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Solving the Simplified MCH Equation and the Combined KdV-mKdV Equations via $\tan\left(\frac{\Phi(\xi)}{2}\right)$-Expansion Method

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Abstract: This paper is concerned with the traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdV-mKdV equations. Based on $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method the solution procedure of nonlinear evolution equations in mathematical physics are investigated. We obtained the exact solutions for the aforementioned nonlinear evolution equations. These solutions containing three types hyperbolic function solution, trigonometric function solution and rational solution. This method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. It is shown that this method, with the help of symbolic computation, provide a straightforward and powerful mathematical tool for solving nonlinear partial differential equations.

Keywords: $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method; simplified MCH equation; combined KdV-mKdV equations.

1 Introduction

In this paper, we consider the simplified modified Cammasa-Holm (MCH) equation [1]

$$\mathrm{u_t} + 2\mathrm{k}\mathrm{u_x} - \mathrm{u_{xxt}} + \beta \mathrm{u}^2\mathrm{u_x} = 0, \quad \beta > 0, \quad \mathrm{k} \in \mathrm{R}, \tag{1}$$

where k and β are constants. Camassa and Holm derived a completely integrable wave equation namely CH equation for water waves by retaining two terms that are usually neglected in the small amplitude, shallow water limit [2]. In [3], modified Camassa-Holm equation has investigated and obtained new peaked solitary wave solutions. New compact and noncompact solutions for two variants of a modified Camassa Holm equation has obtained by Wazwaz [4]. For more details see [5–7]. We next consider the combined KdV-mKdV equation as follows [1]

$$\mathrm{u_t} + \alpha \mathrm{u}\mathrm{u_x} + \beta \mathrm{u}^2\mathrm{u_x} + \mathrm{u_{xxx}} = 0, \tag{2}$$

Eq. 2 is widely used in various fields such as quantum field theory, dust-acoustic waves, ion acoustic waves in plasmas with a negative ion, solid-state physics and fluid dynamics. The KdV and mKdV equations are most popular soliton equations and have been extensively investigated. But nonlinear terms of KdV and mKdV equations often simultaneously exist in practical problems such as fluid physics and quantum field theory [8]. This equations may be described as the wave propagation of the bound particle, sound wave and thermal pulse [9]. Also the combined KdV-mKdV equation has implemented by famous methods as Sub-ODE method [10], homogeneous balance method [11] and algebraic method [12]. Recently, a variety of powerful methods for seeking the explicit and exact solutions of nonlinear evolution equations have been proposed and developed. Among them are the Hirota's bilinear method [13], homotopy analysis method ([14, 15]), variational iteration method [16, 17], homotopy perturbation method [18], sine-cosine method [19], tanh-coth method [20], Bäcklund transformation [21], $(\frac{G'}{G})$-expansion method [22], Exp-function method [23–25], modified simple equation method [26] and so on. Here, we use of an effective method for constructing a range of exact solutions for

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the following nonlinear partial differential equations that in this article we developed solutions as well. The standard tanh method is well-known analytical method which first presented by Malfliet's [27] and developed in [27, 28]. In [20], we applied the generalized tanh-coth method in for solving some nonlinear partial differential equations. Also in [29], the new approach of generalized (G'/G)-expansion method to obtain exact traveling wave solutions of NLEEs is presented. In this paper we explain methods which are called the generalized tanh–coth and generalized (G'/G)-expansion methods are presented to look for travelling wave solutions of nonlinear evolution equations. Authors of [30], obtained exact solutions for the integrable sixth-order Drinfeld-Sokolov-Satsuma-Hirota system by the generalized tanh–coth and generalized (G'/G)-expansion methods. Chand and Malik [31] have applied the (G'/G)-expansion method for finding the exact solutions of some nonlinear evolution equations. For further information in about these methods refer to Ref. [32–36]. The purpose of this paper is to obtain analytical solutions of the simplified MCH equation and the combined KdV-mKdV equations and to determine the accuracy of the $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method in solving these kind of problems. The paper is organized as follows: In Section 2, we describe the $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method. In section 3, we examine the simplified MCH equation and the combined KdV-mKdV equations respectively. Also a conclusion is given in Section 4. Finally some references are given at the end of this paper.

2 Description of $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method

Step 1.We suppose that the given nonlinear fractional partial differential equation for $u(x,t)$ to be in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{3}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u''^2 u'', ...) = 0, \tag{4}$$

the transformation $\xi = x - \mu t$, is wave variable. Also, k, m and n are constants to be determined later.
Step 2. Suppose the traveling wave solution of Eq. (4) can be expressed as follows:

$$u(\xi) = S(\Phi) = \sum_{k=0}^{m} A_k \tan\left(\frac{\Phi(\xi)}{2}\right)^k + \sum_{k=1}^{m} B_k \cot\left(\frac{\Phi(\xi)}{2}\right)^k, \tag{5}$$

where $A_k (0 \le k \le m)$ and $B_k (1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\Phi = \Phi(\xi)$ satisfies the following ordinary differential equation:

$$\Phi'(\xi) = a \sin(\Phi(\xi)) + b \cos(\Phi(\xi)) + c. \tag{6}$$

We will consider the following special solutions of equation (6):
Family 1: **When** $a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$**,**

$$\Phi(\xi) = -2 \arctan\left[-\frac{a}{b-c} + \frac{\sqrt{c^2 - b^2 - a^2}}{b-c} \tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right)\right]. \tag{7}$$

Family 2: **When** $a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$**,**

$$\Phi(\xi) = -2 \arctan\left[-\frac{a}{b-c} - \frac{\sqrt{b^2 + a^2 - c^2}}{b-c} \tanh\left(\frac{\sqrt{b^2 + a^2 - c^2}}{2}(\xi + C)\right)\right]. \tag{8}$$

Family 3: **When** $a^2 + b^2 - c^2 > 0$**,** $b \neq 0$ **and** $c = 0$,

$$\Phi(\xi) = 2 \arctan\left[\frac{a}{b} + \frac{\sqrt{b^2 + a^2}}{b} \tanh\left(\frac{\sqrt{b^2 + a^2}}{2}(\xi + C)\right)\right]. \tag{9}$$

Family 4: **When** $a^2 + b^2 - c^2 < 0$**,** $c \neq 0$ **and** $b = 0$,

$$\Phi(\xi) = 2\arctan\left[-\frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{c}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}(\xi + C)\right)\right]. \tag{10}$$

Family 5: **When** $a^2 + b^2 - c^2 > 0$**,** $b - c \neq 0$ **and** $a = 0$,

$$\Phi(\xi) = 2\arctan\left[\sqrt{\frac{b + c}{b - c}}\tanh\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right)\right]. \tag{11}$$

Family 6: **When** $a = 0$ **and** $c = 0$,

$$\Phi(\xi) = \arctan\left[\frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}\right]. \tag{12}$$

Family 7: **When** $b = 0$ **and** $c = 0$,

$$\Phi(\xi) = \arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)} + 1}, \frac{e^{2a(\xi+C)} - 1}{e^{2a(\xi+C)} + 1}\right]. \tag{13}$$

Family 8: **When** $a^2 + b^2 = c^2$**,**

$$\Phi(\xi) = -2\arctan\left[\frac{(b + c)(a(\xi + C) + 2)}{a^2(\xi + C)}\right], \tag{14}$$

where $A_k(k = 0, 2, ..., m), B_k(k = 1, 2, ..., m), a, b$ and c are constants to be determined later. But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (6). If m is not an integer, then a transformation formula should be used to overcome this difficulty.

Step 3. Substituting (5) into Eq. (4) with the value of m obtained in Step 2. Collecting the coefficients of $\tan\left(\frac{\Phi(\xi)}{2}\right)^k$, $\cot\left(\frac{\Phi(\xi)}{2}\right)^k$ $(k = 0, 1, 2, ...)$ then setting each coefficient to zero, we can get a set of over-determined partial differential equations for $A_0, A_k(k = 1, 2, ..., m), B_k(k = 1, 2, ..., m)$ a, b and c with the aid of symbolic computation Maple 13.

Step 4. Solving the algebraic equations in Step 3, then substituting $A_0, A_1, B_1, ..., A_m, B_m, \mu$ in (5).

3 Illustrative Examples

In this section, we present several examples to illustrate the applicability of $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method to solve nonlinear partial differential equations introduced in Section 1.

3.1 The simplified MCH equation

Consider the nonlinear simplified MCH equation as follow

$$u_t + 2ku_x - u_{xxt} + \beta u^2 u_x = 0, \quad \beta > 0, \quad k \in R, \tag{15}$$

where k and β are constants. Using the wave variable $\xi = x - \mu t$ PDE transforms to an ODE as follows

$$(2k - \mu)u' + \mu u''' + \frac{\beta}{3}(u^3)' = 0, \tag{16}$$

where by integrating Eq. (16) and neglecting the constant of integration we obtain

$$(2k - \mu)u + \mu u'' + \frac{\beta}{3}u^3 + \xi_0 = 0. \tag{17}$$

Balancing the u'' and u^3 by using homogenous principal, we have

$$M + 2 = 3M, \qquad \Rightarrow M = 1. \tag{18}$$

Then the trail solution is

$$u(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right). \tag{19}$$

Substituting (19) and (6) into Eq. (17) and by using the well-known Maple software, we obtain the following sets of non-trivial solutions

Set I:

$$\mu = -\frac{2\beta B_1^2}{3(b+c)^2}, \quad A_0 = \frac{aB_1}{b+c}, \quad A_1 = 0, \quad B_1 = \sqrt{\frac{6k(b+c)^2}{\beta(c^2-2-a^2-b^2)}}, \quad u(\xi) = A_0 + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right), \tag{20}$$

where a, b, c, k and β are arbitrary constants. By using of the (20) and **Family 1** we get

$$u_1(\xi) = \sqrt{\frac{6k(b+c)^2}{\beta(c^2-2-a^2-b^2)}} \left\{\frac{a}{b+c} + \left[\frac{a}{b-c} - \frac{\sqrt{c^2-b^2-a^2}}{b-c} \tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi + C)\right)\right]^{-1}\right\}. \tag{21}$$

By using of the (20) and **Family 2** we get

$$u_2(\xi) = \sqrt{\frac{6k(b+c)^2}{\beta(c^2-2-a^2-b^2)}} \left\{\frac{a}{b+c} + \left[\frac{a}{b-c} + \frac{\sqrt{a^2+b^2-c^2}}{b-c} \tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi + C)\right)\right]^{-1}\right\}. \tag{22}$$

By using of the (20) and **Family 3** we get

$$u_3(\xi) = \sqrt{\frac{-6kb^2}{\beta(2+a^2+b^2)}} \left\{\frac{a}{b} + \left[\frac{a}{b} + \frac{\sqrt{a^2+b^2}}{b} \tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi + C)\right)\right]^{-1}\right\}. \tag{23}$$

By using of the (20) and **Family 4** we get

$$u_4(\xi) = \sqrt{\frac{6kc^2}{\beta(c^2-2-a^2)}} \left\{\frac{a}{c} + \left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c} \tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi + C)\right)\right]^{-1}\right\}. \tag{24}$$

By using of the (20) and **Family 5** we get

$$u_5(\xi) = \sqrt{\frac{6k(b^2-c^2)}{\beta(c^2-2-b^2)}} \coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi + C)\right). \tag{25}$$

By using of the (20) and **Family 6** we get

$$u_6(\xi) = \sqrt{\frac{-6kb^2}{\beta(2+b^2)}} \cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right). \tag{26}$$

By using of the (20) and **Family 8** we get

$$u_7(\xi) = \sqrt{\frac{-3k(b+c)^2}{\beta}} \left[\frac{a}{b+c} - \frac{a^2(\xi + C)}{(b+c)(a(\xi + C)+2)}\right], \tag{27}$$

where $\xi = x - \frac{4k}{a^2+b^2-c^2+2}t$.

Set II:

$$\mu = -\frac{2\beta B_1^2}{3(b+c)^2}, \quad A_0 = \frac{aB_1}{b+c}, \quad A_1 = -\frac{b-c}{b+c}B_1, \quad B_1 = \sqrt{-\frac{6k(b+c)^2}{\beta(2c^2+2+a^2-2b^2)}}, \tag{28}$$

$$\xi_0 = \frac{2a\beta B_1^3(b-c)}{3(b+c)^2}, \quad u(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right), \tag{29}$$

where a, b, c, k and β are arbitrary constants. By using of the (29) and **Family 1** we get

$$u_8(\xi) = \sqrt{-\frac{6k(b+c)^2}{\beta(2c^2+2+a^2-2b^2)}} \left\{ \frac{\sqrt{c^2-b^2-a^2}}{b+c} \tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right) + \left[\frac{a}{b-c} - \frac{\sqrt{c^2-b^2-a^2}}{b-c} \tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-1} \right\}. \tag{30}$$

By using of the (29) and **Family 2** we get

$$u_9(\xi) = \sqrt{-\frac{6k(b+c)^2}{\beta(2c^2+2+a^2-2b^2)}} \left\{ -\frac{\sqrt{a^2+b^2-c^2}}{b+c} \tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right) + \left[\frac{a}{b-c} + \frac{\sqrt{a^2+b^2-c^2}}{b-c} \tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-1} \right\}. \tag{31}$$

By using of the (29) and **Family 3** we get

$$u_{10}(\xi) = \sqrt{-\frac{6kb^2}{\beta(2+a^2-2b^2)}} \left\{ -\frac{\sqrt{a^2+b^2}}{b} \tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right) + \left[\frac{a}{b} + \frac{\sqrt{a^2+b^2}}{b} \tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-1} \right\}. \tag{32}$$

By using of the (29) and **Family 4** we get

$$u_{11}(\xi) = \sqrt{-\frac{6kc^2}{\beta(2c^2+2+a^2)}} \left\{ \frac{\sqrt{c^2-a^2}}{b+c} \tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right) + \left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c} \tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1} \right\}. \tag{33}$$

By using of the (29) and **Family 5** we get

$$u_{12}(\xi) = \sqrt{\frac{3k(b^2-c^2)}{\beta(b^2-c^2-1)}} \left[-\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right) + \coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]. \tag{34}$$

By using of the (29) and **Family 6** we get

$$u_{13}(\xi) = \sqrt{\frac{3kb^2}{\beta(b^2-1)}} \left[-\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right) + \cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]. \tag{35}$$

By using of the (20) and **Family 8** we get

$$u_{14}(\xi) = \sqrt{\frac{-6k(b+c)^2}{\beta(3a^2+2)}} \left[\frac{a}{b+c} + \frac{(b-c)(a(\xi+C)+2)}{a^2(\xi+C)} - \frac{a^2(\xi+C)}{(b+c)(a(\xi+C)+2)}\right], \tag{36}$$

where $\xi = x - \frac{4k}{2c^2+a^2-2b^2+2}t$.

3.2 The (1 + 1)-dimensional combined KdV-mKdV equation

We next consider the The (1 + 1)-dimensional combined KdV–mKdV equation as follow

$$u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0, \tag{37}$$

where k and β are constants. Using the wave variable $\xi = x - \mu t$ PDE transforms to an ODE as follows

$$-\mu u' + \alpha u u'^2 u' + u''' = 0, \tag{38}$$

where by integrating Eq. (38) and neglecting the constant of integration we obtain

$$-\mu u + \frac{\alpha}{2} u^2 + \frac{\beta}{3} u^3 + u'' + \xi_0 = 0. \tag{39}$$

Balancing the u'' and u^3 by using homogenous principal, we have

$$M + 2 = 3M, \qquad \Rightarrow M = 1. \tag{40}$$

Then the trail solution is

$$u(\xi) = A_0 + A_1 \tan\left(\frac{\Phi(\xi)}{2}\right) + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right). \tag{41}$$

Substituting (41) and (6) into Eq. (39) and by using the well-known Maple software, we obtain the following sets of non-trivial solutions

Set I:

$$\mu = -\frac{2\beta(a^2 + b^2 - c^2) + \alpha^2}{4\beta}, \quad A_0 = \mp\frac{3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}}, \quad A_1 = 0, \quad B_1 = \pm(b + c)\sqrt{-\frac{3}{2\beta}}, \tag{42}$$

$$\xi_0 = \frac{6\alpha\beta(a^2 + b^2 - c^2) + \alpha^3}{24\beta^2}, \quad u(\xi) = A_0 + B_1 \cot\left(\frac{\Phi(\xi)}{2}\right),$$

where a, b, c, α and β are arbitrary constants. By using of the (42) and **Family 1** we get

$$u_1(\xi) = \mp\frac{3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm (b + c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b - c} - \frac{\sqrt{c^2 - b^2 - a^2}}{b - c}\tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right)\right]^{-1}. \tag{43}$$

By using of the (42) and **Family 2** we get

$$u_2(\xi) = \mp\frac{3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm (b + c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b - c} + \frac{\sqrt{a^2 + b^2 - c^2}}{b - c}\tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right)\right]^{-1}. \tag{44}$$

By using of the (42) and **Family 3** we get

$$u_3(\xi) = \mp\frac{3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm b\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b} + \frac{\sqrt{a^2 + b^2}}{b}\tanh\left(\frac{\sqrt{a^2 + b^2}}{2}(\xi + C)\right)\right]^{-1}. \tag{45}$$

By using of the (42) and **Family 4** we get

$$u_4(\xi) = \mp\frac{3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm c\sqrt{-\frac{3}{2\beta}}\left[-\frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{c}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}(\xi + C)\right)\right]^{-1}. \tag{46}$$

By using of the (42) and **Family 5** we get

$$u_5(\xi) = -\frac{\alpha}{2\beta} \pm \sqrt{b^2 - c^2}\sqrt{-\frac{3}{2\beta}}\coth\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right). \tag{47}$$

By using of the (42) and **Family 6** we get

$$u_6(\xi) = -\frac{\alpha}{2\beta} \pm b\sqrt{-\frac{3}{2\beta}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}\right]\right). \tag{48}$$

By using of the (42) and **Family 8** we get

$$u_7(\xi) = \mp\frac{3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \mp \sqrt{-\frac{3}{2\beta}}\frac{a^2(\xi + C)}{(a(\xi + C) + 2)}, \tag{49}$$

where $\xi = x + \frac{2\beta(a^2+b^2-c^2)+\alpha^2}{4\beta}t$.

Set II:

$$\mu = -\frac{2\beta(a^2 + b^2 - c^2) + \alpha^2}{4\beta}, \quad A_0 = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}}, \quad B_1 = 0, \quad A_1 = \pm(b - c)\sqrt{-\frac{3}{2\beta}}, \tag{50}$$

$$\xi_0 = \frac{6\alpha\beta(a^2 + b^2 - c^2) + \alpha^3}{24\beta^2}, \quad u(\xi) = A_0 + A_1\tan\left(\frac{\Phi(\xi)}{2}\right),$$

where a, b, c, α and β are arbitrary constants. By using of the (50) and **Family 1** we get

$$u_8(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm (b - c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b - c} - \frac{\sqrt{c^2 - b^2 - a^2}}{b - c}\tan\left(\frac{\sqrt{c^2 - b^2 - a^2}}{2}(\xi + C)\right)\right]. \tag{51}$$

By using of the (50) and **Family 2** we get

$$u_9(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm (b - c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b - c} + \frac{\sqrt{a^2 + b^2 - c^2}}{b - c}\tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right)\right]. \tag{52}$$

By using of the (50) and **Family 3** we get

$$u_{10}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm b\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b} + \frac{\sqrt{a^2 + b^2}}{b}\tanh\left(\frac{\sqrt{a^2 + b^2}}{2}(\xi + C)\right)\right]. \tag{53}$$

By using of the (50) and **Family 4** we get

$$u_{11}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \mp c\sqrt{-\frac{3}{2\beta}}\left[-\frac{a}{c} + \frac{\sqrt{c^2 - a^2}}{c}\tan\left(\frac{\sqrt{c^2 - a^2}}{2}(\xi + C)\right)\right]. \tag{54}$$

By using of the (50) and **Family 5** we get

$$u_{12}(\xi) = -\frac{\alpha}{2\beta} \pm \sqrt{b^2 - c^2}\sqrt{-\frac{3}{2\beta}}\tanh\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right). \tag{55}$$

By using of the (50) and **Family 6** we get

$$u_{13}(\xi) = -\frac{\alpha}{2\beta} \pm b\sqrt{-\frac{3}{2\beta}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right). \tag{56}$$

By using of the (50) and **Family 8** we get

$$u_{14}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \mp \sqrt{-\frac{3}{2\beta}}\frac{(b^2-c^2)(a(\xi+C)+2)}{a^2(\xi+C)}, \tag{57}$$

where $\xi = x + \frac{2\beta(a^2+b^2-c^2)+\alpha^2}{4\beta}t$.

Set III:

$$\mu = -\frac{2\beta(2c^2+a^2-2b^2)+\alpha^2}{4\beta}, \quad A_0 = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}}, \quad A_1 = \pm(b-c)\sqrt{-\frac{3}{2\beta}}, \quad B_1 = \mp(b+c)\sqrt{-\frac{3}{2\beta}},$$

$$\xi_0 = \frac{24a\beta^2(b^2-c^2)\sqrt{-\frac{3}{2\beta}} + 6\alpha\beta(a^2-2b^2+2c^2)+\alpha^3}{24\beta^2}, \quad u(\xi) = A_0 + A_1\tan\left(\frac{\Phi(\xi)}{2}\right) + B_1\cot\left(\frac{\Phi(\xi)}{2}\right), \tag{58}$$

where a, b, c, α and β are arbitrary constants. By using of the (58) and **Family 1** we get

$$u_{15}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm (b-c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b-c} - \frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right] \tag{59}$$

$$\mp(b+c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b-c} - \frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-1}.$$

By using of the (58) and **Family 2** we get

$$u_{16}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm (b-c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b-c} + \frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right] \tag{60}$$

$$\mp(b+c)\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b-c} + \frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right].$$

By using of the (58) and **Family 3** we get

$$u_{17}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \pm b\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b} + \frac{\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right] \tag{61}$$

$$\mp b\sqrt{-\frac{3}{2\beta}}\left[\frac{a}{b} + \frac{\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-1}.$$

By using of the (58) and **Family 4** we get

$$u_{18}(\xi) = \mp\frac{-3a \pm \alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}} \mp c\sqrt{-\frac{3}{2\beta}}\left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right] \tag{62}$$

$$\pm c\sqrt{-\frac{3}{2\beta}}\left[-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1}.$$

By using of the (58) and **Family 5** we get

$$u_{19}(\xi)=-\frac{\alpha}{2\beta}\pm\sqrt{b^2-c^2}\sqrt{-\frac{3}{2\beta}}\left[\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right]. \tag{63}$$

By using of the (58) and **Family 6** we get

$$u_{20}(\xi)=-\frac{\alpha}{2\beta}\pm b\sqrt{-\frac{3}{2\beta}}\left\{\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right. \tag{64}$$
$$\left.-\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right\}.$$

By using of the (58) and **Family 8** we get

$$u_{21}(\xi)=\mp\frac{-3a\pm\alpha\sqrt{-\frac{3}{2\beta}}}{2\beta\sqrt{-\frac{3}{2\beta}}}\mp\sqrt{-\frac{3}{2\beta}}\left[\frac{(b^2-c^2)(a(\xi+C)+2)}{a^2(\xi+C)}+\frac{a^2(\xi+C)}{(a(\xi+C)+2)}\right], \tag{65}$$

where $\xi=x+\frac{2\beta(a^2+2c^2-2b^2)+\alpha^2}{4\beta}t$.

Set IV:

$$\mu=\frac{2a^2-b^2+c^2}{2},\quad A_0=0,\quad A_1=0,\quad B_1=\frac{(b+c)\alpha}{2a\beta},\quad \xi_0=\frac{\alpha(b^2-c^2)}{4\beta},\quad u(\xi)=B_1\cot\left(\frac{\Phi(\xi)}{2}\right), \tag{66}$$

where a, b, c, α and β are arbitrary constants. By using of the (66) and **Family 1** we get

$$u_{22}(\xi)=\frac{(b+c)\alpha}{2a\beta}\left[\frac{a}{b-c}-\frac{\sqrt{c^2-b^2-a^2}}{b-c}\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right]^{-1}. \tag{67}$$

By using of the (66) and **Family 2** we get

$$u_{23}(\xi)=\frac{(b+c)\alpha}{2a\beta}\left[\frac{a}{b-c}+\frac{\sqrt{a^2+b^2-c^2}}{b-c}\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right]^{-1}. \tag{68}$$

By using of the (66) and **Family 3** we get

$$u_{24}(\xi)=\frac{b\alpha}{2a\beta}\left[\frac{a}{b}+\frac{\sqrt{a^2+b^2}}{b}\tanh\left(\frac{\sqrt{a^2+b^2}}{2}(\xi+C)\right)\right]^{-1}. \tag{69}$$

By using of the (66) and **Family 4** we get

$$u_{25}(\xi)=\frac{c\alpha}{2a\beta}\left[-\frac{a}{c}+\frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}(\xi+C)\right)\right]^{-1}. \tag{70}$$

By using of the (66) and **Family 5** we get

$$u_{26}(\xi)=\frac{\alpha\sqrt{b^2-c^2}}{2a\beta}\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right). \tag{71}$$

By using of the (66) and **Family 6** we get

$$u_{27}(\xi)=\frac{(b+c)\alpha}{2a\beta}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right). \tag{72}$$

By using of the (66) and **Family 8** we get

$$\mathrm{u}_{28}(\xi)=\frac{\alpha}{2\beta}\frac{\mathrm{a}(\xi+\mathrm{C})}{(\mathrm{a}(\xi+\mathrm{C})+2)}, \tag{73}$$

where $\xi=\mathrm{x}+\frac{(2\mathrm{a}^2-\mathrm{b}^2+\mathrm{c}^2)}{2}\mathrm{t}$.

Set V:

$$\mu=\mathrm{a}^2\pm\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)},\quad \mathrm{A}_0=-\frac{18\alpha}{\beta[\mathrm{b}^2-\mathrm{c}^2+\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}]},\quad \mathrm{B}_1=\frac{18\alpha[1\pm\sqrt{3(\mathrm{b}^2-\mathrm{c}^2)}]}{(\mathrm{b}-\mathrm{c})\beta[\mathrm{b}^2-\mathrm{c}^2+\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}]},$$
$$\mathrm{A}_1=0,\quad \xi_0=-\frac{6\alpha(3\mathrm{a}^2+\mathrm{c}^2-\mathrm{b}^2)}{\beta[\mathrm{b}^2-\mathrm{c}^2+\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}]},\quad \mathrm{u}(\xi)=\mathrm{A}_0+\mathrm{B}_1\cot\left(\frac{\Phi(\xi)}{2}\right), \tag{74}$$

where $\mathrm{a},\mathrm{b},\mathrm{c},\alpha$ and β are arbitrary constants. By using of the (74) and **Family 1** we get

$$\mathrm{u}_{29}(\xi)=-\frac{18\alpha}{\beta[\mathrm{b}^2-\mathrm{c}^2+\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}]}\left\{1-\frac{[1\pm\sqrt{3(\mathrm{b}^2-\mathrm{c}^2)}]}{\mathrm{b}-\mathrm{c}}\left[\frac{\mathrm{a}}{\mathrm{b}-\mathrm{c}}-\frac{\sqrt{\mathrm{c}^2-\mathrm{b}^2-\mathrm{a}^2}}{\mathrm{b}-\mathrm{c}}\tan\left(\frac{\sqrt{\mathrm{c}^2-\mathrm{b}^2-\mathrm{a}^2}}{2}(\xi+\mathrm{C})\right)\right]^{-1}\right\}. \tag{75}$$

By using of the (74) and **Family 2** we get

$$\mathrm{u}_{30}(\xi)=-\frac{18\alpha}{\beta[\mathrm{b}^2-\mathrm{c}^2+\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}]}\left\{1-\frac{[1\pm\sqrt{3(\mathrm{b}^2-\mathrm{c}^2)}]}{\mathrm{b}-\mathrm{c}}\left[\frac{\mathrm{a}}{\mathrm{b}-\mathrm{c}}+\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}}{\mathrm{b}-\mathrm{c}}\tanh\left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2}}{2}(\xi+\mathrm{C})\right)\right]^{-1}\right\}. \tag{76}$$

By using of the (74) and **Family 3** we get

$$\mathrm{u}_{31}(\xi)=-\frac{18\alpha}{\beta[\mathrm{b}^2+\sqrt{3\mathrm{a}^2\mathrm{b}^2}]}\left\{1-\frac{[1\pm\sqrt{3\mathrm{b}^2}]}{\mathrm{b}}\left[\frac{\mathrm{a}}{\mathrm{b}}+\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{b}}\tanh\left(\frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{2}(\xi+\mathrm{C})\right)\right]^{-1}\right\}. \tag{77}$$

By using of the (74) and **Family 4** we get

$$\mathrm{u}_{32}(\xi)=-\frac{18\alpha}{\beta[-\mathrm{c}^2+\sqrt{-3\mathrm{a}^2\mathrm{c}^2}]}\left\{1+\frac{[1\pm\sqrt{-3\mathrm{c}^2}]}{\mathrm{c}}\left[-\frac{\mathrm{a}}{\mathrm{c}}+\frac{\sqrt{\mathrm{c}^2-\mathrm{a}^2}}{\mathrm{c}}\tan\left(\frac{\sqrt{\mathrm{c}^2-\mathrm{a}^2}}{2}(\xi+\mathrm{C})\right)\right]^{-1}\right\}. \tag{78}$$

By using of the (74) and **Family 8** we get

$$\mathrm{u}_{33}(\xi)=-\frac{18\alpha}{\beta[\mathrm{b}^2-\mathrm{c}^2+\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}]}\left\{1+\frac{\mathrm{a}^2[1\pm\sqrt{3(\mathrm{b}^2-\mathrm{c}^2)}](\xi+\mathrm{C})}{(\mathrm{b}^2-\mathrm{c}^2)[\mathrm{a}(\xi+\mathrm{C})+2]}\right\}, \tag{79}$$

where $\xi=\mathrm{x}-\left(\mathrm{a}^2\pm\sqrt{3\mathrm{a}^2(\mathrm{b}^2-\mathrm{c}^2)}\right)\mathrm{t}$.

4 Conclusions

The new $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion method was successfully used to establish solitary solutions, triangular functions solutions and rational solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear wave equations. The availability of computer systems like Maple facilitates the tedious algebraic calculations. This paper has shown that the method is sufficient incentive to seek more new exact soliton solutions of NEEs in mathematical physics. We found in this work that the obtained results for aforementioned equations give very good results even further of applied method in ([7]). The performance of this method is reliable and effective and gives more solutions. Also, new results are formally developed in this article. It can be concluded that this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.

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