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Picard's Iterative Method for Singular Fractional Differential Equations

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Abstract: This paper is devoted to study a class of initial value problems for singular fractional differential equations. The modified Picard's iterative method is employed to obtain the existence and uniqueness of solutions under the Lipschitz condition, without any restrictions on the Lipschitz constant. An example is presented to illustrate the results.

Keywords: singular fractional differential equations; initial value problems; caputo fractional derivative; picard's iterative method

1 Introduction

This paper is concerned with the existence and uniqueness of solutions for the following singular fractional differential equation with initial condition

$$\begin{cases} {}^{C}D_0^{\alpha}x(t) = f(t, x(t)),\ t \in (0, b],\ 0 < \alpha \leq 1, \\ x(0) = x_0, \end{cases} \tag{1}$$

where ${}^{C}D_0^{\alpha}$ is the Caputo fractional derivative operator of order $0 < \alpha < 1$, and $b > 0$ is a constant. The function f is defined on $\mathbb{R} \times \mathbb{R}$, with $\lim\limits_{t \to 0^+} f(t, \cdot) = \infty$.

Fractional differential equations have gained considerable importance due to their applications in various fields, such as physics, mechanics, chemistry, engineering, etc. It has been found that the differential equations involving fractional derivatives are more realistic and practical to describe many phenomena in nature[1–3]. We also refer the reader to some other works [4–7] on the fractional differential problems. For more details about fractional calculus and fractional differential equations, we refer to the books by Podlubrry [8], Kilbas et al. [9], Lakshmikantham and Vatsala [10] and Agarwal et al. [11]. Recently, there are some works about the existence of solutions for singular fractional differential equations, see [12–16].

The iterative technique is a powerful tool for proving the existence of solutions for differential equations, see, for example, [17–23] and references therein. In [18], Picard's iterative method is employed to obtain the existence of solutions to fractional differential equations with both Riemann-Liouville and Caputo fractional derivatives. Recently, Yang and Liu discussed the singular fractional differential equation (1) in [24]. The existence of local solutions was obtained by u sing Picard's iterative method. However, there is a spurious computation in the proof of Lemma 2.4 in [24], which is essential for the main result.

Motivated by the above comment, in this paper, we discuss the singular fractional differential equation (1). A modified Picard iterative method is employed to investigate the existence and uniqueness of global solutions to (1). Compared with the earlier results, we obtain the global existence results without any restrictions on the Lipschitz constant.

The rest of this paper is organized as follows. In Section 2, we give some definitions and facts about fractional derivative and integral. The main results, the existence and uniqueness of global solutions to (1), is obtained in Section 3. In Section 4, an example is presented to illustrate the main results.

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2 Preliminaries and Lemmas

In this section, we collect some definitions and results needed in our further investigations. Let us denote by $C([a,b])$ the space of all continuous real functions defined on $[a,b]$ endowed with supermium norm, i.e., $\|x\| = \sup\{|x(t)| : t \in [a,b]\}$. Then $C([a,b])$ becomes a Banach space. It is known that the beta and gamma functions are defined respectively by

$$\Gamma(x) = \int_0^{+\infty} s^{x-1}e^{-s}\mathrm{d}s, \qquad \mathbf{B}(x,y) = \int_0^1 (1-s)^{x-1}s^{y-1}\mathrm{d}s, \ \ x>0, y>0.$$

The relation between these functions is

$$\mathbf{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Let us recall the following known definitions about fractional integral and derivatives.

Definition 1 *[18] The Riemann-Liouville fractional integral of order $\alpha > 0$ with the lower limit zero for a function f is defined as*

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \ t>0,$$

provided the right side is point-wisely defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2 *[18] The Caputo fractional derivative of order $\alpha > 0$ with the lower limit zero for a function f is defined as*

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \ t>0,$$

where $\alpha \in (n-1,n), \ n \in \mathbb{N}$.

Lemma 1 *[18] Let $0 < \alpha < 1$. Then the unique solution to the equation ${}^C D_0^\alpha h(t) = 0$ is given by the formula*

$$h(t) = C,$$

for $t > 0$, where $C \in \mathbb{R}$ is a constant, provided $h \in C[(0,b]) \cap L^1_{loc}[0,b)$. Further, if $f \in C([0,b]) \cap L^1_{loc}(a,b)$ such that ${}^C D^\alpha f \in C([0,b]) \cap L^1_{loc}(0,b)$, then

$$I_0^{\alpha C} D_0^\alpha f(t) = f(t) + C$$

for $t > 0$ and some constant $C \in \mathbb{R}$.

3 Main results

In this section, by using the method of modified Picard's iterative, we give the existence and uniqueness results for the initial value problem (1).

For the forthcoming analysis, we need the following hypothesis:

(Hf) $f : (0,b] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(1) $f : (0,b] \times \mathbb{R} \to \mathbb{R}$ is continuous, with $\lim\limits_{t\to 0^+} f(t,x) = \infty$, and there exists a constant $0 < k < \alpha$ such that $t^k f(t,x)$ is a continuous function on $[0,b] \times \mathbb{R}$.

(2) For the k above, there exists $L > 0$ such that

$$t^k |f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|, \tag{2}$$

for all $t \in [0,b]$ and $x_1, x_2 \in \mathbb{R}$.

In the proof of our main results, we need the following Lemma.

Lemma 2 *Suppose that (Hf) holds, the function $x \in C[0,b]$ is a solution of initial value problem* (1) *if and only if it is a solution of the following Volterra integral equation:*

$$x(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s. \tag{3}$$

Proof. We first show that for every $x \in C[0,b]$, the integral $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s$ is convergent and uniformly bounded for $t \in [0,b]$. In fact every $x \in C[0,b]$ is bounded. From $(Hf)(1)$, we know that $t^k f(t,x(t))$ is continuous on $[0,b]$. So there exists a constant $M > 0$ such that $|t^k f(t,x(t))| < M$ for all $t \in [0,b]$. Therefore,

$$
\begin{aligned}
|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s| &\leq |\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-k}s^k f(s,x(s))\mathrm{d}s| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-k} M\mathrm{d}s \\
&= Mt^{\alpha-k}\frac{\mathbf{B}(\alpha,-k+1)}{\Gamma(\alpha)} \\
&\leq Mb^{\alpha-k}\frac{\Gamma(-k+1)}{\Gamma(\alpha-k+1)}.
\end{aligned}
$$

This shows that $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s$ is convergent and uniformly bounded for $t \in [0,b]$.

Suppose that $x \in C[0,b]$ is a solution of initial value problem (1), then it is easy to see from Lemma 1 that $x(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\mathrm{d}s$ for $t \in [0,b]$. So $x \in C[0,b]$ is a solution of (3).

On the other hand, suppose that $x : [0,b] \to R$ is a solution of (3). Then we apply the fractional differential operator ${}^C D_0^\alpha$ to both sides of (3) and immediately obtain that x also solves the differential equation (1). The proof is completed.

Theorem 3 *Suppose that (Hf) holds, then there exists a uniquely defined function $\phi \in C[0,b]$ solving the initial value problem* (1).

Proof. From Lemma 2 we know that it suffices to prove that the integral equation (3) has a unique solution. We first choose $N = [(\frac{2L\Gamma(1-k)}{\Gamma(\alpha-k+1)})^{\frac{1}{\alpha-k}} b] + 1$, and define $h_i = \frac{ib}{N}$ for $i = 0, 1, \cdots N$. Then $0 = h_0 < h_1 < h_2 < \cdots < h_N = b$ and

$$
h := \frac{b}{N} = h_{i+1} - h_i \leq \Big(\frac{\Gamma(\alpha-k+1)}{2L\Gamma(1-k)}\Big)^{\frac{1}{\alpha-k}} \tag{4}
$$

for $i = 1, 2, \cdots, N$.

We first concentrate on the interval $[h_0, h_1]$. We define a function sequence by

$$
\phi_0^{(1)}(t) = x_0, \qquad t \in [h_0, h_1] \tag{5}
$$

and

$$
\phi_n^{(1)}(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi_{n-1}^{(1)}(s))\mathrm{d}s,\ t \in [h_0, h_1] \tag{6}
$$

for $n = 1, 2, \cdots$.

It is easy to see that $\phi_n^{(1)} \in C[h_0, h_1]$. Hence it follows from the proof of Lemma 2 that the functions $\phi_n^{(1)}$ are well-defined. Moreover, according to (Hf) and (4), we deduce that

$$
\begin{aligned}
|\phi_n^{(1)}(t) - \phi_{n-1}^{(1)}(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,\phi_{n-1}^{(1)}(s)) - f(s,\phi_{n-2}^{(1)}(s))|\mathrm{d}s \\
&\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-k}|\phi_{n-1}^{(1)}(s)) - \phi_{n-2}^{(1)}(s))|\mathrm{d}s \\
&\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-k}|\mathrm{d}s\|\phi_{n-1}^{(1)} - \phi_{n-2}^{(1)}\|_\infty \\
&= \frac{L}{\Gamma(\alpha)} t^{\alpha-k}\mathbf{B}(\alpha,-k+1)\|\phi_{n-1}^{(1)} - \phi_{n-2}^{(1)}\|_\infty \\
&\leq \frac{L\Gamma(-k+1)}{\Gamma(\alpha-k+1)} h^{\alpha-k}\|\phi_{n-1}^{(1)} - \phi_{n-2}^{(1)}\|_\infty \\
&\leq \frac{1}{2}\|\phi_{n-1}^{(1)} - \phi_{n-2}^{(1)}\|_\infty
\end{aligned}
$$

for $n = 2, 3, \cdots$, which implies that

$$\|\phi_n^{(1)} - \phi_{n-1}^{(1)}\|_\infty \leq \frac{1}{2^{n-1}}\|\phi_1^{(1)} - \phi_0^{(1)}\|_\infty.$$

Thus, we have that the series $\sum_{n=1}^{\infty}(\phi_n^{(1)} - \phi_{n-1}^{(1)})$ is uniformly convergent on the interval $[h_0, h_1]$. It then follows that $\{\phi_n^{(1)}\}_{n=1}^{\infty}$ is uniformly convergent on $[h_0, h_1]$. Denote by $\phi^{(1)}(t) = \lim_{n\to\infty}\phi_n^{(1)}(t)$. Then $\phi^{(1)} \in C[h_0, h_1]$, since all the $\phi_n^{(1)}$ is continuous on $[h_0, h_1]$. We now prove that $\phi^{(1)}$ is the unique continuous solution of initial value problem (1) on the interval $[h_0, h_1]$. We can get from (2) that

$$t^k|f(t, \phi_n^{(1)}(t)) - f(t, \phi^{(1)}(t)))| \leq L|\phi_n^{(1)}(t) - \phi^{(1)}(t))| \to 0$$

uniformly as $n \to \infty$ on the interval $[h_0, h_1]$. Hence

$$\begin{aligned}
\phi^{(1)}(t) &= \lim_{n\to\infty}\phi_n^{(1)}(t) \\
&= \lim_{n\to\infty}[x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi_{n-1}^{(1)}(s))\mathrm{d}s] \\
&= x_0 + \lim_{n\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-k}s^k f(s, \phi_{n-1}^{(1)}(s))\mathrm{d}s \\
&= x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-k}\lim_{n\to\infty} s^k f(s, \phi_{n-1}^{(1)}(s))\mathrm{d}s \\
&= x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-k}s^k f(s, \phi^{(1)}(s))\mathrm{d}s \\
&= x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi^{(1)}(s))\mathrm{d}s.
\end{aligned}$$

This shows that (3) is satisfied. From Lemma 2, we obtain that $\phi^{(1)}$ is a continuous solution of (1) on $[h_0, h_1]$. Suppose that $\psi \in C([h_0, h_1])$ is also a solution of (1). Then

$$\psi(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \psi(s))\mathrm{d}s,$$

for $t \in [h_0, h_1]$. Since

$$\begin{aligned}
|\phi^{(1)}(t) - \psi(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, \phi^{(1)}(s)) - f(s, \psi(s))|\mathrm{d}s \\
&\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-k}|\phi^{(1)}(s) - \psi(s)|\mathrm{d}s \\
&\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-k}\mathrm{d}s\|\phi^{(1)} - \psi\|_\infty \\
&= \frac{L}{\Gamma(\alpha)} t^{\alpha-k}\mathbf{B}(\alpha, -k+1)\|\phi^{(1)} - \psi\|_\infty \\
&\leq \frac{L\Gamma(-k+1)}{\Gamma(\alpha-k+1)} h^{\alpha-k}\|\phi^{(1)} - \psi\|_\infty \\
&\leq \frac{1}{2}\|\phi^{(1)} - \psi\|_\infty,
\end{aligned}$$

we get that

$$\|\phi^{(1)} - \psi\|_\infty \leq \frac{1}{2}\|\phi^{(1)} - \psi\|_\infty,$$

which implies that $\|\phi^{(1)} - \psi\|_\infty = 0$, and hence $\phi^{(1)} \equiv \psi$ on $[h_0, h_1]$. Therefore, $\phi^{(1)}$ is the unique solution of (1) on $[h_0, h_1]$.

Next we consider the interval $[h_1, h_2]$. Let $g(t) = x_0 + \int_0^{h_1} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi^{(1)}(s))\mathrm{d}s$ for $t \in [h_1, h_2]$. Define a function sequence by

$$\phi_0^{(2)}(t) = g(t) \qquad t \in [h_1, h_2] \tag{7}$$

and

$$\phi_n^{(2)}(t) = g(t) + \int_{h_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi_{n-1}^{(2)}(s))\mathrm{d}s, \qquad t \in [h_1, h_2] \tag{8}$$

for $n = 1, 2, \cdots$. Since $\phi^{(1)}$ is uniquely defined and continuous on $[h_0, h_1]$, g is uniquely defined on $[h_1, h_2]$ and $g(h_1) = \phi^{(1)}(h_1)$. Moreover, one can easily deduce from (Hf) that g is bounded on $[h_1, h_2]$, and hence $\phi_0^{(2)}$ is bounded on $[h_1, h_2]$. Suppose that $\phi_{n-1}^{(2)}$ is bounded on $[h_1, h_2]$, i.e., $\phi_{n-1}^{(2)}(t) \le M_1$ for some constant $M_1 > 0$ and every $t \in [h_1, h_2]$. Then $|f(s, \phi_{n-1}^{(2)}(s))| \le M$ for some constant $M > 0$ and all $s \in [h_1, h_2]$, due to the fact that f is continuous on $[h_1, h_2] \times [-M_1, M_1]$. Therefor,

$$\begin{aligned}
|\phi_n^{(2)}(t)| &\le \|g\|_\infty + \int_{h_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, \phi_{n-1}^{(2)}(s))|\mathrm{d}s \\
&\le \|g\|_\infty + M \int_{h_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathrm{d}s \\
&\le \|g\|_\infty + \frac{Mh^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

for every $t \in [h_1, h_2]$. This shows that $\phi_n^{(2)}$ is also bounded on $[h_1, h_2]$. So we can deduce by mathematical inductive that $\phi_n^{(2)}$ is bounded and continuous on $[h_1, h_2]$ for every $n \in N$, and hence $\{\phi_n^{(2)}\}$ is well-defined. We now prove that $\{\phi_n^{(2)}\}$ is convergent uniformly for $t \in [h_1, h_2]$. In fact, since

$$\begin{aligned}
|\phi_n^{(2)}(t) - \phi_{n-1}^{(2)}(t)| &\le \frac{1}{\Gamma(\alpha)} \int_{h_1}^t (t-s)^{\alpha-1} |f(s, \phi_{n-1}^{(2)}(s)) - f(s, \phi_{n-2}^{(2)}(s))|\mathrm{d}s \\
&\le \frac{L}{\Gamma(\alpha)} \int_{h_1}^t (t-s)^{\alpha-1} s^{-k} |\phi_{n-1}^{(2)}(s)) - \phi_{n-2}^{(2)}(s))|\mathrm{d}s \\
&\le \frac{L}{\Gamma(\alpha)} \int_{h_1}^t (t-s)^{\alpha-1} (s-h_1)^{-k} \mathrm{d}s \|\phi_{n-1}^{(2)} - \phi_{n-2}^{(2)}\|_\infty \\
&\le \frac{L}{\Gamma(\alpha)} (h_2 - h_1)^{\alpha-k} \mathbf{B}(\alpha, 1-k) \|\phi_{n-1}^{(2)} - \phi_{n-2}^{(2)}\|_\infty \\
&= \frac{L\Gamma(-k+1)}{\Gamma(\alpha-k+1)} h^{\alpha-k} \|\phi_{n-1}^{(2)} - \phi_{n-2}^{(2)}\|_\infty \\
&\le \frac{1}{2} \|\phi_{n-1}^{(2)} - \phi_{n-2}^{(2)}\|_\infty
\end{aligned}$$

for $n = 2, 3, \cdots$, similar with the proof of interval $[h_0, h_1]$, we know that $\{\phi_n^{(2)}\}$ is convergent uniformly for $t \in [h_1, h_2]$. We denote by $\phi^{(2)}(t) = \lim_{n\to\infty} \phi_n^{(2)}(t)$. Then $\phi^{(2)} \in C[h_1, h_2]$ since each $\phi_n^{(2)} \in C[h_1, h_2]$, $n = 1, 2, \cdots$. Using the same arguments as above, we can obtain that $\phi^{(2)}$ is the unique continuous function satisfying

$$\begin{aligned}
\phi^{(2)}(t) &= g(t) + \int_{h_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi^{(2)}(s))\mathrm{d}s \\
&= x_0 + \int_0^{h_1} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi^{(1)}(s))\mathrm{d}s + + \int_{h_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, \phi^{(2)}(s))\mathrm{d}s,
\end{aligned}$$

for $t \in [h_1, h_2]$, which is the unique solution to (1) on $[h_1, h_2]$.

Taking the next interval $[h_2, h_3]$, repeat this process, and we conclude that there exists a unique solution $\phi^{(i)}$ to the integral equation (3) on every interval $[h_i, h_{i+1}]$. Now we set

$$\phi(t) = \begin{cases} \phi^{(1)}(t), & t \in [h_0, h_1], \\ \phi^{(2)}(t), & t \in [h_1, h_2], \\ \cdots \\ \phi^{(N)}(t), & t \in [h_{N-1}, h_N]. \end{cases} \tag{9}$$

The fact that $\phi^{(i)} \in C[h_{i-1}, h_i]$ $(i = 1, 2, \cdots N)$, as well as the definition of $\phi^{(i)}, i = 1, 2, \cdots, N$, shows that $\phi \in C[0, b]$ and is actually the unique solution of initial value problem (1) defined on $[0, b]$. This completes the proof.

Remark 4 *From Theorem 3 we can see that there is no restriction on the constant b or the Lipschitz constant L. That is to say, for any $b > 0$, there is a unique solution to* (1) *on* $[0.b]$. *This means that actually we have obtained the following global existence of the initial value problem* (1).

Theorem 5 *Suppose that (Hf) holds. Then the initial value problem* (1) *has a unique solution on* $[0, \infty)$.

Remark 6 *From the proof of Lemma 2 and Theorem 3, we can see that the assumption that f is joint continuous is just used to guarantee the boundedness of f on some relevant subsets. For this reason, this condition can be replaced by the following boundedness assumption, which is employed in [24].*

(H'f) $f : (0, b] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
(1) $f(t, \cdot)$ is continuous for all $t \in (0, b]$, and $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$.
(2) there exists a constant $0 < k < \alpha$, $M \geq 0$ and $L > 0$ such that

$$t^k |f(t, x)| \leq M, \tag{10}$$

and

$$t^k |f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|, \tag{11}$$

for all $t \in [0, b]$, $x, x_1, x_2 \in \mathbb{R}$.

Theorem 7 *Suppose that $(H'f)$ holds, then there exists a uniquely defined function $\phi \in C[0, b]$ solving the initial value problem* (1).

The proof is Similar to that of Theorem 3, so we omit it.

Remark 8 *In [24], using the Picard's iterative method, the authors obtained the unique continuous solution of initial value problems* (1) *on the interval $[t_0, t_0 + h]$, where $h < b$ satisfies some extra conditions. Unfortunately, there is a spurious computation in the proof of Lemma 2.4 in [24] (see line 10 to line 14 of page 7 of [24]), which plays a key role in the proof of the main result, Theorem 2.1. Under the same assumptions of Theorem 2.1 of [24], we get the global existence of the initial value problem* (1) *in Theorem 7, without any other assumptions on the Lipschitz constant L.*

4 An Example

In this section, as an application of our main results, an example is presented. We consider the following fractional differential equation.

$$\begin{cases} {}^C D_0^{\frac{2}{3}} x(t) = t^{-\frac{1}{2}}(1 + 2x(t)), & t \in (0, 10], \\ x(0) = 4, \end{cases} \tag{12}$$

where $\alpha = \frac{2}{3}$, $x_0 = 4$, $f(t, x) = t^{-\frac{1}{2}}(1 + 2x)$ for $(t, x) \in (0, 10] \times R$, and $\lim\limits_{t \to 0^+} f(t, \cdot) = \infty$. Set $k = \frac{1}{2}$, then $t^k f(t, x) = 1 + 2x$ is continuous on $[0, 10]$. So $H(f)(1)$ is satisfied. Since $|f(t, x_1) - f(t, x_2)| = 2t^{-k}|x_1 - x_2|$, $H(f)(2)$ is also satisfied with $L = 2$. Therefore, all the assumptions in Theorem 3 are fulfilled.

Proposition 9 *The initial value problem (12) has a uniquely continuous solution on* $[0, 10]$.

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References

[1] Q. Dong. Existence and continuous dependence for weighted fractional differential equations with infinite delay. *Adv. Differ. Equ.* 251(2014): 1-12.

[2] T. Poinot and J. Trigeassou. Identification of fractional systems using an output-error technique. *Nonlinear Dyn.* 38(2004): 133-154.

[3] Y. Rossikhin and M. Shitikova. Application of fractional dericatives to the analysis of damped vibrations of viscoelastic single mass system. *Acta Mech.* 120(1997): 109-125.

[4] S. Wang and Y. Yu. Application of multistage homotopy-perturbation method for the solutions of the chaotic fractional order systems. *Int. J. Nonlinear Sci.* 13(2012): 3-14.

[5] Z. Fan. A short note on the solvability of impulsive fractional differential equations with Caputo derivatives. *Appl. Math. Lett.* 38(2014): 14-19.

[6] G. Gautam and J. Dabas. Mild solution for nonlocal fractional functional differential equation with not instantaneous impulse. *Int. J. Nonlinear Sci.* 21(2016): 151-160.

[7] S. Ji. Approximate controllability of semilinear nonlocal fractional differential systems via an approximating method. *Appl. Math. Comput.* 236(2014): 43-53.

[8] I. Podlubny. Fractional differential equations. *Academic Press, San Diego.* (1999).

[9] A. Kilbas et al. Theory and applications of fractional differential equations. *Elsevier, Amsterdam.* (2006).

[10] V. Lakshmikantham and A. Vatsala. Basic theory of fractional differential equations. *Nonlinear Anal.* 69(2008): 2677-2682.

[11] R. Agarwal et al. On the concept of solutions for fractional differential equations with uncertainty. *Nonlinear Anal.* 72(2010): 2859-2862.

[12] D. Baleanu et al. The existence of solutions for a nonlinear mixed problem of singular fractional differential equations. *Adv. Differ. Equ.* 359(2013): 1-12.

[13] W. Jiang et al. The existence of positive solutions for the singular fractional differential equation. *J. Appl. Math. Comput.* 41(2013): 171-182.

[14] S. Vong. Positive solutions of singular fractional differential equations with integral boundary conditions. *Math. comput. Model.* 57(2013): 1053-1059.

[15] Y. Wang et al. Existence and uniqueness of a positive solution to singular fractional differential equations. *Bound. Value. probl.* 20(2012): 1-12.

[16] X. Zhang et al. Multiple positive solutions of a singular fractional differential equation with negatively perturbed term. *Math. comput. Model.* 55(2012): 1263-1274.

[17] Z. Dahmani and A. Anber. The variational iteration method for solving the fractional foam drainage equation. *Int. J. nonlinear Sci.* 10(2010) :39-45.

[18] K. Diethelm. The analysis of fractional differential equations. *Lecture Notes in Mathmatics, Springer-Verlag, Berlin.* (2010).

[19] V. Lakshmikantham and A. Vatsala. General uniqueness and monotone iterative technique for fractional differential equations. *Appl. Math. Lett.* 21(2008): 828-834.

[20] J. Ramos. Iterative and non-iterative methods for non-linear volterra integro-differential equations. *Appl. Math. Comput.* 214(2009): 287-296.

[21] J. Ramos. Picard's iterative method for nonlinear advection-reaction-diffusion equations. *Appl. Math. Comput.* 215(2009): 1526-1536.

[22] J. Wang et al. Picard and weakly Picard technique for nonlinear differential equations in Banach spaces. *J. Math. Anal. Appl.* 389(2012): 261-274.

[23] S. Zhang. Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives. *Nonlinear Anal.* 71(2009): 2087-2093.

[24] X. Yang and Y. Liu. Picard iterative processes for initial value problems of singular fractional differential equations. *Adv. Differ. Equ.* 102(2014): 3-17.