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Multistep Methods for System of Nonlinear Volterra Integral Equations of the First Kind

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Abstract: This paper is concerned with deriving numerical methods based on multistep methods for solving systems of nonlinear Volterra integral equations (NVIEs) of the first kind. Here, the convergence and stability of the method is also investigated and finally the accuracy and efficiency of the method is verified by solving some examples.

Keywords: system of nonlinear Volterra integral equations; integral equations of the first kind; multistep methods.

1 Introduction

In general, the integral equations of the first kind are the ill-posed problems, it means that, a small perturbation in the right hand side of the equation causes a large perturbation in the solution [1]. But many of this equations can not be usually solved exactly and they should be solved by numerical methods. Therefore, giving suitable numerical methods for these equations is very worthwhile. Up to now, many works have been done about them. For example, the block-by-block methods have been developed in [2]. The numerical solution of these equations has been presented in [1] by taking the Chebyshev expansion for solution and using the Galerkin method. The Fredholm integral equations have been considered by regularised collocation method in [3]. In [4], two methods have been proposed, which are based on the Chebyshev interpolation of the solution and integrand term for the numerical solution of the VIEs of the first kind. A Galerkin method in [5] and an algorithm based on the regularization in [6] have been studied to solve the first kind Fredholm integral equations. To see other numerical methods for the integral equations of the first kind, we refer to [7,8]. On the other hand, the multistep methods have been developed to solve integral equations in some interesting works. For example, Holyhead, Mckee and Taylor in [9] and Holyhead, Mckee in [10] studied the multistep methods and their stability and convergence. Also, in [11] Andrade and Mckee discussed on the optimal accuracy linear multistep methods for the first kind VIEs.

In this paper, we consider the system of nonlinear Volterra integral equations (NVIEs) of the first kind of the form:

$$\int_0^t K_i(t,s)G_i(u_1(s),\dots,u_n(s))ds = f_i(t), \quad i=1,\dots,n,\ t\in[0,T], \tag{1}$$

where for $i=1,\dots,n$, G_i is a nonlinear and continuous function, f_i is a continuous function on $[0,T]$ and the kernel $k_i(t,s)$ is continuous on $[0,T]\times[0,T]$.

It is clear that for solvability of the equation (1), it is necessary that $f_i(0)=0$, $i=1,\dots,n$. For more details about the solvability of the Volterra integral equations of the first kind see [12].

We first set $g_i(s) := G_i(u_1(s),\dots,u_n(s))$, $i=1,\dots,n$, to convert the system (1) to a linear system of the form:

$$\int_0^t K_i(t,s)g_i(s)ds = f_i(t),\ i=1,\dots,n. \tag{2}$$

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To discuss the existence and uniqueness of the solution of (2), by differentiating of (2) with respect to t, we obtain

$$K_i(t,t)g_i(t) + \int_0^t \frac{\partial K_i(t,s)}{\partial t} g_i(s)ds = f_i'(t),\ i = 1, \ldots, n. \tag{3}$$

If $K_i(t,t) \neq 0$ for every $t \in [0,T]$, then (2) is equivalent to the Volterra integral equation of the second kind:

$$g_i(t) + \int_0^t \frac{\partial K_i(t,s)/\partial t}{K_i(t,t)} g_i(s)ds = \frac{f_i'(t)}{K_i(t,t)},\ i = 1, \ldots, n. \tag{4}$$

We know that this system has a unique solution under the suitable conditions, see [13,14]. Thus, the system (2) has a unique solution. For more study about existence and uniqueness of the solution of integral equations see [12].

2 Description of the method

In this section, we present the method of this paper. To this end, consider the ith equation of the system (2) as:

$$\int_0^t K(t,s)y(s)ds = f(t), \tag{5}$$

where the subscribe i is suppressed for simplicity.

For a given positive integer N, let $t_i = ih,\ i = 0, 1, \ldots, N$ with $h = T/N$. Setting $t = t_i$ in (5) implies

$$\int_0^{t_i} K(t_i,s)y(s)ds = f(t_i),\ i = 1, \ldots, n. \tag{6}$$

Now, we use a quadrature rule to approximate the integral part of this equation. At each step we determine the approximate value of y_i, and at the next step, we determine the next value of the unknown function, namely y_{i+1}, using previous values y_j, $j = 0, \ldots, i$. Note that, in the initial steps, choosing of the quadrature rules is restricted by the number of mesh points.

Assume that we are given r starting values which may be obtained from a multistep method or we know them.

So we define [9]:

$$[\varphi_h y]_i = \begin{cases} h(y_i - \widehat{y}_i), & i = 0, 1, \ldots, r-1 \\ \\ h\sum_{j=0}^{i} w_{ij}k_{ij}y_j - f_i, & i = r, \ldots, N. \end{cases} \tag{7}$$

where, for $i = 0, 1, \ldots, r-1$, $\widehat{y}_i$s are the starting values and for $i = r, \ldots, N$, f_i , k_{ij} denote $f(t_i)$, $k(t_i, t_j)$ respectively. Also w_{ij}s are the weights of the quadrature rules and for $i = 0, \ldots, N$, y_i denotes the approximate of $y(t_i)$.

In matrix notation we have

$$\varphi_h y = hU_h y - g$$

where

$$U_h = (U_{ij}) \quad , \quad U_{ij} = \begin{cases} \delta_{ij}, & 0 \leq i,j \leq r-1 \\ w_{ij}k_{ij}, & r \leq j \leq i \\ 0, & i < j \end{cases} \tag{8}$$

and

$$y = (y_0, \ldots, y_{r-1}, y_r, \ldots, y_N)^T,$$

$$g = (h\widehat{y}_0, \ldots, h\widehat{y}_{r-1}, f_r, \ldots, f_N)^T, \tag{9}$$

where in (8), $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$.

For example, for $N = 6,\ r = 2$ and Simpson's rule as the main repeated rule, 3/8th rule as the end formula and for $k \equiv 1$ the method has the following representation:

$$h \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1/3 & 4/3 & 1/3 & & & & \\ 3/8 & 9/8 & 9/8 & 3/8 & & & \\ 1/3 & 4/3 & 2/3 & 4/3 & 1/3 & & \\ 1/3 & 4/3 & 1/3+3/8 & 9/8 & 9/8 & 3/8 & \\ 1/3 & 4/3 & 2/3 & 4/3 & 2/3 & 4/3 & 1/3 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} h\widehat{y}_0 \\ h\widehat{y}_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

As we mentioned previously, we require some starting values. Here we give a method to obtain the exact first starting value, for this purpose, differentiating of equation (5) yields:

$$y(t) + \int_0^t \left(\frac{\partial k}{\partial t}(t,s)/k(t,t)\right)ds = \frac{f'(t)}{k(t,t)} \tag{10}$$

and by assuming $k(t,t) \neq 0$ and by setting $t = 0$ in (10):

$$\widehat{y}_0 = y(0) = \frac{f'(0)}{k(0,0)} \tag{11}$$

The next starting values can be obtained by some quadrature rules, for example, Trapezoidal rule, Simpson's rule and etc from the main integral equation.

The following algorithm can be given to the presented method.

Algorithm of the method:

Step 1: For a given positive integer N, set $h = T/N$ and $t_i = ih$ for $i = 0, \ldots, N$.
Step 2: Select main repeated formula and end formula(e) and form the matrix U_h from (8).
Step 3: Obtain $y_0 = \widehat{y}_0$ exactly from (11).
Step 4: Obtain the approximate starting values $\widehat{y}_1, \ldots, \widehat{y}_{r-1}$ by a suitable method (of order p).
Step 5: Form the vector g from (9).
Step 6: Solve the lower triangular system $hU_h y = g$.

3 Convergence and stability analysis

In this section, we present the convergence and stability analysis of the method. To this end, we introduce the operator Δ_h as:

$$\Delta_h : C[0,T] \to R^{N+1}$$
$$\Delta_h f(t) = [f(0), f(h), \ldots, f(T)]^T.$$

We also need to some definitions as following.

Definition 1 *[11] The discretisation φ_h is said to be consistent of order $p \geq 1$, if for $i = r, r+1, \ldots, N$, there exists a constant C, independent of h, such that*

$$\left| [hU_h\Delta_h y(t)]_i - \int_0^{t_i} K(t_i,s)y(s)ds \right| \leq Ch^p. \tag{12}$$

It is clear that, if we choose main repeated formulae of local order $p+1$ and end formulae of local order at least p, then the method will be consistent of order p. Furthermore, it is obvious that consistency is independent of the kernel of the integral if the kernel is sufficiently smooth.

Definition 2 *[11] The discretisation φ_h is said to be stable if there exists a constant C, independent of h, such that*

$$\| U_h^{-1} \|_\infty < C. \tag{13}$$

Thus, according to this definition, to show the stability of the presented method it is sufficient that the coefficient matrix U_h has the bounded inverse under the infinity norm.

Definition 3 *[11] The discretisation φ_h is said to be convergent of order p, if for any set of starting values satisfying $\mid \widehat{y}_j - y(t_j) \mid < C_1 h^p$, $j = 0, 1, \ldots, r-1$ we have*

$$\parallel y - \Delta_h y(t) \parallel_\infty < Ch^p, \tag{14}$$

where C_1 and C are constants, independent of h.

The proof of following theorem can be found in [9].

Theorem 1 *If the discretisation φ_h has starting methods convergent of order p, which is stable and consistent of order p, then it is convergent of order $p-1$.*

In this paper, we apply the 3/8th-rule as the main repeated formula which is of order $p = 4$, and the methods 4-step and 5-step as end formulae which are of order at least $p = 4$. We also apply the fourth order runge-kutta method as starting formula which has $p = 4$. On the other hand, we know, from [9], for these choices of formulae the method is stable. Therefore, by theorem 1 the presented method is of order $p - 1 = 3$.

4 Numerical examples

In this section, we give some numerical examples to clarify the efficiency of the presented method.

As mentioned previously, we shall use the 3/8th-rule as the main repeated formula combined with a 4-step and a 5-step end formula over even and odd subintervals, respectively.

We show the coefficients of 4-step and 5-step rules with α_i and β_i, respectively. Choosing the α_i and β_i such that the quadrature rules are exact for polynomials of degree three, we will have:

$$\begin{cases} \alpha_0 = 0.39740084, \\ \alpha_1 = 1.07706330, \\ \alpha_2 = 1.05107170, \\ \alpha_3 = 1.07706330, \\ \alpha_4 = 0.39740084. \end{cases} \quad and \quad \begin{cases} \beta_0 = 0.43284615, \\ \beta_1 = 1.02180185, \\ \beta_2 = 0.95933115, \\ \beta_3 = 1.24606743, \\ \beta_4 = 0.87843368, \\ \beta_5 = 0.46151978. \end{cases}$$

Example 1. As the first example, consider the system of nonlinear VIEs:

$$\begin{cases} \int_0^t cos(t-s)u(s)v(s)ds = sin(t) \\ \\ \int_0^t (s^3+1)(u(s)+v(s))^3 ds = 2t^4 + 8t, \end{cases}$$

on $[0, 1]$, with exact solutions $u(t) = v(t) = 1$.

First, we set $g_1(s) = u(s)v(s)$ and $g_2(s) = (u(s) + v(s))^3$. Then by solving the linearised integral equations by the presented method, we obtain the values of $g_i(s), i = 1, 2$ at mesh points. Finally, by solving the nonlinear algebraic system

$$\begin{cases} u(s_j)v(s_j) = g_1(s_j) \\ \\ (u(s_j) + v(s_j))^3 = g_2(s_j), \end{cases}$$

for $j = 1, \ldots, n$, by Newton method we determine the approximate values of unknown functions at mesh points.

In this example, we use three exact starting values, that is $u(t_j) = v(t_j) = 1$ for $j = 0, 1, 2$.

The numerical results are reported in Table 1, which includes the errors $e_j = |u(t_j) - u_j|$ and $\varepsilon_j = |v(t_j) - v_j|$ for some mesh points t_j.

Table 1: Computational results of Example 4.1 for $n = 20$ at some nodes.

t	e_j	ε_j
0.2	3.259e-10	3.259e-10
0.4	1.910e-8	1.910e-8
0.6	1.386e-8	1.386e-8
1.0	1.966e-8	1.966e-8

Example 2. Consider the system of nonlinear VIEs:

$$\begin{cases} \int_0^t (3u^2(s) + v^2(s))ds = t^3 + t \\ \\ \int_0^t exp(t-s)(u(s)v(s))ds = exp(t) - t - 1 \end{cases}$$

on $[0, 1]$, with exact solutions $u(t) = t, v(t) = 1$.

The Table 2 shows the approximate values at some mesh points, with three exact starting values.

Table 2: Computational results of Example 4.2 for $n = 20$ at some nodes.

t	e_j	ε_j
0.2	1.913e-6	1.145e-6
0.4	6.770e-5	8.116e-5
0.6	2.923e-4	5.265e-4
1.0	1.564e-5	4.697e-5

Example 3. This example is proposed in [15]:

$$\begin{cases} \int_0^t (1 - t^2 + s^2)(u(s) + v^3(s))ds = -1/12t^6 - 2/15t^5 + 1/4t^4 + 1/3t^3 \\ \\ \int_0^t (5 + t - s)(u^3(s) - v(s))ds = -5/2t^2 - 1/6t^3 + 5/7t^7 + 1/56t^8 \end{cases}$$

on $[0, 1]$, with exact solutions $u(t) = t^2, v(t) = t$.

We proceed similar to previous examples and obtain the numerical results reported in Tables 3 and 4.

Example 4. As the last example, consider the system of nonlinear VIEs:

$$\begin{cases} \int_0^t 2tsin(2s)(u^2(s) - v^2(s))ds = t(1 - cos4t)/4 \\ \\ \int_0^t 2(t + s)u(s)v(s)ds = t/2 + (sin2t)/4 - tcos2t \end{cases}$$

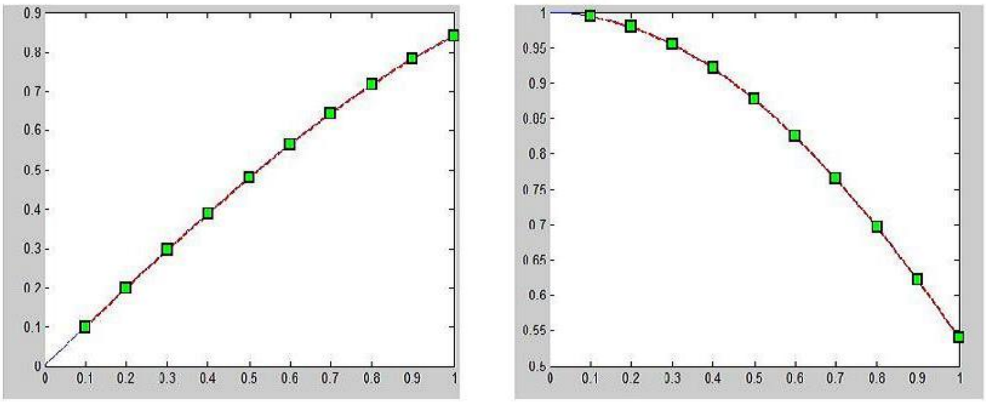
on $[0, 1]$, with exact solutions $u(t) = cost, v(t) = sint$.

Table 3: Computational results of Example 4.3 for $n = 20$ at some nodes.

t	e_j	ε_j
0.2	5.87e-6	1.625e-6
0.4	2.11e-4	1.169e-6
0.6	5.33e-4	1.359e-4
1.0	3.65e-4	2.633e-5

Table 4: Computational results of Example 4.3 for $n = 80$ at some nodes.

t	e_j	ε_j
0.2	1.81e-6	3.23e-6
0.4	2.05e-6	8.14e-7
0.6	2.96e-6	9.58e-7
1.0	2.26e-6	3.15e-7



(a) (b)

Figure 1: Numerical results of Example 4.4: $v(t) = sint$ (a) and $u(t) = cost$ (b).

We show the numerical results of this example in the Figure 1 at some nodes.

The above results show the efficiency of the presented method. Also, by comparing two last tables we see that we can obtain higher accuracy of the solution by increasing number of the mesh points.

5 Conclusion

In this paper, we extended the multistep methods to a class of nonlinear systems of the Volterra integral equations of the first kind. We converted the system of NVIE of the first kind to LVIE of the first kind and then by using multistep methods we solved the resulted system. Comparison with available papers shows that the presented method gives results with high accuracy. We hope that the suggested method in this paper can be applied to the two-dimensional VIEs of the first kind.

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