ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.22(2016) No.1, pp.9-18

Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces Using Common Limit in the Range Property

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(Received 9 July 2014, accepted 15 August 2015)

Abstract: The aim of this work is to introduce the new property which is so called common limit in the range for four self-mappings and give some examples of mappings which satisfy this property. Moreover, we establish some new existence of a common fixed point theorem for generalized contractive mappings in fuzzy metric spaces by using new property and give some examples. Our results do not require the condition of closeness of range and so our theorems generalize, unify and extend many results in the literature.

Keywords: weakly compatible Maps; fuzzy metric space; property (E.A); common property (E.A); CLRg property; common limit in range property.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [28] in 1965. In 1975, Kramosil and Michalek [14] gave the notion of fuzzy metric spaces which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [17]. On the other hand, fixed point theory is one of the most famous mathematical theories with applications in several branches of science. Fixed point theory in fuzzy metric spaces has been developed starting with the work of Heilpern [9]. He introduced the concept of fuzzy contraction mappings and proved some fixed point theorems for fuzzy contraction mappings in metric linear spaces, which is a fuzzy extension of the Banach contraction principle. In [6, 7], George and Veeramani introduced and studied the notion of fuzzy metric spaces which constitutes a modification of the one due to Kramosil and Michalek. From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. Many authors have contributed to the development of this theory and apply to fixed point theory, for instance [1,3-5,8,10,11,15,16,18- 20,22,24-27].

In 1976, Jungck [12] introduced the notion of commuting mappings. Afterward, Sessa [23] gave the notion of weakly commuting mappings. Jungck [13] defined the notion of compatible mappings to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true. The concept of property (E.A) in metric space has been recently introduced by Aamri and El Moutawakil [2]. The concept of property (E.A) allows us to replace the completeness requirement of the space with a more natural condition of closeness of the range. In 2009, M. Abbas et al. [1] introduced the notion of common property (E.A).

Recently in 2011, Sintunavarat and Kumam [25] introduced the concept of the common limit in the range property and also established the existence of common fixed point theorems for generalized contractive mappings satisfy this property in fuzzy metric spaces. The aim of this work is to introduce the new property which is the so called common limit in the range?for four self-mappings and give some examples of mappings which satisfy this property. Moreover, we establish some new existence of a common fixed point theorem for generalized contractive mappings in fuzzy metric spaces by using new property and give some examples. Ours results do not require the condition of closeness of range and so our theorems generalize, unify and extend many results in the literature.

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IJNS.2016.08.15/917

2 Preliminaries

The concept of triangular norms (t-norms) is originally introduced by Menger [19] in study of statistical metric spaces.

Definition 1 [23]. *A binary operation* $*: [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangular norm* (t*-norm) if it satisfies the following conditions:*

(i) $*$ *is associative and commutative;*

(ii) $*$ *is continuous;*

(iii) $a * 1 = a$ *for all* $a \in [0,1]$*;*

(iv) $a * b \leq c * d$*, whenever* $a \leq c$ *and* $b \leq d$*, for all* $a, b, c, d \in [0,1]$.

Examples of continuous t-norms are $a * b = \min\{a, b\}, a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$.

Definition 2 [16]. *A fuzzy metric space is a triple* $(X, M, *)$*, where* X *is a non-empty set,* $*$ *is a continuous* t*-norm and* M *is a fuzzy set on* $X \times X \times [0, +\infty)$*, satisfying the following properties:*

1. *(KM-1)* $M(x, y, 0) = 0$ *for all* $x, y \in X$*;*
2. *(KM-2)* $M(x, y, t) = 1$ *for all* $t > 0$ *iff* $x = y$*;*
3. *(KM-3)* $M(x, y, t) = M(y, x, t)$ *for all* $x, y \in X$ *and for all* $t > 0$*;*
4. *(KM-4)* $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ *is left continuous for all* $x, y \in X$*;*
5. *(KM-5)* $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ *for all* $x, y, z \in X$ *and for all* $t, s > 0$.

The function $M(x, y, t)$ denote the degree of nearness between x and y w.r.t. t respectively.

Lemma 1 [16]. *In a* KM*-fuzzy metric space* $(X, M, *)$*,* $M(x, y, \cdot)$ *is non-decreasing for all* $x, y \in X$.

Definition 3 [7].*Let* $(X, M, *)$ *be a fuzzy metric space. Then a sequence* $\{x_n\}_{n \in \mathbb{N}}$ *is said to be convergent to* $x \in X$*, that is,* $\lim_{n \to +\infty} x_n = x$*, if* $\lim_{n \to +\infty} M(x_n, x, t) = 1$ *for all* $t > 0$.

Definition 4 [15]. *Two self-mappings* f *and* g *of a fuzzy metric space* $(X, M, *)$ *are said to be compatible if* $\lim_{n \to +\infty} M(fgx_n, gfx_n) = 1$ *for all* $t > 0$*, whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n \to +\infty} f(x_n) = \lim_{n \to +\infty} gx_n = z$ *for some* $z \in X$.

Definition 5 [15]. *Two self-mappings* f *and* g *of a fuzzy metric space* $(X, M, *)$ *are said to be non-compatible if there exists at least one sequence* $\{x_n\}$ *in* X *such that* $\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = z$ *for some* $z \in X$*, but for some* $t > 0$*, either* $\lim_{n \to +\infty} M(fgx_n, gfx_n) \neq 1$ *or the limit does not exist.*

Definition 6 [9]. *A pair* (f, g) *of self-mappings of a non-empty set* X *is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if* $fz = gz$ *for some* $z \in X$*, then* $fgz = gfz$.

If two self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible then they are weakly compatible but the converse need not be true.

Definition 7 [2]. *A pair* (f, g) *of self-mappings of a fuzzy metric space* $(X, M, *)$ *is said to satisfy the property (E.A) if there exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = z,$$

for some $z \in X$.

From Definition 7, it is easy to see that any two non-compatible self-mappings of a fuzzy metric space $(X, M, *)$ satisfy the property (E.A) but the reverse need not be true.

Definition 8 [1]. *Two pairs (A, S) and (B, T) of self-mappings of a fuzzy metric space $(X, M, *)$ are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that*

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

for some $z \in X$.

Definition 9 [27]. *A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to satisfy the common limit in the range of g property (CLR_g, for short) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = gz,$$

for some $z \in X$.

Inspired by Sintunavarat et al. [27], We introduce the following notion:

Definition 10 *Two pairs (A, S) and (B, T) of self-mappings of a fuzzy metric space $(X, M, *)$ are said to share the common limit in the range of S property if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that*

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = Sz,$$

for some $z \in X$.

Example 2 *Let $(X, M, *)$ be a fuzzy metric space with $X = [-1, 1]$ and for all $x, y \in X$*

$$M(x, y, t) = \begin{cases} e^{\frac{-|x-y|}{t}}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

Define self mappings A, B, S and T on X by $Ax = \frac{x}{3}, Bx = \frac{-x}{3}, Sx = x, Tx = -x$ for all $x \in X$. Then with sequences $x_n = \frac{1}{n}$ and $y_n = \frac{-1}{n}$ in X, one can easily verify that
$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = S(0)$.
This shows that the pairs (A, S) and (B, T) share the common limit in the range of S property.

Definition 11 [9]. *Two families of self-mappings $\{A_i\}$ and $\{S_j\}$ are said to be pairwise commuting if:*

1. $A_iA_j = A_jA_i$, $i, j \in \{1, 2, \ldots, m\}$,
2. $S_iS_j = S_jS_i$, $i, j \in \{1, 2, \ldots, n\}$,
3. $A_iS_j = S_jA_i$, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$.

Definition 12 *Let Ψ_4 be the set of all continuous functions $F : [0, 1]^4 \to R$(set of real numbers) satisfying the following conditions:*
(F_1) *For every $u > 0, v \geq 0$ with $F(u, v, u, v) \geq 0$ or $F(u, v, v, u) \geq 0$, we have $u > v$,*
(F_2) *$F(u, u, 1, 1,) < 0$ for all $u \in (0, 1)$.*

Example 3 *Define $F : [0, 1]^4 \to R$ as $F(t_1, t_2, t_3, t_4) = t_1 - \varphi(min\{t_2, t_3, t_4\})$ where $\varphi : [0, 1] \to [0, 1]$ is increasing and continuous function such that $\varphi(t) > t$ for all $t \in (0, 1)$.*

Example 4 *Define $F : [0, 1]^4 \to R$ as $F(t_1, t_2, t_3, t_4) = \int_0^{t_1} \phi(s)\, ds - \varphi(\int_0^{\min\{t_2,t_3,t_4\}} \phi(s)\, ds)$*
where $\varphi : [0, 1] \to [0, 1]$ is increasing and continuous function such that $\varphi(t) > t$ for all $t \in (o, 1)$ and $\phi : R^+ \to R^+$ is a Lebesgue integrable function which is summable and satisfies $0 < \int_0^{\epsilon} \phi(s)\, ds < 1$ for all $0 < \epsilon < 1$ and $\int_0^1 \phi(s)\, ds = 1$.

Example 5 *Define $F : [0, 1]^4 \to R$ as*
$F(t_1, t_2, t_3, t_4) = \int_0^{t_1} \phi(s)\, ds - \varphi(min\{\int_0^{t_2} \phi(s)\, ds, \int_0^{t_3} \phi(s)\, ds, \int_0^{t_4} \phi(s)\, ds\})$
where $\varphi : [0, 1] \to [0, 1]$ is increasing and continuous function such that $\varphi(t) > t$ for all $t \in (0, 1)$ and $\phi : R^+ \to R^+$ is a Lebesgue integrable function which is summable and satisfies $0 < \int_0^{\epsilon} \phi(s)\, ds < 1$ for all $0 < \epsilon < 1$ and $\int_0^1 \phi(s)\, ds = 1$

3 Main Results

Lemma 6 *Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying the following:*
(i) the pair (A, S) satisfies the common limit in the range of S property or ((B, T) satisfies the common limit in the range of T property);
(ii) for any $x, y \in X$, F in Ψ_4 and $t > 0$ such that
(3.1) $F(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)) \geq 0$;
(iii) $A(X) \subseteq T(X)(or B(X) \subseteq S(X))$.
Then the pairs (A, S) and (B, T) share the common limit in the range of S property (or T property).

Proof. Suppose that the pair (A, S) satisfies the common limit in the range of S property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Sz$ for some point $z \in X$. Then as $A(X) \subseteq T(X)$, hence for each $\{x_n\}$, there exist $\{y_n\}$ in X such that $A(x_n) = T(y_n)$. Therefore,$\lim_{n\to\infty} Ty_n = Sz$. Thus in all, we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = Sz$. Now, we assert that $\lim_{n\to\infty} By_n = Sz$. Suppose not, then applying inequality 6, we get
$F(M(Ax_n, By_n, t), M(Sx_n, Ty_n, t), M(Sx_n, Ax_n, t), M(Ty_n, By_n, t)) \geq 0$
taking $n \to \infty$, we obtain
$F(M(Sz, \lim_{n\to\infty} By_n, t), M(Sz, Sz, t), M(Sz, Sz, t), M(Sz, \lim_{n\to\infty} By_n, t)) \geq 0$
$F(M(Sz, \lim_{n\to\infty} By_n, t), 1, 1, M(Sz, \lim_{n\to\infty} By_n, t)) \geq 0$
By F_2, we get $M(Sz, \lim_{n\to\infty} By_n, t) > 1$, which is contradiction and therefore, $\lim_{n\to\infty} By_n = Sz$. Then the pairs (A, S) and (B, T) share the common limit in the range of S property. (Similarly, we prove that if (B, T) satisfies common limit in the range of T property then the pairs (A, S) and (B, T) share the common limit in the range of T property.)
■

Theorem 7 *Let A, B, S and T be self mappings of a Fuzzy metric space $(X, M, *)$ satisfying inequality 6. Suppose that (i) the pair (A, S) satisfies the common limit in the range of S property or (or (B, T) satisfies the common limit in the range of T property);*
(ii) $A(X) \subseteq T(X)(or B(X) \subseteq S(X))$.
Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. In view of Lemma 6, the pairs (A, S) and (B, T) share the common limit in the range of S property, that is there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Sz$
for some $z \in X$.
Firstly, we assert that $Az = Sz$.
Suppose not, then by 6, we have
$F(M(Az, By_n, t), M(Sz, Ty_n, t), M(Sz, Az, t), M(Ty_n, By_n, t)) \geq 0$
taking $n \to \infty$, we obtain
$F(M(Az, Sz, t), M(Sz, Sz, t), M(Sz, Az, t), M(Sz, Sz, t)) \geq 0$
$F(M(Az, Sz, t), 1, M(Sz, Az, t), 1) \geq 0$
By F_1, we get $M(Sz, Az, t) > 1$ which gives contradiction and therefore, $Az = Sz$. Since, $A(X) \subseteq T(X)$, there exist $v \in X$ such that $Az = Tv$.
Secondly, we assert that $Bv = Tv$. Suppose not, then by 6, we get
$F(M(Az, Bv, t), M(Sz, Tv, t), M(Sz, Az, t), M(Tv, Bv, t)) \geq 0$
$F(M(Tv, Bv, t), M(Tv, Tv, t), M(Tv, Tv, t), M(Tv, Bv, t)) \geq 0$
$F(M(Tv, Bv, t), 1, 1, M(Tv, Bv, t)) \geq 0$
Again by using F_1, we get $M(Tv, Bv, t) > 1$, a contradiction and therefore $Tv = Bv = Az = Sz$. Since the pairs (A, S) and (B, T) are weakly compatible and $Az = Sz$ and $Tv = Bv$, therefore, $ASz = SAz = AAz = SSz, BTv = TBv = TTv = BBv$.
Finally, we assert that $AAz = Az$. Suppose not, then again by 6, we have
$F(M(AAz, Bv, t), M(SAz, Tv, t), M(SAz, AAz, t), M(Tv, Bv, t)) \geq 0$
$F(M(AAz, Bv, t), M(AAz, Bv, t), M(AAz, AAz, t), M(Bv, Bv, t)) \geq 0$
$F(M(AAz, Az, t), M(AAz, Az, t), 1, 1) \geq 0$
which is contradiction to F_2 and therefore $AAz = Az = SAz$ which gives, Az is common fixed point of A and S.

Similarly, one can easily prove that $BBv = Bv = TBv$, that is Bv is common fixed point of B and T. As $Az = Bv$, therefore Az is common fixed point of A, B, S and T. The uniqueness of common fixed point is an easy consequence of inequality 6.

■ By choosing A, B, S and T suitably, one can derive corollaries involving two or three mappings.

Corollary 8 *Let A and S be self mappings of a Fuzzy metric space $(X, M, *)$ satisfying:*
(i) the pair (A, S) satisfies the common limit in the range of S property;
(ii) $A(X) \subseteq T(X)$;
(iii)$F(M(Ax, Ay, t), M(Sx, Sy, t), M(Sx, Ax, t), M(Sy, Ay, t)) \geq 0$.
Then A and S have a point of coincidence. Moreover, A and S have a unique common fixed point provided that A and S are weakly compatible.

Proof. Taking $B = A$ and $T = S$ in Theorem 7. ■

Corollary 9 *Let A, B, S and T be self mappings of a Fuzzy metric space (X,M,*) satisfying inequality 6. Suppose that*
(i) the pairs (A, S) and (B, T) satisfies the common limit in the range of S property;
(ii) $A(X) \subseteq T(X)(or B(X) \subseteq S(X))$. Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Proof easily follows on same lines of Theorem 7, using Lemma 6. ■

Theorem 10 *Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying inequality 6. Suppose that*
(i) the pair (A, S) (or (B, T) satisfies the common limit in the range of T) satisfies property (E.A.) and $S(X)$ is a closed subspace of X;
(ii) $A(X) \subseteq T(X)(or B(X) \subseteq S(X))$.
Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Suppose pair (A, S) satisfy property (E.A.), there exist a sequence x_n in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some $p \in X$. It follows from $S(X)$ is a closed subspace of X that $p = Sz$ for some $z \in X$ and then the pair (A, S) satisfies the common limit in the range of S property. By Theorem 7, we get A, B, S and T have a unique common fixed point. ■

Corollary 11 *Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying inequality 6. Suppose that*
(i) the pairs (A, S) and (B, T) satisfies common property (E.A.) and $S(X)$ is a closed subspace of X;
(ii) $A(X) \subseteq T(X)(or B(X) \subseteq S(X))$.
Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) satisfies common property (E.A.), there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = p$
for some $p \in X$. It follows from $S(X)$ is a closed subspace of X that $p = Sz$ for some $z \in X$ and then the pairs (A, S) and (B, T) share the common limit in the range of S property. By Theorem 7, we get A, B, S and T have a unique common fixed point. ■ Since the pair of non compatible mappings implies to the pair satisfying property (E.A.), we get the following corollary.

Corollary 12 *Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying inequality 6. Suppose that*
(i) the pair (A, S) (or (B, T)) is non compatible mappings and $S(X)$ is a closed subspace of X;
(ii) $A(X) \subseteq T(X)(or B(X) \subseteq S(X))$.
Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pair (A, S) is non compatible mappings, we get A and S satisfy property (E.A.). Therefore, by Theorem 10, we get A, B, S and T have a unique common fixed point in X.

■ Although property (E.A.) (common property (E.A.)) is an essential tool to claim the existence of common fixed points of some mappings. However this property require the condition of closedness of $S(X)$. Note that Theorem 7

weaken the condition of closed sunspace of $S(X)$. Therefore it is most interesting to used common limit in the range of S property as another auxiliary tool to claim the existence of a common fixed point. However, all the main results in this paper are some of the choices for claim that the existence of common fixed point in Fuzzy metric spaces. Our result may be the motivation to other authors for extending and improving these results to suitable tools for these problems.
As an application of Theorems 7, we prove a common fixed point theorem for four finite families of mappings on fuzzy metric spaces. While proving our result, we utilize Definition 11 which is a natural extension of commutativity condition to two finite families.

Theorem 13 *Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_n\}$, $\{S_1, S_2, \ldots, S_p\}$ and $\{T_1, T_2, \ldots, T_q\}$ be four finite families of self-mappings of a KM-fuzzy metric space $(X, M, *)$ such that $A = A_1A_2 \cdots A_m$, $B = B_1B_2 \cdots B_n$, $S = S_1S_2 \cdots S_p$ and $T = T_1T_2 \cdots T_q$ satisfy the condition 6 of Theorem 7. Then*

(a) the pairs (A, S) and (B, T) have a point of coincidence each;

(b) $\{A_i\}$, $\{B_j\}$, $\{S_k\}$ and $\{T_r\}$ have a unique common fixed point provided that the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_j\}, \{T_r\})$ commute pairwise, for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, p$ and $r = 1, \ldots, q$.

Proof. Since the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_j\}, \{T_r\})$ commute pairwise, we first show that $AS = SA$. In fact, we have

$$\begin{aligned} AS &= (A_1A_2 \cdots A_m)(S_1S_2 \cdots S_p) \\ &= (A_1A_2 \cdots A_{m-1})(A_mS_1S_2 \cdots S_p) \\ &= (A_1A_2 \cdots A_{m-1})(S_1S_2 \cdots S_pA_m) \\ &= (A_1A_2 \cdots A_{m-2})(A_{m-1}S_1S_2 \cdots S_pA_m) \\ &= (A_1A_2 \cdots A_{m-2})(S_1S_2 \cdots S_pA_{m-1}A_m) \\ &= \ldots \\ &= A_1(S_1S_2 \cdots S_pA_2 \cdots A_m) \\ &= (S_1S_2 \cdots S_p)(A_1A_2 \cdots A_m) = SA. \end{aligned}$$

Similarly one can prove that $BT = TB$, and hence, obviously the pair (A, S) is compatible and (B, T) is weakly compatible. Now, using Theorem 7, we conclude that A, S, B and T have a unique common fixed point in X, say z.
Now, we need to prove that z remains the fixed point of all the component mappings. To this aim, consider

$$\begin{aligned} A(A_iz) &= ((A_1A_2 \cdots A_m)A_i)z = (A_1A_2 \cdots A_{m-1})(A_mA_i)z \\ &= (A_1A_2 \cdots A_{m-1})(A_iA_m)z = (A_1A_2 \cdots A_{m-2})(A_{m-1}A_iA_m)z \\ &= (A_1A_2 \cdots A_{m-2})(A_iA_{m-1}A_m)z = \cdots = A_1(A_iA_2 \cdots A_m)z \\ &= (A_1A_i)(A_2 \cdots A_m)z = (A_iA_1)(A_2 \cdots A_m)z \\ &= A_i(A_1A_2 \cdots A_m)z = A_iAz = A_iz. \end{aligned}$$

Similarly, one can prove that $A(S_kz) = S_k(Az) = S_kz$, $S(S_kz) = S_k(Sz) = S_kz$, $S(A_iz) = A_i(Sz) = A_iz$, $B(B_jz) = Bj(Bz) = B_jz$, $B(T_rz) = T_r(Bz) = T_rz$, $T(T_rz) = T_r(Tz) = T_rz$ and $T(B_jz) = B_j(Tz) = B_jz$, which show that (for all i, j, k and r) A_iz and S_kz are other fixed points of the pair (A, S) whereas B_jz and T_rz are other fixed points of the pair (B, T). Since A, B, S and T have a unique common fixed point, then we get $z = A_iz = S_kz = B_jz = T_rz$, for all $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, p$ and $r = 1, \ldots, q$. Thus z is the unique common fixed point of $\{A_i\}$, $\{B_j\}$, $\{S_k\}$ and $\{T_r\}$. ■

Our next result involves a lower semi continuous function $\psi : [0, 1] \to [0, 1]$ such that $\psi(t) > t$ for all $t \in (0, 1)$ along with $\psi(0) = 0$ and $\psi(1) = 1$.

Theorem 14 *Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$ satisfying the conditions $\{i\}$ and $\{ii\}$ of Theorem 7 and for all $x, y \in X, t > 0$*
(3.2)

$$\int_0^{M(Ax,By,t)} \phi(s)\, ds \geq \psi(\int_0^{\min\{M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t)\}} \phi(s)\, ds)$$

where $\phi : R^+ \to R^+$ is a Lebesgue integrable function which is summable and satisfies $0 < \int_0^\epsilon \phi(s)\,ds < 1$ for all $0 < \epsilon < 1$ and $\int_0^1 \phi(s)\,ds = 1$.
Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Suppose that the pair (A, S) satisfies the common limit in the range of S property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Sz$ for some point $z \in X$. Then as $A(X) \subseteq T(X)$, hence for each $\{x_n\}$, there exist $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, $\lim_{n\to\infty} Ty_n = Sz$. Thus in all, we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = Sz$. Now, we assert that $\lim_{n\to\infty} By_n = Sz$. Suppose not, then applying inequality 14, we get

$$\int_0^{M(Ax_n,By_n,t)} \phi(s)\,ds \geq \psi(\int_0^{\min\{M(Sx_n,Ty_n,t),M(Sx_n,Ax_n,t),M(Ty_n,By_n,t)\}} \phi(s)\,ds)$$

taking $n \to \infty$, we obtain

$$\begin{aligned}\int_0^{M(Sz,\lim_{n\to\infty} By_n,t)} \phi(s)\,ds &\geq \psi(\int_0^{\min\{M(Sz,Sz,t),M(Sz,Az,t),M(Sz,\lim_{n\to\infty} By_n,t)\}} \phi(s)\,ds)\\ &= \psi(\int_0^{\min\{1,1,M(Sz,\lim_{n\to\infty} By_n,t)\}} \phi(s)\,ds)\\ &= \psi(\int_0^{M(Sz,\lim_{n\to\infty} By_n,t)} \phi(s)\,ds)\\ &> \int_0^{M(Sz,\lim_{n\to\infty} By_n,t)} \phi(s)\,ds\end{aligned}$$

a contradiction, therefore, $\lim_{n\to\infty} By_n = Sz$. Then the pairs (A, S) and (B, T) share the common limit in the range of S property , that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz$ for some $z \in X$.
Firstly, we assert that$Az = Sz$. Suppose not, then by 14, we have

$$\int_0^{M(Az,By_n,t)} \phi(s)\,ds \geq \psi(\int_0^{\min\{M(Sz,Ty_n,t),M(Sz,Az,t),M(Ty_n,By_n,t)\}} \phi(s)\,ds)$$

taking $n \to \infty$,

$$\begin{aligned}\int_0^{M(Az,Sz,t)} \phi(s)\,ds &\geq \psi(\int_0^{\min\{M(Sz,Sz,t),M(Sz,Az,t),M(Sz,Sz,t)\}} \phi(s)\,ds)\\ &= \psi(\int_0^{\min\{1,M(Sz,\lim_{n\to\infty} By_n,t),1\}} \phi(s)\,ds)\\ &= \psi(\int_0^{M(Sz,Az,t)} \phi(s)\,ds)\\ &> \int_0^{M(Sz,Az,t)} \phi(s)\,ds\end{aligned}$$

which is contradiction and therefore, $Az = Sz$.
Since, $A(X) \subseteq T(X)$, there exist $v \in X$such that $Az = Tv$.
Secondly, we assert that$Bv = Tv$. Suppose not, then by 14, we get

$$\int_0^{M(Az,Bv,t)} \phi(s)\,ds \geq \psi(\int_0^{\min\{M(Sz,Tv,t),M(Sz,Az,t),M(Tv,Bv,t)\}} \phi(s)\,ds)$$

$$\int_0^{M(Tv,Bv,t)} \phi(s)\,ds \geq \psi(\int_0^{\min\{M(Tv,Tv,t),M(Tv,Tv,t),M(Tv,Bv,t)\}} \phi(s)\,ds)$$

$$= \psi(\int_0^{\min\{1,1,M(Tv,Bv,t)\}} \phi(s)\, ds)$$

$$= \psi(\int_0^{M(Tv,Bv,t)} \phi(s)\, ds)$$

$$> \int_0^{M(Tv,Bv,t)} \phi(s)\, ds$$

a contradiction and therefore $Tv = Bv = Az = Sz$.
Since the pairs (A, S) and (B, T) are weakly compatible and $Az = Sz$ and $Tv = Bv$, therefore, $ASz = SAz = AAz = SSz, BTv = TBv = TTv = BBv$.
Finally, we assert that$AAz = Az$. Suppose not, then again by 14, we have

$$\int_0^{M(AAz,Bv,t)} \phi(s)\, ds \geq \psi(\int_0^{\min\{M(SAz,Tv,t),M(SAz,AAz,t),M(Tv,Bv,t)\}} \phi(s)\, ds)$$

$$\int_0^{M(AAz,Az,t)} \phi(s)\, ds \geq \psi(\int_0^{\min\{M(AAz,Bv,t),M(AAz,AAz,t),M(Bv,Bv,t)\}} \phi(s)\, ds)$$

$$= \psi(\int_0^{\min\{M(AAz,Az,t),1,1\}} \phi(s)\, ds)$$

$$= \psi(\int_0^{M(AAz,Az,t)} \phi(s)\, ds)$$

$$> \int_0^{M(AAz,Az,t)} \phi(s)\, ds$$

which is contradiction and therefore $AAz = Az = SAz$ which gives, Az is common fixed point of A and S. Similarly, one can easily prove that $BBv = Bv = TBv$, that is Bv is common fixed point of B and T. As $Az = Bv$, therefore Az is common fixed point of A, S, B and T. The uniqueness of common fixed point is an easy consequence of inequality 14.
■

Remark 15 *Notice that results similar to Theorems 10, 13 and corollaries 8, 9, 11, 12 can also be outlined in respect to Theorem 14, but we may omit the details with a view to avoid any repetition.*

Corollary 16 *The conclusions of Theorem 7 remain true if condition 6 is replaced by one of the following conditions:*
(i) $M(Ax, By, t) \geq \varphi(min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\})$*, where* $\varphi : [0, 1] \to [0, 1]$ *is increasing and continuous function such that* $\varphi(t) > t$ *for all* $t \in (0, 1)$*;*
(ii) $M(Ax, By, t) \geq k(min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\})$ *where* $k > 1$*;*
(iii) $M(Ax, By, t) \geq kM(Sx, Ty, t) + min\{M(Sx, Ax, t), M(Ty, By, t)\}$ *where* $k > 0$*;*
(iv) $M(Ax, By, t) \geq aM(Sx, Ty, t) + bM(Sx, Ax, t) + cM(Ty, By, t)$ *where* $a > 1$*,* $b, c \geq 0$ *and* $b, c \neq 1$*;*
(v) $M(Ax, By, t) \geq aM(Sx, Ty, t) + b[M(Sx, Ax, t) + M(Ty, By, t)]$ *, where* $a > 1$ *and* $0 \leq b < 1$*;*
(vi) $M(Ax, By, t) \geq kM(Sx, Ty, t)M(Sx, Ax, t)M(Ty, By, t)$ *where* $k > 1$.

Corollary 17 *Let* A, B, S *and* T *be self mappings of a fuzzy metric space* $(X, M, *)$ *satisfying the conditions* $\{i\}$ *and* $\{ii\}$ *of Theorem 7 and for all* $x, y \in X, t > 0$
(3.3)

$$\int_0^{\varphi\{M(Ax,By,t),M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t)\}} \phi(s)\, ds \geq 0$$

where $\phi : R^+ \to R^+$ *is a Lebesgue integrable function which is summable and satisfies* $0 < \int_0^\epsilon \phi(s)\, ds < 1$ *for all* $0 < \epsilon < 1$ *and* $\int_0^1 \phi(s)\, ds = 1$ *and a function* $\varphi : [0, 1] \to [0, 1]$ *such that*
$\int_0^{\varphi(u,1,u,1)} \phi(s)\, ds \geq 0$ *or* $\int_0^{\varphi(u,1,1,u)} \phi(s)\, ds \geq 0$ *or* $\int_0^{\varphi(u,u,1,1)} \phi(s)\, ds \geq 0$ *implies* $u = 1$.
Then the pairs (A, S) *and* (B, T) *have a point of coincidence each. Moreover,* A, B, S *and* T *have a unique common fixed point provided that both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Let Φ be the set of all continuous functions $\varphi : [0,1]^4 \to R$ satisfying the following conditions:
(i)for every $u, v \geq 0$ with $\varphi(u, v, v, u) \geq 0$ or $\varphi(u, v, u, v) \geq 0$, we have $u \geq v$;
(ii) $\varphi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 18 *Define $\varphi : [0,1]^4 \to R$ as $\varphi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$. Then $\varphi \in \Phi$.*

Corollary 19 *Let A, B, S and T be self mappings of a Fuzzy metric space $(X, M, *)$ satisfying the conditions $\{i\}$ and $\{ii\}$ of Theorem 7 and for all $x, y \in X, t > 0$*
(3.4)

$$\int_0^{\varphi\{M(Ax,By,t),M(Sx,Ty,t),M(Sx,Ax,t),M(Ty,By,t)\}} \phi(s)\, ds \geq 0$$

where $\phi : R^+ \to R^+$ is a Lebesgue integrable function which is summable and satisfies $0 < \int_0^{\epsilon} \phi(s)\, ds < 1$ for all $0 < \epsilon < 1$ and $\int_0^1 \phi(s)\, ds = 1$
and $\varphi \in \Phi$. Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible.

Example 20 *Let $(X, M, *)$ be a fuzzy metric space where $X = [0, 2)$ with a $t-$ norm defined by $a * b = min\{a, b\}$ and*

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

Define $F : [0,1]^4 \to R$ as $F(t_1, t_2, t_3, t_4) = t_1 - \varphi(min\{t_2, t_3, t_4\})$ where $\varphi : [0,1] \to [0,1]$ is increasing and continuous function such that $\varphi(t) > t$ for all $t \in (o, 1)$. Define A, B, S and T by $Ax = Bx = 1$, $S(x) = \begin{cases} 1, & \text{if } x \in Q; \\ \frac{2}{3}, & \text{if } otherwise \end{cases}$ and $T(x) = \begin{cases} 1, & \text{if } x \in Q; \\ \frac{1}{3}, & \text{if } otherwise \end{cases}$.

Clearly, the pairs (A, S) and (B, T) satisfies all conditions of Theorem 7 and shares the common limit in the range of S property and $A(X) \subseteq T(X)$.Hence, A, B, S and T have a unique common fixed point $x = 1$.

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