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On the Shape of Limit Cycles of LÜ system and Moon-Rand System

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Abstract: This paper studies the asymptotic expressions of the limit cycles for a new modified four-dimensional Lü system and three-dimensional Moon-Rand system. Firstly, by applying center manifold theory, we reduce the original systems to simplified two-dimensional systems. Secondly, with the help of the stability of the focus and Poincaré-Bendixson Theorem, we prove the existence of limit cycles in the new modified four-dimensional Lü system and the Moon-Rand system. Thirdly, we apply the method of Friedrich to get the first several terms of the asymptotic expansions of the limit cycles of the above two high dimension nonlinear differential systems.

Keywords: central manifold, Hopf bifurcation, limit cycle, stability of focus, Poincaré-Bendixson theorem.

1 Introduction

As we all know, many authors have paid intense attention to the number of limit cycles emerging from the bifurcation in the nonlinear dynamical systems in the past decades [1–5]. When solving problems of limit cycles of high dimensional dynamical systems, it is complex to study them directly. Based on that, a series of methods are introduced to solve these problems. For example, with the help of normal form theory and central manifold theory, we can simplify the system to a simple one in the neighborhood of the equilibrium point. The idea of center manifold theory is to simplify the system by reducing the original system to a center manifold which has smaller dimension than that of the original system. The main idea of normal form theory [6] is to utilize necessary coordinate transformations to construct a form of the original system as simple as possible. The new form is equivalent to the original system which is easy for us to discuss the properties of solutions of the original system. In details, for a high dimensional system whose Jacobian matrix of equilibrium only has negative eigenvalues and a pair of pure imaginary eigenvalues, we can obtain a two dimensional system with the aid of center manifold [8]. Poincaré-Bendixson Theorem (see [7] for details) is of great use in proving the existence of limit cycles in the planar system. The method of Fredich (see [8] for details) plays important role in solving the problems of existence and expressions of the period solutions of planar periodic differential equations.

In [9, 10], the following modified Lü system

$$\begin{cases} \dot{x} = -ax + ay + ayz, \\ \dot{y} = by + u - xz, \\ \dot{z} = -cz + xy, \\ \dot{u} = -(b^2 - ab + \varepsilon)x, \end{cases} \tag{1}$$

is considered, where x, y, z, u are state variables and a, b, c, ε are real parameters. The system (1) has only one equilibrium O(0, 0, 0, 0) when $b^2 - ab + \varepsilon \neq 0$. The authors [10] studied Hopf bifurcation under the conditions that the trace of Jacobian matrix of the linearized system of system (1) at equilibrium $O(0, 0, 0, 0)$ was not always zero, and utilized formulae given in [8] directly to find the bifurcation parameter and obtain the asymptotic expansion of limit cycle up to $O(\epsilon)$ order. In this paper, we study the shape of limit cycle bifurcated from the equilibrium points of the modified Lü system by combining the center manifold Theorem, Poincaré-Bendixson Theorem and method of Fredich, and give the asymptotic expansion of limit cycle up to $O(\epsilon^{3/2})$ order.

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In [11, 12], the number of limit cycles bifurcated from the equilibrium point of a generalized Moon-Rand system by two different approaches. In this paper, we study the shape of limit cycle bifurcated from the equilibrium point of the following generalized Moon-Rand system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - xz, \\ \dot{z} = -z + c_0x^2 + c_1xy + c_2y^2, \end{cases} \tag{2}$$

where $c_i,\ i = 0, 1, 2$ are parameters. By noting that the trace of Jacobian matrix of the linearized system of (2) at the equilibrium $O(0, 0, 0)$ is always equal to zero, we can not directly apply the formulae given [8] to get the asymptotic expansion of limit cycle of system (2) as the in [10].

The rest of this paper is organized as follows. In Section 2, we give the asymptotic expansion of the center manifold of the equilibrium of system (1), and center manifold theory is used to simplify the system, the method of Friedch is used to obtain the shape of the periodic solution of the modified Lü system (1). In Section 3, we compute the first Liapunov constants of the reduced Moon-Rand system on the center manifold, and give the condition of existence of the limit cycle of the Moon-Rand system first. Then we compute the asymptotic expansion of the period solutions of the generalized Moon-Rand system and plot its graph.

2 Shape of Limit Cycle of the Modified Lü System

2.1 Computation of Center Manifold in the Modified Lü System

The Jacobian matrix of the system (1) at O is as follows

$$\begin{bmatrix} -a & a & 0 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & -c & 0 \\ -(b^2 - ab + \varepsilon) & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation of the above Jacobian matrix is

$$f(\lambda) = (\lambda + c)(\lambda^3 + (a - b)\lambda^2 - ab\lambda + a(b^2 - ab + \varepsilon)) = 0. \tag{3}$$

In the following, we always assume that $a > 0$, $b < 0$ and $c > 0$. By computing, we get the eigenvalues of system (1) $\lambda_{1,2} = \pm\omega i$, $\lambda_3 = b - a$, $\lambda_4 = -c$, and their corresponding eigenvectors

$$v_{1,2} = \begin{bmatrix} 1 \\ \frac{a\pm\omega i}{a} \\ 0 \\ \frac{\pm\omega i(a-b)}{a} \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ \frac{b}{a} \\ 0 \\ -b \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

where $\omega = \sqrt{-ab}$.

Let $X = PY$, where

$$X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & \frac{\omega}{a} & \frac{b}{a} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\omega(a-b)}{a} & -b & 0 \end{bmatrix}, Y = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ u_1 \end{bmatrix}.$$

By calculating, we obtain the inverse matrix of P

$$P^{-1} = \begin{bmatrix} \frac{b(2a-b)}{(a^2+b^2-3ab)} & \frac{a(a-b)}{a^2+b^2-3ab} & 0 & \frac{a}{a^2+b^2-3ab} \\ \frac{a^2b}{(a^2+b^2-3ab)\omega} & \frac{-a^2b}{(a^2+b^2-3ab)\omega} & 0 & \frac{a(a-b)}{(a^2+b^2-3ab)\omega} \\ \frac{a(a-b)}{(a^2+b^2-3ab)} & \frac{-a(a-b)}{(a^2+b^2-3ab)} & 0 & \frac{a)}{(a^2+b^2-3ab)} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then system (1) is transformed to the following system

$$\left\{\begin{array}{lll} \dot{x_1} & = & \omega y_1 + \frac{-b(2a-b)(ax_1+y_1\omega+bz_1)u_1 - a(a-b)(x_1+z_1)u_1 + a\varepsilon x_1}{a^2+b^2-3ab}, \\ \dot{y_1} & = & -\omega x_1 + \frac{a^2 b(ax_1+y_1\omega+bz_1)u_1 + a^2 b(x_1+z_1)u_1 - a(a-b)\varepsilon x_1}{\omega(a^2+b^2-3ab)}, \\ \dot{z_1} & = & (b-a)z_1 + \frac{a(a-b)(ax_1+y_1\omega+bz_1)u_1 + a(a-b)(x_1+z_1)u_1 - ax_1\varepsilon}{(a^2+b^2-3ab)}, \\ \dot{u_1} & = & -cu_1 + \frac{(x_1+z_1)(ax_1+y_1\omega+bz_1)}{a}. \end{array}\right. \tag{4}$$

For convenience, we put $\dot{\varepsilon} = 0$ into the above equation to calculate the center manifold of $O(0,0,0,0)$ of system (4).

From [8], let $z_1 = H(x_1, y_1, \varepsilon) = h_{200}x_1^2 + h_{020}y_1^2 + h_{002}\varepsilon^2 + h_{110}x_1y_1 + h_{101}x_1\varepsilon + h_{011}y_1\varepsilon + o(x_1^2 + y_1^2 + \varepsilon^2)$, $u_1 = K(x_1, y_1, \varepsilon) = k_{200}x_1^2 + k_{020}y_1^2 + k_{002}\varepsilon^2 + k_{110}x_1y_1 + k_{101}x_1\varepsilon + k_{010}y_1\varepsilon + o(x_1^2 + y_1^2 + \varepsilon^2)$ be the center manifold of system (4) at the origin, where h_{ijk}, k_{ijk}, $i, j, k = 0, 1, 2$ are undetermined constants, then put them to the system (4) and through the comparison of coefficients of the different power of ε, we obtain the following lemma.

Lemma 1 *The center manifold of system (4) at the equilibrium point $O(0,0,0,0)$ has the following expression*

$$\begin{array}{l} H(x_1, y_1, \varepsilon) = \frac{a\omega\varepsilon y_1 - a(a-b)\varepsilon x_1}{-6ab^3+11a^2b^2+b^4-6a^3b+a^4} + o(x_1^2 + y_1^2 + \varepsilon^2), \\ G(x_1, y_1, \varepsilon) = \frac{(c^2-bc-2ab)x_1^2 + (bc-2ab)y_1^2}{(c^2-4ab)c} + \frac{\omega(c-2a)}{(c^2-4ab)a}x_1y_1 + o(x_1^2 + y_1^2 + \varepsilon^2). \end{array} \tag{5}$$

Then by substituting $z_1 = H(x_1, y_1, \varepsilon)$ and $u_1 = K(x_1, y_1, \varepsilon)$ into the first two equations of the system (4), we obtain a two-dimensional system

$$\left\{\begin{array}{ll} \dot{x_1} = & a\varepsilon x_1 + \omega y_1 + \frac{b(2a-b)}{\alpha_1}(ax_1 + y_1\omega + b\phi_1(x_1, y_1))\phi_2(x_1, y_1) \\ & -a(a-b)(x_1 + \phi_1(x_1, y_1))\phi_2(x_1, y_1) + o((x^2 + y^2 + \varepsilon^2)^{3/2}), \\ \dot{y_1} = & -(a(a-b)\varepsilon + \omega)x_1 + \frac{a\omega}{\alpha_1}(ax_1 + y_1\omega + \phi_1(x_1, y_1))\phi_2(x_1, y_1) \\ & +a^2b(x_1 + \phi_1(x_1, y_1))\phi_2(x_1, y_1) + o((x^2 + y^2 + \varepsilon^2)^{3/2}), \end{array}\right. \tag{6}$$

where $\alpha_1 = \frac{1}{a^2+b^2-3ab}$, $\phi_1(x_1, y_1) = \frac{\omega a\varepsilon y_1 - a(a-b)x_1\varepsilon}{-6ab^3+11a^2b^2+b^4-6a^3b+a^4}$, $\phi_2(x_1, y_1) = \frac{1}{ac(c^2-4ab)}(a(c^2 - 2ab - bc)x_1^2 + ab(c - 2a)y_1^2 + c\omega(c - 2a)x_1y_1)$.

2.2 Asymptotic Expansion of Limit Cycle of the modified Lü system and Its Graphs

By direct computation, we get the divergence quantity of equilibrium $(0,0)$ of system (6) is $v_1 = a\epsilon$. To study the bifurcation problem near the equilibrium and discuss on what conditions system (6) will have a limit cycle, we need to determine the stability of equilibrium $(0,0)$ when $v_1 = 0$. By applying Liapunove constants formula given in [13], as $v_1 = 0$ we get the first Liapunove constant of the equilibrium $(0,0)$

$$v_3 = \frac{(-2a^3b^2c + 4a^3bc^2 - 4a^2bc^2 - 5a^2b^2c^2 + 2b^2a^2c + 3a^3c^2 + 8a^3b^3 + 8a^3b^2 + ab^3c^2 - 8a^4b - 8a^4b^2 + 2a^4bc)}{8(-3ab+b^2+a^2)(-c^2+4ab)\omega ca}. \tag{7}$$

Lemma 2 *For system (6), the following conclusions hold.*

(1) If $v_1 = 0$, $v_3 > 0$, then as $\varepsilon < 0$ and small, system (6) has a unstable limit cycle.

(2) If $v_1 = 0$, $v_3 < 0$, then as $\varepsilon > 0$ and small, system (6) has a stable limit cycle.

Proof. When $v_1 = 0$ and $v_3 > 0$, we know that the equilibrium $(0,0)$ of the system (6) is unstable. Here we change the parameter ε slightly to satisfy that $v_1 < 0$. That means the stability of the equilibrium $(0,0)$ of system (6) has been changed from unstable to stable. With the help of Poincaré-Bendixson Theorem, we obtain that system (6) has a unstable limit cycle in the neighborhood of the origin. Then the proof of conclusion (1) is completed. ■

Similarly, we can prove conclusion (2).

Remark 3 *As the parameters $a = 2$, $b = -1$, $c = 2$ given in [10], from Lemma 2 we get $v_1 = 0$ and $v_3 > 0$, that means the equilibrium $(0,0)$ of reduced system (6) on the center manifold is unstable. Therefore, as $\varepsilon > 0$ system (1) can not have a stable limit cycle bifurcated from the equilibrium $O(0,0,0,0)$ as simulated in [10].*

To compute the asymptotic expansion of limit cycle of system (6) under the conditions of Lemma 2, we need the following lemma given in [8].

Lemma 4 *Consider the following planar system*

$$\begin{cases} \frac{dx}{dt} = -y + f(t), \\ \frac{dy}{dt} = x + g(t), \end{cases} \tag{8}$$

where $f(t)$ and $g(t)$ are continuous functions with period of 2π. Then the sufficient and necessary conditions for the system (8) has periodic solutions is that

$$\begin{cases} \int_0^{2\pi}[f(t)\cos t + g(t)\sin t]dt = 0, \\ \int_0^{2\pi}[-f(t)\sin t + g(t)\cos t]dt = 0. \end{cases} \tag{9}$$

Then, our main result in this section is the following theorem.

Theorem 5 *Suppose that the parameters a, b, c, ε satisfy conditions of Lemma 2, let the parameter μ satisfies $\varepsilon = \mu^2 d_1 + o(\mu^3)$, where $d_1 = \frac{-2ab^2c+4abc^2+5ab^2c^2-4a^2bc^2-8a^2b^2+2a^2b^2c-2}{4ac(4ab-c^2)}$. Then the periodic solution of system (1) has the following asymptotic expansion*

$$X(s) = \begin{pmatrix} \mu\cos(\omega s) + \mu^3(\frac{-a\omega d_1\sin(\omega s)+a(a-b)d_1\cos(\omega s)}{\alpha_2} + \psi_{21}(s)) + o(\mu^3) \\ \mu(\frac{a\cos(\omega s)-\omega\sin(\omega s)}{a}) + \mu^3(\frac{-b\omega d_1\sin(\omega s)+(a-b)\cos(\omega s)}{\alpha_2} + \frac{\omega}{a}\psi_{22}(s)) + o(\mu^3) \\ \mu^2(\frac{(c^2-bc-2ab)\sin^2(\omega s)+(bc-2ab)\cos^2(\omega s)+\omega(c-2a)\sin(\omega s)\cos(\omega s)}{(c^2-4ab)ca}) + o(\mu^3) \\ \mu(\frac{\omega(b-a)\cos(\omega s)-a^3b\sin(\omega s)}{a}) + \mu^3(\frac{a\omega d_1\sin(\omega s)-(a-b)\cos(\omega s)}{\alpha_2} + \frac{b}{a}\psi_{22}(s)) + o(\mu^3) \end{pmatrix}, \tag{10}$$

where $s = \frac{t}{1+\sigma_1\mu+o(\mu)}$, $\sigma_1 = \frac{8a^2b-2a^2bc+2abc+2ab^2c-3ac^2-abc^2+b^2c^2}{abc(4ab-c^2)}$, $\alpha_2 = -6ab^3 + 11a^2b^2 + b^4 - 6a^3b + a^4$, and $\psi_{21}(s)$, $\psi_{22}(s)$are given by

$$\begin{aligned} \psi_{21}(s) = & \cos(\omega s)(\int_0^s[\cos(\omega s)R_1(s) - \sin(\omega s)R_2(s)]ds + m_1) \\ & + \sin(\omega t)(\int_0^s[\sin(\omega s)R_1(s) + \cos(\omega s)R_2(s)]ds + m_2), \\ \psi_{22}(s) = & \cos(\omega t)(\int_0^s[\sin(\omega s)R_1(s) + \cos(\omega s)R_2(s)]ds + m_2 \\ & - \sin(\omega t)(\int_0^s[\cos(\omega s)R_1(s) - \sin(\omega s)R_2(s)]ds + m_1), \end{aligned} \tag{11}$$

$R_1(s) = a\alpha_1 d_1\cos(\omega s) - \sigma_1\omega\sin(\omega s) - b(2a-b)\alpha_1(a\cos(\omega s) - \omega\sin(\omega s))\phi_1(\cos(\omega s), -\sin(\omega s)))$ $- \alpha_1 a(a-b)\cos(\omega s)\phi_1(\cos(\omega s), -\sin(\omega s))$, $R_2(s) = -(\frac{a(a-b)d_1\alpha_1}{\omega} + \sigma_1\omega)\cos(\omega s)$ $+ ba^2\alpha_1((a+1)\cos(\omega s) - \omega\sin(\omega s))\phi_1(\cos(\omega s), -\sin(\omega s))$, *and* $m_1 = \frac{1}{\alpha_3}(a^2b^2(20a - 8b + 6ac + 2bc + 24ab - 12b^2 - 4a^2) + b^2c^2(6a - 17a^2 - b^2) + abc(-11ac + 4a^2 + 2a^3 - 2b^2 - 2b^3 + 10b^2 + 2a^2c) - 8a^4b + 3a^3c^2)$, $m_2 = \frac{\omega}{\alpha_3}(4abc^2 - 4a^2b^3 - 4a^2b^2 + 4a^3b + 3ab^2c^2 + 2ab^3c - 4a^2b^2c - b^3c^2 - 6a^2bc - 6a^3bc - a^2c^2 + 4a^2bc^2 - 4a^3b)$, $\alpha_3 = 8abc(3abc^2 - b^2c^2 - a^2c^2 - 12a^2b^2 + 4ab^3 + 4a^3b)$.

Proof. From lemma 2, we know that under the conditions of Theorem 5 system (6) has a limit cycle, denoted by Γ : $x_1 = x_1(t)$, $y_1 = y_1(t)$. Suppose that the limit cycle Γ passes the initial point $(x_1(0), y_1(0)) = \mu(1, 0)$, and let T_μ be the period of the limit cycle Γ, where μ is a small real number satisfying $\varepsilon = \mu d(\mu)$, and $d(\mu)$ is a function of μ. Then, we make the following scaling: $s = \frac{T_0 t}{T_\mu}$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mu\begin{pmatrix} \eta_{01} \\ \eta_{02} \end{pmatrix} + \mu^2\begin{pmatrix} \eta_{11} \\ \eta_{12} \end{pmatrix} + \mu^3\begin{pmatrix} \eta_{21} \\ \eta_{22} \end{pmatrix} + o(\mu^3),$$

where $T_0 = \frac{2\pi}{\omega}$, $T_\mu = T_0(1 + \sigma(\mu))$, $\sigma(\mu) = \sigma_0 + \sigma_1\mu + \sigma_2\mu^2 + o(\mu^3)$, $d(\mu) = d_0 + d_1\mu + d_2\mu^2 + o(\mu^3)$.

To determine the expression of σ_i, d_i and $(\eta_{i1}, \eta_{i2})^T$, $i = 0, 1, 2$, we substitute the above into the system (6), and obtain the equations that $(\eta_{i1}, \eta_{i2})^T$, $i = 0, 1, 2$ satisfy by comparing the coefficient of μ^i, $i = 0, 1, 2$ with the help of Maple 17.

$$\begin{pmatrix} \dot\eta_{01} \\ \dot\eta_{02} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} \eta_{01} \\ \eta_{02} \end{pmatrix}, \begin{pmatrix} \eta_{01}(0) \\ \eta_{02}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{12}$$

By solving the above equation, we get that

$$\begin{pmatrix} \eta_{01}(s) \\ \eta_{02}(s) \end{pmatrix} = \begin{pmatrix} \cos(\omega s) \\ -\sin(\omega s) \end{pmatrix}.$$

By comparing the coefficient of μ^1, we get

$$\begin{pmatrix} \dot{\eta_{11}} \\ \dot{\eta_{12}} \end{pmatrix} = \begin{pmatrix} \omega\eta_{12} + \sigma_0\omega\eta_{02}(s) + \frac{ad_0}{a^2+b^2-3ab} \\ -\omega\eta_{11} - \sigma_0\omega\eta_{01}(s) + \frac{a(b-a)d_0}{\omega(a^2+b^2-3ab)} \end{pmatrix}, \begin{pmatrix} \eta_{11}(0) \\ \eta_{12}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{13}$$

Due to the period of $(\eta_{11}(s), \eta_{12}(s))^T$ is T_0, by applying Lemma 4, we get that the nonhomogenous terms of equation (13) should satisfy the following conditions of orthogonality

$$\begin{cases} \int_0^{T_0}(r_1(s)\cos\omega s - r_2(s)\sin\omega s)ds &= 0, \\ \int_0^{T_0}(r_1(s)\sin\omega s + r_2(s)\cos\omega s)ds &= 0, \end{cases} \tag{14}$$

where $r_1(s) = \sigma_0\omega\eta_{02} + \frac{ad_0}{a^2+b^2-3ab}$, $r_2(s) = -\sigma_0\omega\eta_{01} + \frac{a(b-a)d_0}{\omega(a^2+b^2-3ab)}$.

From equation (24), we obtain that $\sigma_0 = 0$, $d_0 = 0$, and the particular solution of the above initial value problem is

$$\begin{pmatrix} \eta_{11}(s) \\ \eta_{12}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Next, compare the coefficient of μ^2, we get

$$\begin{pmatrix} \dot{\eta_{21}} \\ \dot{\eta_{22}} \end{pmatrix} = \begin{pmatrix} \omega\eta_{22} + G_1(\eta_{01}\eta_{02}), \\ -\omega\eta_{21} + G_2(\eta_{01}, \eta_{02}) \end{pmatrix}, \begin{pmatrix} \eta_{11}(0) \\ \eta_{12}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

where $G_1(\eta_{01}, \eta_{02}) = a\alpha_1\alpha_2 d_1\eta_{01} + \omega\sigma_1\eta_{02} - b(2a-b)\alpha_1(a\eta_{01} + \omega\eta_{02})\phi_2(\eta_{01}, \eta_{02}) - a(a-b)\alpha_1\eta_{01}\phi_2(\eta_{01}, \eta_{02})$, $G_2(\eta_{01}, \eta_{02}) = -\omega\sigma_1\eta_{01} - a(a-b)d_1\alpha_1\alpha_2\eta_{01} + a^2 b\phi_2(\eta_{01}, \eta_{02})(a\eta_{01} + \omega\eta_{02} + \eta_{01})$.

By applying Lemma 4 and conditions of orthogonality, we can obtain the expression of d_1, σ_1 which are given in Theorem 1.

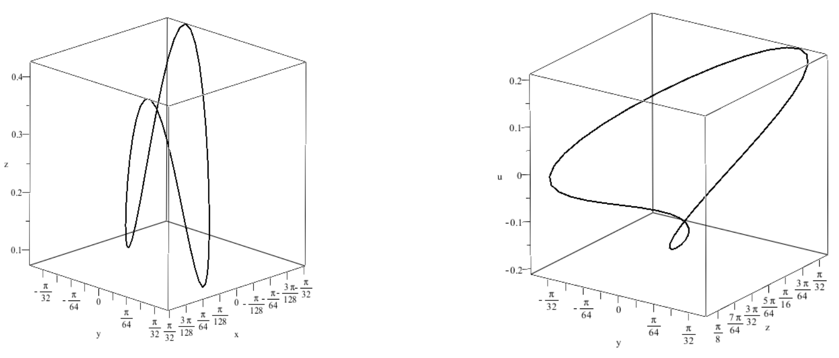
Take them to system (15) and obtain the solution $\psi_{21}(s), \psi_{22}(s)$ which are given in (11).

Then by applying the asymptotic expansion of the center manifold given in Lemma 1 and the transformation $X = PY$, we obtain the result of Theorem 1.

The proof of Theorem 5 is completed. ■

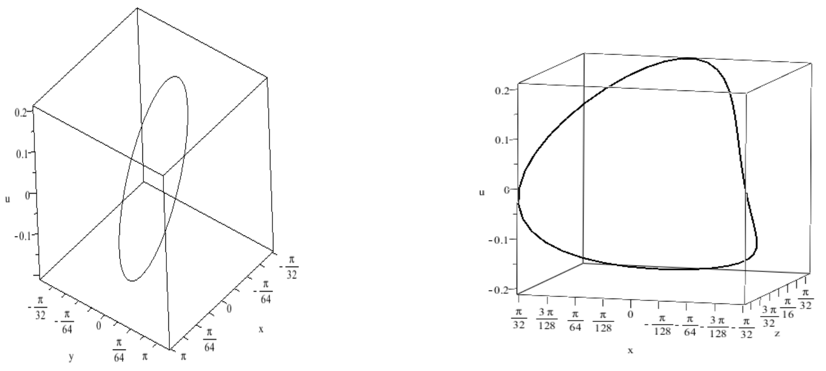
Remark 6 *Compare with the result in [10], my computation approximate to the higher order, which is very useful to our simulation.*

Here, to plot the graph of limit cycle of system (1), we choose one group parameter $a = 2$, $b = -1/2$, $c = 1$, $\varepsilon = 0.01$, then obtain the following shapes of limit cycle of system (1) in different coordinate space with the aid of Maple17.



(a) The plot of limit cycle in (x,y,z) space (b) The graph of limit cycle in (z,y,u) space

Figure 1: The plot of limit cycle of Lü system



(a) The graph of limit cycle in (x,y,u) space (b) The plot of limit cycle in (x,z,u) space

Figure 2: The plot of limit cycle of Lü system

3 Shape of Limit Cycle of the Moon-Rand System

3.1 Computation of Center Manifold of the Moon-Rand System

Consider the modified Moon-rand system (2). Since the system (2) has been already in normal form in the neighborhood of $O(0,0,0)$, we assume that the center manifold has the Taylor expansion at $(x,y)=(0,0)$ by applying center manifold Theorem in [8]. Put it to the system (2) and compare the coefficients of each term, we obtain the the asymptotic expansion of the center manifold given in the following lemma.

Lemma 7 *The center manifold $z=\tilde{H}(x,y)$ of system (2) at the equilibrium $O(0,0,0)$ is given by*

$$\tilde{H}(x,y)=\tilde{h}_{20}x^2+\tilde{h}_{11}xy+\tilde{h}_{02}y^2+\tilde{h}_{40}x^4+\tilde{h}_{13}xy^3+\tilde{h}_{22}x^2y^2+\tilde{h}_{31}x^3y+\tilde{h}_{04}y^4+o(x^4+y^4), \quad (16)$$

where $\tilde{h}_{20}=\frac{1}{5}(2c_2+c_1+3c_0)$, $\tilde{h}_{11}=\frac{1}{5}(2c_2+c_1-2c_0)$, $\tilde{h}_{02}=\frac{1}{5}(3c_2-c_1+2c_0)$, $\tilde{h}_{40}=\frac{1}{2125}(364c_0c_2+628c_2^2+188c_2c_1+12c_1^2+47c_0c_1-142c_0^2)$, $\tilde{h}_{13}=\frac{1}{2125}(384c_0c_2+18c_2^2-222c_2c_1+22c_1^2-268c_0c_1-448c_0^2)$, $\tilde{h}_{22}=\frac{1}{425}(12c_0c_2+144c_2^2+9c_2c_1+6c_1^2+66c_0c_1-156c_0^2)$, $\tilde{h}_{31}=\frac{1}{2125}(194c_0c_2+288c_2^2-152c_2c_1-73c_1^2-38c_0c_1+368c_0^2)$, $\tilde{h}_{04}=\frac{1}{2125}(-384c_0c_2-18c_2^2+222c_2c_1-22c_1^2+268c_0c_1+448c_0^2)$.

Then by substituting $z=\tilde{H}(x,y)$ into the first two equations of the system (2), we obtain a two-dimensional system

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-x-x\tilde{H}(x,y). \end{cases} \quad (17)$$

3.2 Existence of limit cycle of system (2)

By direct computation, we get the divergence quantity of equilibrium $O(0,0)$ of the reduced system (17) is always equal to 0. To determined the stability of the equilibrium $O(0,0)$, we rewrite system (17) into complex form and apply Liapunove constants formula given in [13], we get the following first two Liapunove constants of system (17)

$$\tilde{v}_3=\frac{1}{40}(-2c_2-c_1+2c_0), \quad (18)$$

$$\tilde{v}_5=-\frac{117c_2c_0}{1000}-\frac{93c_2^2}{2000}+\frac{219c_2c_1}{4000}-\frac{3c_1^2}{2000}+\frac{261c_1c_0}{4000}-\frac{273c_0^2}{2000}. \quad (19)$$

Lemma 8 *Consider the system (17). Let $c_1=2c_0-2c_2+\varepsilon$, then $\tilde{v}_5=-\frac{3c_2^2}{20}-\frac{3c_2c_0}{20}$ and we have the following conclusions.*

(1) As $\tilde{v}_3=0$, $\tilde{v}_5>0$, then as $\varepsilon>0$, system (17) has a unstable limit cycle.

(2) As $\tilde{v}_3=0$, $\tilde{v}_5<0$, then as $\varepsilon<0$, system (17) has a stable limit cycle.

Proof. When $\tilde{v}_3 = 0,\ \tilde{v}_5 > 0$, we know that the equilibrium point $O(0,0)$ of system (17) is unstable. Here we change the value of parameter ε slightly to satisfy that $\tilde{v}_3 < 0$. With the help of Poincaré-Bendixson Theorem, we obtain that system (5) will have a unstable limit cycle in a small neighborhood of the origin. Then the proof of conclusion (1) is completed. ■

Similarly, we can show conclusion (2) of the Lemma 8.

3.3 Asymptotic Expansion of Limit Cycle of the Moon-Rand System and Its Graph

From Lemma 8, we know that under conditions of Lemma 8 system (2) has a limit cycle on the center manifold of equilibrium $O(0,0,0)$. Here, we give the main result in this section.

Theorem 9 *Suppose that the parameters $c_0,\ c_1,\ c_2,\ \varepsilon$ of system (2) satisfy conditions of Lemma 5, let the parameter ν satisfies $\varepsilon = \tilde{d}_1\nu^2 + o(\nu^3)$, where $\tilde{d}_1 = -c_2c_0 - c_2^2$. Then the limit cycle of the Moon-Rand system (2) has the following asymptotic expansion*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \nu\cos\tilde{s} + \frac{\nu^3}{|\tilde{d}_1|^{3/2}}\tilde{\psi}_{21} + o(\nu^3), \\ -\nu\sin\tilde{s} + \frac{\nu^3}{|\tilde{d}_1|^{3/2}}\tilde{\psi}_{22} + o(\nu^3), \\ \nu^2(\tilde{h}_{20}\cos^2\tilde{s} - \tilde{h}_{11}\cos\tilde{s}\sin\tilde{s} + \tilde{h}_{02}\sin^2\tilde{s}) + o(\nu^3), \end{pmatrix}, \tag{20}$$

where $\tilde{h}_{20},\ \tilde{h}_{11},\ \tilde{h}_{02}$ are given in Lemma 4, $\tilde{s} = \frac{t}{1+\frac{-c_2-3c_0}{8}\nu + o(\nu)}$, and functions $\tilde{\eta}_{21},\ \tilde{\eta}_{22}$ are given by

$$\begin{pmatrix} \tilde{\psi}_{21} \\ \tilde{\psi}_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{8}(c_0 - c_2)\sin^2\tilde{s}\cos\tilde{s} \\ -\frac{1}{8}\sin\tilde{s}(2c_0 + 2c_2 + (3c_0 - 3c_2)\cos^2\tilde{s}) \end{pmatrix}. \tag{21}$$

Proof. From lemma 8, we know that under the conditions of Theorem 5 system (17) has a limit cycle, denoted by $\tilde{\Gamma}:\ x = x(t),\ y = y(t)$. Suppose that the limit cycle $\tilde{\Gamma}$ passes the initial point $(x_1(0), y_1(0)) = \nu(1,0)$, and let T_ν be the period of the limit cycle $\tilde{\Gamma}$, where ν is a small real number satisfying $\varepsilon = \nu d(\nu)$, and $d(\nu)$ is a function of ν. Then, we make the following scaling $\tilde{s} = \frac{2\pi t}{T_\nu}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \nu\begin{pmatrix} \tilde{\eta}_{01} \\ \tilde{\eta}_{02} \end{pmatrix} + \nu^2\begin{pmatrix} \tilde{\eta}_{11} \\ \tilde{\eta}_{12} \end{pmatrix} + \nu^3\begin{pmatrix} \tilde{\eta}_{21} \\ \tilde{\eta}_{22} \end{pmatrix} + o(\nu^3),$$

where $T_\nu = 2\pi(1 + \tilde{\sigma}(\nu))$, $\tilde{\sigma}(\nu) = \tilde{\sigma}_0 + \tilde{\sigma}_1\nu + \tilde{\sigma}_2\nu^2 + O(\nu^3)$, $\tilde{d}(\nu) = \tilde{d}_0 + \tilde{d}_1\nu + \tilde{d}_2\nu^2 + O(\nu^3)$.

To determine the expression of $\tilde{\sigma}_i,\ \tilde{d}_i$ and $(\eta_{i1}, \eta_{i2})^T,\ i = 0,1,2$, we substitute the above into the system (17), and obtain the equations that $(\eta_{i1}, \eta_{i2})^T$ satisfy by comparing the coefficient of $\nu^i,\ i = 0,1,2$ with the help of Maple 17.

$$\begin{pmatrix} \dot{\tilde{\eta}}_{01} \\ \dot{\tilde{\eta}}_{02} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\eta}_{01} \\ \tilde{\eta}_{02} \end{pmatrix},\ \begin{pmatrix} \tilde{\eta}_{01}(0) \\ \tilde{\eta}_{02}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{22}$$

By solving the above equation, we get that

$$\begin{pmatrix} \tilde{\eta}_{01}(\tilde{s}) \\ \tilde{\eta}_{02}(\tilde{s}) \end{pmatrix} = \begin{pmatrix} \cos\tilde{s} \\ -\sin\tilde{s} \end{pmatrix}.$$

By comparing the coefficient of ν^1, we get

$$\begin{pmatrix} \dot{\tilde{\eta}}_{11} \\ \dot{\tilde{\eta}}_{12} \end{pmatrix} = \begin{pmatrix} -\tilde{\sigma}_0\tilde{\eta}_{02} + \tilde{\eta}_{01} \\ \tilde{\sigma}_0\tilde{\eta}_{01} - \tilde{\eta}_{11} \end{pmatrix},\ \begin{pmatrix} \tilde{\eta}_{11}(0) \\ \tilde{\eta}_{12}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{23}$$

Due to the period of $\tilde{\eta}_{11},\ \tilde{\eta}_{12}$ are both 2π, from lemma 3 , we know that the nonhomogenous terms of (23) should satisfy the conditions of orthogonality

$$\begin{cases} \int_0^{2\pi}[(-\tilde{\sigma}_0\tilde{\eta}_{02} + \tilde{\eta}_{01})\cos\tilde{s} - (\tilde{\sigma}_0\tilde{\eta}_{01} - \tilde{\eta}_{11})\sin\tilde{s}]d\tilde{s} = 0, \\ \int_0^{2\pi}[(-\tilde{\sigma}_0\tilde{\eta}_{02} + \tilde{\eta}_{01})\sin\tilde{s} + (\tilde{\sigma}_0\tilde{\eta}_{01} - \tilde{\eta}_{11})\cos\tilde{s}]d\tilde{s} = 0. \end{cases} \tag{24}$$

From (24), we obtain that $\tilde{\sigma}_0 = 0$, and the particular solution of the above initial value problem (23) is

$$\begin{pmatrix} \tilde{\eta}_{11} \\ \tilde{\eta}_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Secondly, compare the coefficient of ν^2, we get

$$\begin{pmatrix} \dot{\tilde{\eta}}_{21} \\ \dot{\tilde{\eta}}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{\eta}_{22} + \tilde{\sigma}_1\tilde{\eta}_{02} \\ -\tilde{\eta}_{21} - \tilde{\eta}_{01}(\tilde{\sigma}_2\tilde{\eta}_{02}^2 + \tilde{\sigma}_0\tilde{\eta}_{01}^2) - \tilde{\sigma}_1\tilde{\eta}_{01} \end{pmatrix}, \begin{pmatrix} \tilde{\eta}_{21}(0) \\ \tilde{\eta}_{22}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{25}$$

By applying Lemma 4, we obtain

$$\tilde{\sigma}_1 = \frac{-c_2 - 3c_0}{8}, \tilde{d}_0 = 0, \tilde{d}_1 = -c_2c_0 - c_2^2$$

■

Take them to system (25) and obtain the solution $\tilde{\psi}_{21}(s), \tilde{\psi}_{22}(s)$ which are given in (21).

Then by applying the asymptotic expansion of the center manifold given in Lemma 8, we obtain the result of Theorem 9.

To plot the graph of limit cycle of system (2), we choose one group parameter $c_0 = 2$, $c_2 = -1$, $c_1 = 6$, and $\varepsilon = \tilde{d}_1\nu^2 + o(\nu^3) = 0.01$, then obtain the following shape of limit cycle of system (2) with the aid of Maple 17.

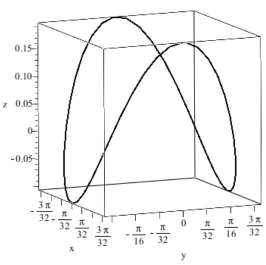


Figure 3: The graph of limit cycle of Moon-Rand system (2) in (x,y,z) space

4 Conclusion

The purpose of this paper is to obtain the asymptotic expression of the limit cycles for a new modified four-dimensional Lü system and three-dimensional Moon-Rand system. We apply central manifold theory to get a two-dimensional system and with the help of the stability of the focus and Poincaré-Bendixson Theorem to prove the existence of limit cycles in the above two systems. At last, we apply the method of Friedrich to get the first several expansion terms of the limit cycles.

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