b, bq^{1-n}/\sqrt{a}c, -bq^{1-n}/\sqrt{a}c, q^{-2n}/bc; q, p)_n}. \tag{3.8}$$

Combining (3.7) and (3.8), we see that

$$\sum_{k=0}^{n} \frac{\theta(-bcq^{2k}; p)(-bc, b^2q/ac^2; q, p)_k (ac^4q^{2n}, q^{-2n}; q^2, p^2)_k}{\theta(-bc; p)(q, -ac^3/b; q, p)_k (b^2c^2q^{2n+2}, b^2q^{2-2n}/ac^2; q^2, p^2)_k} q^k$$
$$= \frac{(ac^3/b, ac^3q/b, ac^3/bp, ac^3pq/b, b^2q^2/ac^2, b^2c^2q^2; q^2, p^2)_n}{(bcq^2, bcq, bcpq^2, bcq/p, a^2c^6/b^2, ac^2/b^2; q^2, p^2)_n}, \tag{3.9}$$

which yields that the identity (2.3) holds for

$$a_n = \frac{(ac\sqrt{c}/\sqrt{b}, ac\sqrt{cq}/\sqrt{b}, ac\sqrt{c}/\sqrt{bp}; q, p)_n}{(q, \sqrt{bcq}, \sqrt{bcq}, \sqrt{bcpq}; q, p)_n} \frac{(ac\sqrt{cpq}/\sqrt{b}, bq/ac; q, p)_n (bc; q, p)_{2n}}{(\sqrt{bcq/p}, a^2c^3/b; q, p)_n (ac^2q^n; q, p)_n} c^{-n} q^{-\binom{n}{2}}$$

and

$$b_n = \frac{\theta(\sqrt{bc}; \sqrt{p})(-\sqrt{bc}, b\sqrt{q}/ac; \sqrt{q}, \sqrt{p})_n}{\theta(\sqrt{bc}q^n; \sqrt{p})(\sqrt{q}, -ac\sqrt{c}/\sqrt{b}; \sqrt{q}, \sqrt{p})_n} \left(\frac{a}{b\sqrt{q}}\right)^n q^{-\binom{n}{2}},$$

where the inverse pair $f_{n,k}$ and $f_{n,k}^{-1}$ are given by (2.6) and (2.7), respectively. This implies (2.2) holds for the above values a_n and b_n, i.e.,

$$\sum_{k=0}^{n} \frac{\theta(acq^{2k}; p)(ac^2, ac\sqrt{c}/\sqrt{b}, ac\sqrt{cq}/\sqrt{b}, ac\sqrt{c}/\sqrt{bp}, ac\sqrt{cpq}/\sqrt{b}, bq/ac, bcq^n, q^{-n}; q, p)_k}{\theta(ac^2; p)(q, \sqrt{bcq}, \sqrt{bcq}, \sqrt{bcpq}, \sqrt{bcq/p}, a^2c^3/b, acq^{1-n}/b, ac^2q^{n+1}; q, p)_k} q^k$$
$$= \frac{\theta(\sqrt{bc}; \sqrt{p})(-\sqrt{q}, b\sqrt{q}/ac; \sqrt{q}, \sqrt{p})_n (ac^2q; q, p)_n}{\theta(\sqrt{bc}q^n; \sqrt{p})(\sqrt{bc}, -ac\sqrt{c}/\sqrt{b}; \sqrt{q}, \sqrt{p})_n (b/ac; q, p)_n}.$$

Making the simultaneous changes $bc \to a^2$, $ac\sqrt{c}/\sqrt{b} \to b$, $q \to q^2$ and $p \to p^2$ yields (1.4). ∎

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