ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.22(2016),No.2,pp.67-72

Pricing of an Option with the Aid of Lie Symmetry Methods

Anupma Bansal[1] *, Meenakshi Mittal[2]
[1] Department of Mathematics, D.A.V College for Women, Ferozepur Cantt-152001 (Punjab), India
[2] Department of Commerce, D.A.V College for Women, Ferozepur Cantt-152001 (Punjab), India
(Received 8 March 2015, accepted 15 August 2016)

Abstract: The pricing of options is a central problem in financial investment. It is of both theoretical and practical importance since the use of options thrives in the financial market. In option pricing theory, one of the most effective models for pricing options is the Black-Scholes model. In this direction, this manuscript is an attempt to apply Lie group symmetries to Black-Scholes equation which helps us to reduce the studied equation to ordinary differential equations (ODE) in various cases. We have obtained the solutions in each case and in addition to this, we have applied our results to Indian Treasury Bills of March 2013-Feb 2014.

Keywords: financial engineering; lie Group method; black-Scholes equation

1 Introduction

Financial mathematical models are always remain a challenge for researchers because of their complexity and due to the fact that they cannot be reduced to completely solvable equations. Similarily, with the study of other dynamical processes, the evolution of the value of financial derivatives, stock price patterns, critical crashes etc. are strongly nonlinear and present random behaviour. Financial economics, however, only came of age in 1973 [1] with the publication of the preference-free option pricing formula by Fischer Black, Myron Scholes and Robert Merton. Their following model established the everyday use of mathematical models as essential tools in the world of finance, both in the classroom and on the trading floor:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \tag{1}$$

where r is the risk free rate, σ is the volatility of returns of the underlying asset, t is the time in years and x is the price of stock. The key insight behind the equation is that one can perfectly hedge the option by buying and selling the underlying asset in just the right way.

Over the last few years, there has been considerable development in the mathematical analysis of partial differential equations which arise in Finance. Finding special solutions of such equations and investigating the corresponding properties of solutions are very important in both practice and theory for understanding such problems. One of the most general technique to find such solutions is to apply Lie symmetry methods [2-5] on such equations which have applications in almost all branches of theoretical physics, chemistry, engineering, economics etc. One reason for the overall prominence of the concept of symmetry is its nativeness and its simplicity. Intutively speaking, a symmetry is a transformation of an object leaving this object invariant. This is clearly such a general property that it can be recovered almost everywhere in nature and correspondingly, in numerous areas of science and art.

Till now, many scholars have worked on this model for its mathematical and financial perspectives. In [6], D. Silverman showed explicity how to obtain solutions of Black-Scholes (BS) equation for call option pricing using methods available to mathematics, physics and engineering students i.e.Green's function method for the diffusion equation. N. Sukhomlin and J. Ortiz [7] constructed a class of new solutions for both the Black-Scholes and the Diffusion Equations using similarities between them. Moreover, they constructed two important new solutions: one that generalizes both terms of the Black-Scholes classical solution and another with paradoxical properties. By using the Adomian approximate decomposition technique, the analytical solutions of the Black-Scholes equation are obtained in [8]. Dimas et al. [9]

* **Corresponding author**. *E-mail address*: anupma2512@yahoo.co.in

IJNS.2016.10.15/924

considered some well-known partial differential equations that arise in Financial Mathematics, namely the Black-Scholes-Merton, Longstaff, Vasicek, Cox-Ingersoll-Ross and Heath equations to discover any underlying connections taking into account the Lie remarkability property of the heat equation. In this paper, we will study the exact solutions of BS equation with the aid of symmetry group. Dimas et al. [9] have studied the equation (1) by using the Lie properties of heat equation. But, here we will discuss the reductions and solutions of BS equation in certain cases and will try to analyse our obtained solutions by considering Indian Treasury Bills of March 2013-Feb 2014.

2 Lie symmetries and group invariant solutions

In order to apply the classical method to the BS equation (1) with two independent variables and one field, we consider the following one parameter Lie group of infinitesimal tansformations in x, t, u:

$$\begin{aligned} u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \end{aligned} \tag{2}$$

where ϵ is the group parameter. It is therefore, necessary that this one transformation leaves the set of solutions of (2) invariant. This yields an overdetermined linear system of equations for the infinitesimals $\eta(x,t,u), \xi(x,t,u), \tau(x,t,u)$, whose obtained solution [9] is as follows:

$$\begin{aligned} &\eta = C_3 u + \tfrac{C_4}{\kappa+\lambda}(\lambda^2 t + 2\kappa \log x)u + \tfrac{C_5}{\kappa+\lambda}(2\log x + \kappa t)u + f(t,x) \\ &\tfrac{C_6}{\kappa+\lambda}\left((2\log x + \kappa t)^2 + \lambda t(\lambda t - 2) - \kappa t(\kappa t + 2)\right)u, \\ &\xi = C_2 x + 2C_4 x \log x + C_5 t x + 4C_6 t^2 x \log x, \\ &\tau = C_1 + 4C_4 t + 4C_6 t^2, \end{aligned} \tag{3}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are arbitrary constants and $f(t,x)$ is any solution of (1) and $(\kappa, \lambda) = (\sigma^2 - 2r, \sigma^2 + 2r)$. We have considered here in this paper, the two cases for symmetry reductions and exact solutions and the remaining can be considered for the future references.

2.1 Case 1: $C_2 = 0, C_4 = 0, C_5 = 0, C_6 = 0, f(t,x) = 0$

In this subsection, we will make use of symmetries

$$\eta = C_3 u, \;\; \xi = 0, \;\; \tau = C_1, \tag{4}$$

to reduce the equation (1) to ODE. On using the charactersitc equation

$$\frac{du}{\eta} = \frac{dx}{\xi} = \frac{dt}{\tau}, \tag{5}$$

the similarity variables and similarity solutions are given as

$$u(x,t) = e^t F(\zeta), \quad \zeta = x, \tag{6}$$

which reduces the BS equation to the following ODE:

$$F(\zeta) + \frac{\sigma^2}{2}\xi^2 F''(\zeta) + r\xi F'(\zeta) - rF(\zeta) = 0. \tag{7}$$

On solving the reduced ODE, we obtain the following general solution of BS equation:

$$u(x,t) = e^t \left(C_1\, x^{1/2\, \frac{-2\,r+\sigma^2+\sqrt{4\,r^2+4\,r\sigma^2+\sigma^4-8\,\sigma^2}}{\sigma^2}} + C_2\, x^{-1/2\, \frac{2\,r-\sigma^2+\sqrt{4\,r^2+4\,r\sigma^2+\sigma^4-8\,\sigma^2}}{\sigma^2}} \right), \tag{8}$$

where C_1, C_2 are arbitrary constants. The evolution of solution (8) with $C_1 = 2, C_2 = 3, r = 2$ has been shown in Fig. 3 for $t = 1$ (thick line), $t = 2$ (thin line), $t = 3$ (dotted line).

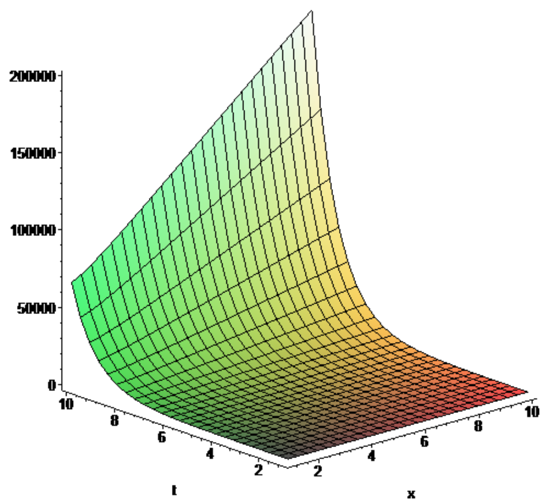

Figure 1: The figure of solution (8) for $C_1 = 2, C_2 = 2, r = 3, t = 4$

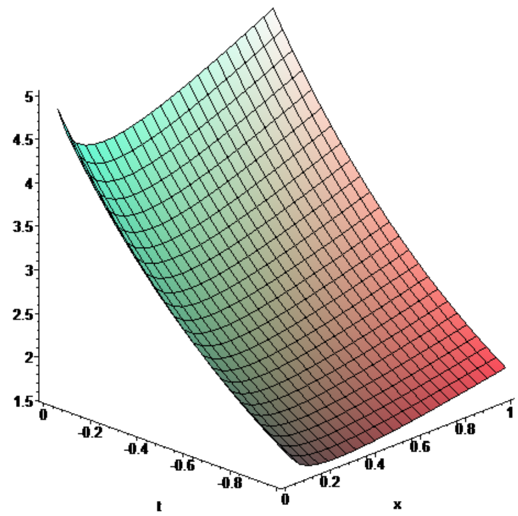

Figure 2: The figure of solution (8) for $C_1 = 2, C_2 = 3, r = 2, t = 4$

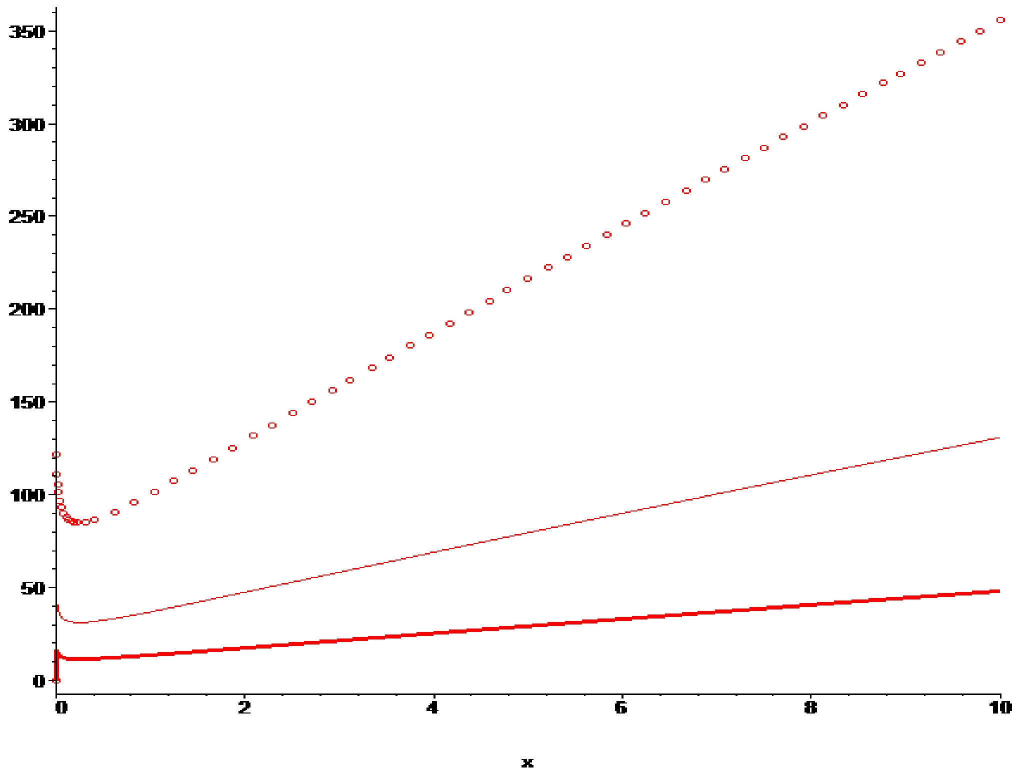

Figure 3: The Black-Scholes solution for varying asset price x and time $t = 1, 2, 3$

2.2 Case 2: $C_2 \neq 0, C_4 = 0, C_5 = 0, C_6 = 0, f(t, x) = 0$

Here, our symmetries (3) will take form as under:

$$\eta = C_3 u, \;\; \xi = C_2 x, \;\; \tau = C_1. \tag{9}$$

For this case, on solving the equations (5) and (9), we get

$$u(x,t) = e^{\frac{C_3}{C_1} t} F(\zeta), \quad \zeta = \log x - \frac{C_2}{C_1} t. \tag{10}$$

The reduced ODE in this case will be

$$(C_3 - C_1 r)F(\zeta) + (C_1 r - C_2)F'(\zeta) + \frac{\sigma^2}{2} C_1 (F''(\zeta) - F'(\zeta)) = 0. \tag{11}$$

The BS equation possesses the following solution corresponding to above reduced ODE:

$$\begin{aligned} &u(x,t) = \\ &e^{\frac{C_3 t}{C_1}} \left(C_1 e^{-1/2 \frac{\left(2 C_1 r - 2 C_2 - \sigma^2 C_1 - \sqrt{4 {C_1}^2 r^2 - 8 C_1 r C_2 + 4 {C_1}^2 r \sigma^2 + 4 {C_2}^2 + 4 C_2 \sigma^2 C_1 + \sigma^4 {C_1}^2 - 8 \sigma^2 C_1 C_3}\right)(\log(x) C_1 - C_2 t)}{\sigma^2 {C_1}^2}} \right) \\ &+ e^{\frac{C_3 t}{C_1}} \left(C_2 e^{-1/2 \frac{\left(2 C_1 r - 2 C_2 - \sigma^2 C_1 + \sqrt{4 {C_1}^2 r^2 - 8 C_1 r C_2 + 4 {C_1}^2 r \sigma^2 + 4 {C_2}^2 + 4 C_2 \sigma^2 C_1 + \sigma^4 {C_1}^2 - 8 \sigma^2 C_1 C_3}\right)(\log(x) C_1 - C_2 t)}{\sigma^2 {C_1}^2}} \right) \end{aligned} \tag{12}$$

3 Application of black-scholes solution to Indian treasury bills

Any solution to some equation becomes more effective when it is applicable to certain phenomenon. In this manuscript, our main aim is not only to find solutions of BS equation but also to apply those solutions to financial market. We have tried to apply our obtained results to Indian Treasury Bills of Mar 2013-Feb 2014.

Treasury bills are instrument of short-term borrowing by the Government of India, issued as promissory notes under discount. The interest received on them is the discount which is the difference between the price at which they are issued and their redemption value. They have assured yield and negligible risk of default. Under one classification, treasury bills are categorised as ad hoc, tap and auction bills and under another classification it is classified on the maturity period like 91-days TBs, 182-days TBs, 364-days TBs and two types of 14-days TBs. In the recent times, the Reserve Bank of India has been issuing only 91-day and 364-day treasury bills. The auction format of 91-day treasury bill has changed from uniform price to multiple price to encourage more responsible bidding from the market players. The bills are of two kinds-Adhoc and Regular. The Adhoc bills are issued for investment by the state governments, semi government departments and foreign central banks for temporary investment. They are not sold to banks and general public. The treasury bills sold to the public and banks are called regular treasury bills. They are freely marketable. Here, we have considered the case of interbank rate which is the rate of interest charged on short-term loans made between banks. Interbank Rate in India is increased to 9.15 percent in February of 2014 from 8.90 percent in January of 2014. Interbank Rate in India averaged 7.55 Percent from 1993 until 2014, reaching an all time high of 12.97 Percent in October of 1995 and a record low of 3.10 Percent in June of 2009. The following table provides the actual values of Indian Treasury Bill Yield from Mar 2013-Feb 2014:

Table 1: Indian treasury bill yield

$Mar.$	$Apr.$	$May.$	$Jun.$	$Jul.$	$Aug.$	$Sep.$	$Oct.$	$Nov.$	$Dec.$	$Jan.$	$Feb.$
8.19	7.56	7.31	7.48	11.26	12.02	9.57	8.6	8.94	8.73	8.9	9.15

Treasury Bills, known as gilt edged security is basically a riskless security in Indian money market. As these bills are issued by government of our economy, these are generally safe. Applying this concept to BS equation, we have considered variable x i.e. price of stock on horizontal axis with r, rate of return to be fixed. In Figures 4,5,6,7, we have plotted graphs of Treasury yield of months Mar 13-May 13, Jun 13-Aug 13, Sep 13-Nov 13, Dec 13-Feb 14. Technically, the Treasury yield curve can change in various ways: it can move up or down (a parallel shift), become flatter or steeper (a shift in slope), or become more or less humped (a change in curvature). The brief explanation of figures 4,5,6,7 is given as follows:

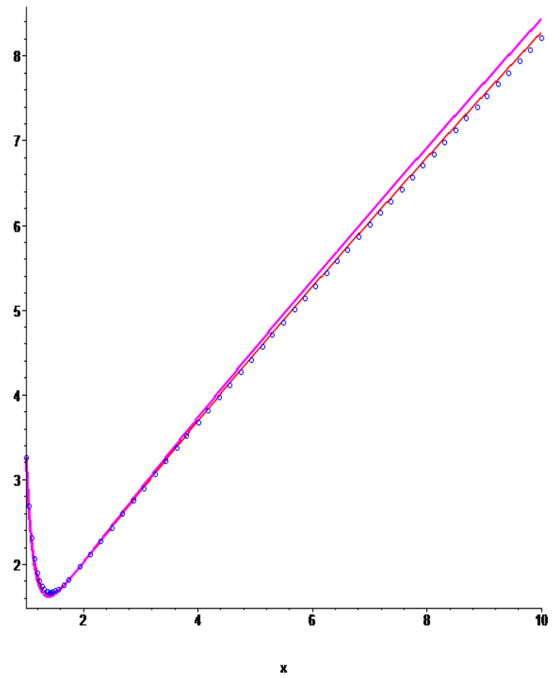

Figure 4: The BS solution (8) with $C_1 = 1, C_2 = 2$ for Mar 13-May 13

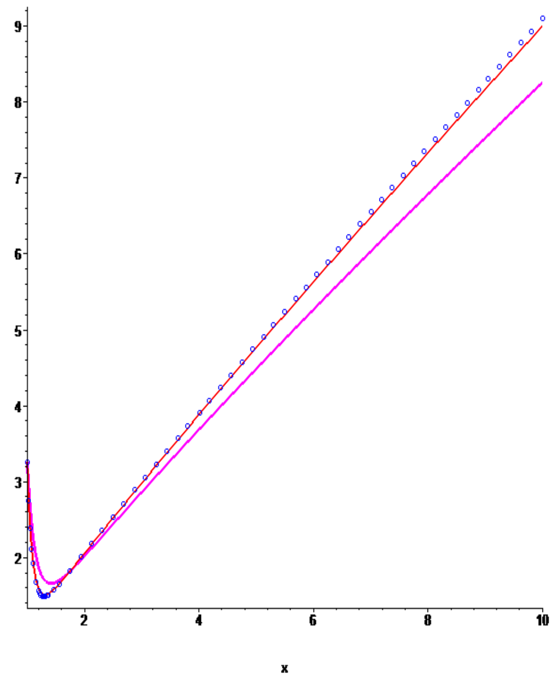

Figure 5: The effect of varying rate of interest for months Jun 13-Aug 13 can be seen in this graph

- In Fig. 4, thick line indicates the value of stock with investment of Rs. 1-10 at the rate of 8.19 with time period one month (Mar 13); thin line depicts the value of stock with investment of Rs. 1-10 at the rate of 7.56 for one month (Apr 13); whereas, dotted line indicates the value of stock with the investment of Rs. 1-10 at the rate of 7.31 for the month of May 13 (Similar type of lines are considered for the next cases). This figure collectively indicates that if rate of interest falls down, value of stock will also fall down with time period remaining the same (Here, in all the cases time is considered to be fixed i.e. one month).
- In Fig. 5, rate of interest from Jun 13, Jul 13 and Aug 13 varies from 7.48 to 11.26 and then to 12.02. Here, we noticed that the more deeper gap in rate of interest will yield more line of difference in stock value and the lesser gap will generate thin line of difference in value of stock.
- Here, in Fig 6, the interest rate changes from 9.57 to 8.6 and then to 8.94. We observed that, as the gap between three interest rates is very marginal, the graphs are trying to overlap with each other at some point (as is indicated by thick, thin and dotted graph lines).
- In the months of Dec 13, Jan 14, Feb 14 the rate of interest is 8.73, 8.9 and 9.15. Here, BS solution represents the scenario of market as if the interest rate tends to be the same, the graph lines also tend to collide with each other.

4 Conclusion

In this paper, we have applied Lie symmetry methods to one of the most important equation of Finance i.e. Black-Scholes equation. With the help of already deduced symmetries, we have reduced BS equation to Ordinary differential Equations in two different cases and also, obtained the general solutions of that reduced ODEs. In the end, we have also applied our results to Indian Treasury Bills.

Remark 4.1: Figure 3 depicts the effect of change in time and asset price on value of option price.

Remark 4.2: In Figures 4, 5, 6, 7, we can easily see that how the riskfree rate effects the final value of option price.

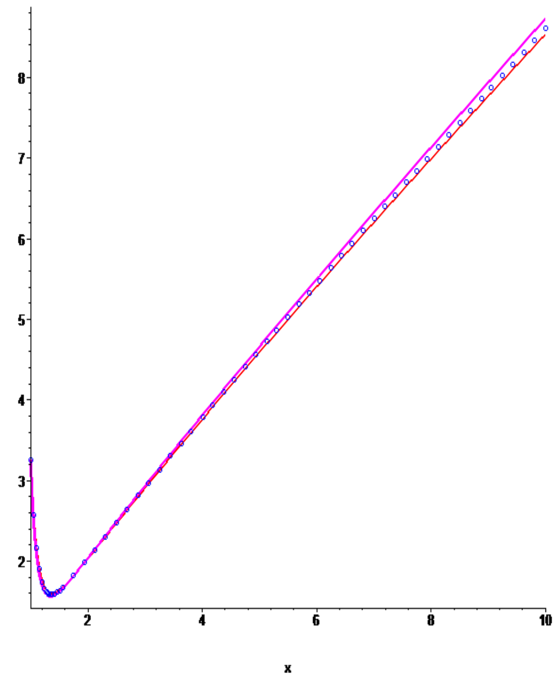

Figure 6: This graphs depicts the different values of stock with change in rate of interest for Sep 13-Nov 13

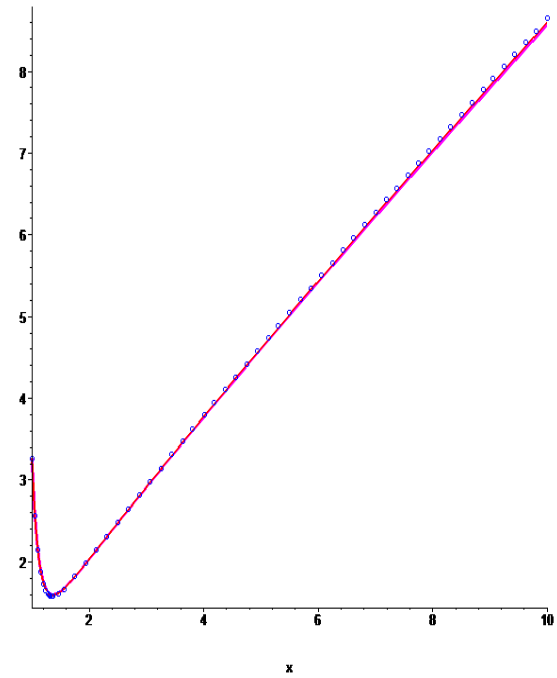

Figure 7: This graph shows as interest rate comes closer, so is the stock value for months Dec 13-Feb 14

References

[1] F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(1973): 637-654.

[2] P. J. Olver. Applications of Lie Groups to Differential Equations. *Springer Verlag, New York.* 107(1993).

[3] G. W. Bluman and J. D. Cole. Similarity Methods for Differential Equations. *Springer Verlag, New York.* (1974).

[4] L. V. Ovsiannikov. Group Analysis of Differential Equations. *Academic Press, New York.* (1982).

[5] A. Bansal and R. K. Gupta. Lie Point Symmetries and Similarity Solutions of the Time Dependent Coefficients Calogero Degasperis Equation. *Physica Scripta*. 86(2012).

[6] D. Silverman. Solution of the Black Scholes Equation Using the Green's Function of Diffusion Equation. *Department of Physics and Astronomy*. (1999): 1-7.

[7] N. Sukhomlin and J. Ortiz. Equivalence and New Exact Solutions to the Black Scholes And Diffusion Equations. *Applied Mathematics E-Notes*. 7(2007): 206-213.

[8] M. Bohner and Y. Zheng. On analytical solutions of the Black-Scholes equation. *Applied Mathematics Letters*. 22(2009): 309-313.

[9] S. Dimas et al. Complete Specification of Some Partial Differential Equations that Arise in Financial Mathematics. *Journal of Nonlinear Mathematical Physics*. 16(2009): 73–92.