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Mild Solution for Nonlocal Fractional Functional Differential Equation with not Instantaneous Impulse

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Abstract: This paper is concerned with the problem of the existence results of the mild solution for non-local fractional differential equations with state dependent delay subject to not instantaneous impulse. The existence results are proved with the help of fixed point theorems. One example involving partial differential equations is presented to illustrate the existence result .

Keywords: Fractional order differential equation; Functional differential equations; Impulsive conditions; Fixed point theorem;

1 Introduction

In the last few decades fractional differential equations are more interested in developing the numerical methods and theoretical analysis, because it is recently proved that it has memory and hereditary property which is valuable in serval field of engineering and sciences. For some recent development theory and applications of fractional differential equations reader can see the monographs [1–5] and with various conditions such as initial, impulse and nonlocal, one can see the papers [6–12, 14, 21, 22, 26, 28–30] and reference therein.

It has been seen that the theory of existence result for fractional differential equations is not yet sufficiently elaborated, compared to that of theory of ordinary differential equations. Due to this fact, it is important and necessary to study the existence result for semi-linear functional fractional differential equations. The spacial type of functional differential equations is delay differential equations in which delay may state-dependent or constant with different type of conditions. See for more details of relevant update theory of state dependent delay in the cited papers [13, 15–20, 23–25, 27].

Zhou et al. [11] study the following Cauchy problem

$$^{c}D_t^q[x(t) - h(t, x_t)] + Ax(t) = f(t, x_t),\ t \in (0, a] \tag{1}$$
$$x_0(\vartheta) + (g(x_{t_1}, \cdots, x_{t_p}))(\vartheta) = \varphi(\vartheta),\ \vartheta \in [-r, 0], \tag{2}$$

and prove the existence and uniqueness result of mild solutions by using Krasnoselskii's fixed point theorem. Chauhan et al. [6] find the definition of mild solution with the help of Laplace transformation and established the existence and uniqueness results of a mild solution applying the Banach and Krasnoselskii's Fixed Point Theorems for the following model equation

$$\frac{d^\alpha}{dt}x(t) + Ax(t) = f(t, x, x(a_1(t)), \cdots, x(a_m(t))),\ t \in J = [0, T],\ t \neq t_k \tag{3}$$
$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \ldots m, \tag{4}$$
$$x(0) + g(x) = x_0. \tag{5}$$

Bahuguna [7] establish the existence, uniqueness and continuation of a mild solution of problem (3) with out impulsive condition and taking the order $\alpha = 1$ and condition (5). They proved some regularity results under different conditions.

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Author [8] also study the problem (3) with out impulsive condition and taking the nonlocal condition (5) and established existence results.

Hernandez et al. [9] use first time not instantaneous impulsive condition for semi-linear abstract differential equation of the form

$$u'(t) = Au(t) + f(t, u(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{6}$$

$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N, u(0) = u_0, \tag{7}$$

and introduced the concepts of mild and classical solution. They established the existence results by using fixed point theorems. Further, Pierri et al. [10] extend the results of [9] in the study of the problem (6)-(7) using the theory of analytic semigroup and fractional power of closed operators and established the existence results of solutions. Kumar et al. [28] have studied the the following fractional order problem with not instantaneous impulse

$$^C D_t^\beta u(t) + Au(t) = f(t, u(t), g(u(t))),\ t \in (s_i, t_{i+1}],\ i = 0, 1, \cdots, N, \tag{8}$$

$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N,\ u(0) = u_0 \in H, \tag{9}$$

by using the Banach fixed point theorem with condensing map established the existence and uniqueness results.

Motivated by the above said work, we consider the following fractional functional differential equations of the form

$$^C D_t^\alpha u(t) = Au(t) + J^{1-\alpha} f(t, u_{\rho(t,u_t)}, u(a_1(t)), \cdots, u(a_m(t))),\ t \in (s_i, t_{i+1}] \subset J,\ i = 0, 1, \cdots, N, \tag{10}$$

$$u(t) = g_i(t, u(t)),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, N, \tag{11}$$

$$u(t) + h(u_{t_1}, \cdots, u_{t_p})(t) = \phi(t),\ t \in (-\infty, 0], \tag{12}$$

where $^C D_t^\alpha$ is Caputo's fractional derivative of order $0 < \alpha \leq 1$, $J^{1-\alpha}$ is Riemann-Liouville fractional integral operator and $J = [0, T]$ is operational interval. The map $A : D(A) \subset X \to X$ is a closed linear sectorial operator defined on a Banach space $(X, \|\cdot\|_X), 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T$, are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, \cdots, N$. $f : J \times \mathfrak{B}_h \times X^m \to X$, for each of $j = 1, 2, \cdots, m$, the map a_j is defined on $[0, T]$ into $(-\infty, T]$ satisfying some properties, $h : \mathfrak{B}_h^p \to X$ such that $0 < t_1 < t_2 < \cdots < t_p < T$ and $\rho : J \times \mathfrak{B}_h \to (-\infty, T]$ are appropriate functions. The history function $u_t : (-\infty, 0] \to X$ is element of $\mathfrak{B}_h$ and defined by $u_t(\theta) = u(t + \theta),\ \theta \in (-\infty, 0]$. Here impulses are not instantaneous means these impulses start abruptly at the points t_i and their action continues on the interval $[t_i, s_i]$ and $\mathfrak{B}_h$ is a phase space defined in next section.

To the best of our knowledge, system (10)-(12) is an untreated topic yet in the literature and this fact is the motivation of the present work. Further, this paper has four sections, second section provides some basic definitions, theorems, notations and lemma. Third section is equipped with existence results of the mild solution of the considered problem in this paper and fourth section is concerned with an example.

2 Preliminaries

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|u\|_X = \sup_{t\in J}\{|u(t)| : u \in X\}$ and $L(X)$ denotes the Banach space of all bounded linear operators from X into X equipped with its natural topology. Due to infinite delay we use abstract phase space $\mathfrak{B}_h$ as defined in [12] details are as follow:

Assume that $h : (-\infty, 0] \to (0, \infty)$ is a continuous functions with $l = \int_{-\infty}^0 h(s)ds < \infty, s \in (-\infty, 0]$. For any $a > 0$, we define

$$\mathfrak{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

and equipped the space $\mathfrak{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \forall\, \psi \in \mathfrak{B}$. Let us define

$$\mathfrak{B}_h = \{\psi : (-\infty, 0] \to X, \text{ s.t. for any } a \geq c > 0, \psi\,|_{[-c,0]} \in \mathfrak{B}\ \&\ \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}ds < \infty\}.$$

If $\mathfrak{B}_h$ is endowed with the norm $\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}ds,\ \forall\ \psi \in \mathfrak{B}_h$, then it is clear that $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a complete Banach space.

We consider the another space

$$\mathfrak{B}'_h := PC((-\infty, T]; X),\ T < \infty,$$

be a Banach space of all such functions $u : (-\infty, T] \to X$, which are continuous every where except for a finite number of points $t_i \in (0,T),\ i = 1, 2, \ldots, N$, at which $u(t_i^+)$ and $u(t_i^-)$ exists and endowed with the norm

$$\|u\|_{\mathfrak{B}_h'} = \sup\{\|u(s)\|_X : s \in [0,T]\} + \|\phi\|_{\mathfrak{B}_h}, u \in \mathfrak{B}_h^{'},$$

where $\|\cdot\|_{B_h'}$ to be a semi-norm in $\mathfrak{B}_h^{'}$.

For a function $u \in \mathfrak{B}_h'$ and $i \in \{0, 1, ..., N\}$, we introduce the function $\bar{u}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\bar{u}_i(t) = \begin{cases} u(t), \text{ for } t \in (t_i, t_{i+1}], \\ u(t_i^+), \text{ for } t = t_i. \end{cases}$$

If $u : (-\infty, T] \to X$ is a function s.t. $u \in \mathfrak{B}_h'$ then for all $t \in J$, the following conditions hold:

(C_1) $u_t \in \mathfrak{B}_h$.

(C_2) $\|u(t)\|_X \le H\|u_t\|_{\mathfrak{B}_h}$.

(C_3) $\|u_t\|_{\mathfrak{B}_h} \le K(t)\sup\{\|u(s)\|_X : 0 \le s \le t\} + M(t)\|\phi\|_{\mathfrak{B}_h}$, where $H > 0$ is constant; $K, M : [0,\infty) \to [0,\infty),\ K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $u(t)$.

(C_{4_ϕ}) The function t$\to \phi_t$ is well defined and continuous from the set

$$\Re(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0,T] \times \mathfrak{B}_h\}$$

into $\mathfrak{B}_h$ and there exists a continuous and bounded function $J^\phi : \Re(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathfrak{B}_h} \le J^\phi(t)\|\phi\|_{\mathfrak{B}_h}$ for every $t \in \Re(\rho^-)$.

Lemma 1 *([13])Let $u : (-\infty, T] \to X$ be function such that $u_0 = \phi, u\mid_{J_k} \in C(J_k, X)$ and if (C_{4_ϕ}) hold, then*

$$\|u_s\|_{\mathfrak{B}_h} \le (M_b + J^\phi)\|\phi\|_{\mathfrak{B}_h} + K_b \sup\{\|u(\theta)\|_X;\ \theta \in [0, \max\{0, s\}]\},\ s \in \Re(\rho^-) \cup J,$$

where $J^\phi = \sup_{t\in\Re(\rho^-)} J^\phi(t),\ M_b = \sup_{s\in[0,T]} M(s)$ and $K_b = \sup_{s\in[0,T]} K(s)$.

Definition 1 Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds =_a J_t^{n-\alpha} f^{(n)}(t),$$

where $a \ge 0,\ n \in N$. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{{}_0^C D_t^\alpha f(t); \lambda\} = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^k(0); \quad n-1 < \alpha \le n.$$

Definition 2 A two parameter function of the Mittag-Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_c \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - y} d\mu,\ \alpha,\ \beta > 0,\ y \in C,$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |y|^{\frac{1}{\alpha}}$ counter clockwise. The Laplace integral of this function given by

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega},\ Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega > 0.$$

For more details on the above definitions one can see the monographs of I. Podlubny [5].

Definition 3 *[14]* A closed and linear operator A is said to be sectorial if there are constants $\omega \in R,\ \theta \in [\frac{\pi}{2}, \pi],\ M > 0$, such that the following two conditions are satisfied:

(1) $\sum_{(\theta,\omega)} = \{\lambda \in C : \lambda \neq \omega,\ |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$

(2) $\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda-\omega|},\ \lambda \in \sum_{(\theta,\omega)}$,

where X is the complex Banach space with norm denoted $\|.\|_X$.

Definition 4 *[15]* Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}u = \int_0^\infty e^{-\lambda t} S_\alpha(t)udt,\ Re\lambda > \omega,\ u \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Lemma 2 Let f satisfies the uniform Holder condition with exponent $\beta \in (0, 1]$ and A is a sectorial operator. Consider the fractional equations of order $0 < \alpha < 1$

$${}_a^C D_t^\alpha u(t) = Au(t) + J^{1-\alpha} f(t),\ t \in J = [a, T],\ a \geq 0,\ u(a) = u_0. \tag{13}$$

Then a function $u(t) \in C([a, T], X)$ is the solution of the equation (13) if it satisfies the following integral equation

$$u(t) = S_\alpha(t - a)u_0 + \int_a^t S_\alpha(t - s)f(s)ds, \tag{14}$$

where $S_\alpha(t)$ is solution operator generated by A defined as

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} d\lambda,$$

Γ is a suitable path lying on $\sum_{\theta,\omega}$.

Proof. Let $t = w + a$, then the problem (13) translated into the form

$$D_w^\alpha \tilde{u}(w) = A\tilde{u}(w) + J^{1-\alpha} \tilde{f}(w),$$

$$\tilde{u}(0) = u_0.$$

Now, applying the Laplace transform, we have

$$\lambda^\alpha L\{\tilde{u}(w)\} - \lambda^{\alpha-1}\tilde{u}(0) = AL\{\tilde{u}(w)\} + L\{J^{1-\alpha}\tilde{f}(w)\}.$$

We get

$$\lambda^\alpha L\{\tilde{u}(w)\} - \lambda^{\alpha-1}\tilde{u}(0) = AL\{\tilde{u}(w)\} + \frac{1}{\lambda^{1-\alpha}} L\{\tilde{f}(w)\}. \tag{15}$$

Since $(\lambda^\alpha I - A)^{-1}$ exists, that is $\lambda^\alpha \in \rho(A)$, from (15), we obtain

$$L\{\tilde{u}(w)\} = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}\tilde{u}(0) + (\lambda^\alpha I - A)^{-1} \times \frac{1}{\lambda^{1-\alpha}} L\{\tilde{f}(w)\}.$$

Therefore, by taking the inverse Laplace transformation, we have

$$\tilde{u}(w) = E_{\alpha,1}(Aw^\alpha)\tilde{u}(0) + \int_0^w E_{\alpha,1}(A(w - \tau)^\alpha)\tilde{f}(\tau)d\tau.$$

Let $w = t - a$, we obtain

$$u(t) = E_{\alpha,1}(A(t - a)^\alpha)u_0 + \int_0^{t-a} E_{\alpha,1}(A(t - a - \tau)^\alpha)f(\tau)d\tau.$$

This is the same as

$$u(t) = E_{\alpha,1}(A(t-a)^{\alpha})u_0 + \int_a^t E_{\alpha,1}(A(t-s)^{\alpha})f(s)ds.$$

Let $S_\alpha(t) = E_{\alpha,1}(At^{\alpha})$, then we have

$$u(t) = S_\alpha(t-a)u_0 + \int_a^t S_\alpha(t-s)f(s)ds.$$

This completes the proof of the Lemma. ■

Following definition of mild solution is based on definition 2.1 in [9].

Definition 5 A function $u : (-\infty, T] \to X$ s.t. $u \in \mathfrak{B}'_h$ is called a mild solution of the problem (10)-(12) if $u(0) = \phi(0),\ u(t) = g_j(t, u(t))$ for $t \in (t_j, s_j]$ and each $j = 1, 2, \cdots, N$, satisfies the following integral equation

$$u(t) = \begin{cases} S_\alpha(t)(\phi(0) - h(u_{t_1}, \cdots, u_{t_p})(0)) \\ + \int_0^t S_\alpha(t-s)f(s, u_{\rho(s,u_s)}, u(a_1(s)), \cdots, u(a_m(s)))ds, & \text{for all } t \in [0, t_1], \\ S_\alpha(t-s_i)g_i(s_i, u(s_i)) \\ + \int_{s_i}^t S_\alpha(t-s)f(s, u_{\rho(s,u_s)}, u(a_1(s)), \cdots, u(a_m(s)))ds, & \text{for all } t \in [s_i, t_{i+1}], \end{cases}$$

and every $i = 1, 2, \cdots, N$.

In fact, from the lemma 2 it is easy to see that definition 5 holds, so the proof is omitted.

Theorem 3 (Theorem 3.4 [6]) Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that (i) $Pu + Qw \in B$, whenever $u, w \in B$. (ii) P is compact and continuous. (iii) Q is a contraction mapping. Then there exists $z \in B$ such that $z = Pz + Qz$.

3 Main Result

This section is equipped with existence results of mild solutions for the nonlocal impulsive system (10)-(12). If $A \in \mathcal{A}^{\alpha}(\theta_0, \omega_0)$, then strongly continuous $\|S_\alpha(t)\| \le Me^{\omega t}$. Let $\widetilde{M_S} := \sup_{0\le t\le T} \|S_\alpha(t)\|_{L(X)}$. So we have $\|S_\alpha(t)\|_{L(X)} \le \widetilde{M_S}$.

To prove our results we assume the function $\rho : [0, T] \times \mathfrak{B}_h \to (-\infty, T]$ is continuous and $\phi \in \mathfrak{B}_h$. If $u \in \mathfrak{B}_h$ we defined $\bar{u} : (-\infty, T) \to X$ as the extension of u to $(-\infty, T]$ such that $\bar{u}(t) = \phi$. We defined $\widetilde{u} : (-\infty, T) \to X$ such that $\widetilde{u} = u + x$ where $x : (-\infty, T) \to X$ is the extension of $\phi \in \mathfrak{B}_h$ such that $x(t) = S_\alpha(t)\phi(0)$ for $t \in J$. Now, we introduce the following axioms:

(H_1) There exist positive constant L_f such that

$$\|f(t, \varphi, u_1 \cdots, u_m) - f(t, \xi, w_1 \cdots, w_m)\|_X \le L_f[\|\varphi - \xi\|_{\mathfrak{B}_h} + \sum_{i=1}^{m} \|u_i - w_i\|_X],$$

for $t \in J,\ \varphi, \xi \in \mathfrak{B}_h,\ (u_1 \cdots, u_m), (w_1 \cdots, w_m) \in X^m$.

(H_2) There exist positive constant L_h such that

$$\|h(\phi(t_1), \cdots, \phi(t_p)) - h(\varphi(t_1), \cdots, \varphi(t_p))\|_X \le L_h\|\phi - \varphi\|_{\mathfrak{B}_h},$$

for all $t_i \in J$ and $\phi, \varphi \in \mathfrak{B}_h$.

(H_3) There exist positive constant L_{g_i} such that

$$\|g_i(t, y) - g_i(t, z)\|_X \le L_{g_i}\|y - z\|_X,$$

for all $t \in J$ and $y, z \in X$.

(H_4) f is continuous function and there exist a positive constant M such that

$$\|f(t,\varphi,u_1\cdots,u_m)\|_X \le M_1, \text{ for } (u_1\cdots,u_m)\in X^m, \varphi\in\mathfrak{B}_h \text{ and } t\in J.$$

(H_5) There exists positive constants M_2, M_3 such that

$$\|h((\phi(t_1),\cdots,\phi(t_p))\|_X \le M_2;\ \|g_i(t,y)\|_X\le M_3, \text{ for } y\in X,\ (\phi(t_1),\cdots,\phi(t_p))\in\mathfrak{B}_h^p.$$

Theorem 4 Let the assumptions $(H_1)-(H_3)$ hold and the constant

$$\delta=\max\{\widetilde{M_S}(\|K\|_\infty L_h+TL_f(K_b+m)),\widetilde{M_S}(L_{g_i}+TL_f(K_b+m))\}<1,$$

for $i=1,2,\cdots,N$. Then there exists a unique mild solution $u(t)$ on J of the system (10)-(12).

Proof. Let $\bar\phi:(-\infty,T)\to X$ be the extension of ϕ to $(-\infty,T]$ such that $\bar\phi(t)=\phi(0)$ on J. Consider the space $\mathfrak{B}''_h=\{u\in\mathfrak{B}'_h: u(0)=\phi(0)\}$ and $u(t)=\phi(t)$, for $t\in(-\infty,0]$ endowed with the uniform convergence topology. Let us consider a operator $P:\mathfrak{B}''_h\to\mathfrak{B}''_h$ defined as $Pu(0)=\phi(0),\ Pu(t)=g_i(t,\bar u(t))$ for $t\in(t_i,s_i]$ and

$$Pu(t)=\begin{cases} S_\alpha(t)(\phi(0)-h(\bar u_{t_1},\cdots,\bar u_{t_p})(0)) \\ +\int_0^t S_\alpha(t-s)f(s,\bar u_{\rho(s,\bar u_s)},\bar u(a_1(s)),\cdots,\bar u(a_m(s)))ds, & t\in[0,t_1], \\ S_\alpha(t-s_i)g_i(s_i,\bar u(s_i)) \\ +\int_{s_i}^t S_\alpha(t-s)f(s,\bar u_{\rho(s,\bar u_s)},\bar u(a_1(s)),\cdots,\bar u(a_m(s)))ds, & t\in[s_i,t_{i+1}],\end{cases}$$

where $\bar u:(-\infty,T]\to X$ is such that $\bar u(0)=\phi$ and $\bar u=u$ on J. It is obvious that P is well defined. Let $u(t),u^*(t)\in\mathfrak{B}''_h$ for $t\in[0,t_1]$, we have

$$\begin{aligned}\|Pu(t)-Pu^*(t)\|_X &\le \|S_\alpha(t)\|_{L(X)}(\|h(\bar u_{t_1},\cdots,\bar u_{t_p})(0))-h(\bar u^*_{t_1},\cdots,\bar u^*_{t_p})(0))\|_X \\ &\quad+\int_0^t\|S_\alpha(t-s)\|_{L(X)}\|f(s,\bar u_{\rho(s,\bar u_s)},\bar u(a_1(s)),\cdots,\bar u(a_m(s))) \\ &\quad-f(s,\bar u^*_{\rho(s,\bar u^*_s)},\bar u^*(a_1(s)),\cdots,\bar u^*(a_m(s)))\|_X ds \\ &\le \widetilde{M_S}(\|K\|_\infty L_h+TL_f(K_b+m))\|u-u^*\|_{\mathfrak{B}''_h}.\end{aligned}$$

For $t\in[s_i,t_{i+1}]$, we estimate as

$$\begin{aligned}\|Pu(t)-Pu^*(t)\|_X &\le \|S_\alpha(t-s_i)\|_{L(X)}\|g_i(s_i,\bar u(s_i))-g_i(s_i,\bar u^*(s_i))\|_X \\ &\quad+\int_{s_i}^t\|S_\alpha(t-s)\|_{L(X)}\|f(s,\bar u_{\rho(s,\bar u_s)},\bar u(a_1(s)),\cdots,\bar u(a_m(s))) \\ &\quad-f(s,\bar{u^*}_{\rho(s,\bar u^*_s)},\bar u^*(a_1(s)),\cdots,\bar u^*(a_m(s)))\|_X ds \\ &\le \widetilde{M_S}(L_{g_i}+TL_f(K_b+m))\|u-u^*\|_{\mathfrak{B}''_h}.\end{aligned}$$

For $t\in(t_j,s_j]$, we get

$$\|Pu(t)-Pu^*(t)\|_X\le L_{g_j}\|u-u^*\|_{\mathfrak{B}''_h},\qquad j=1,2,\cdots,N.$$

Gathering above results, we obtain

$$\|Pu(t)-Pu^*(t)\|_X\le\delta\|u-u^*\|_{\mathfrak{B}''_h}.$$

Since $\delta<1$, which implies that P is a contraction map and there exists a unique fixed point which is the mild solution of system (10)-(12). This completes the proof of the theorem. ■

Theorem 5 Let the assumptions $(H_2)-(H_5)$ hold and

$$\max\{\widetilde{M_S}\|K\|_\infty L_h,\widetilde{M_S}L_{g_i}\}<1.$$

Then system (10)-(12) has atleast one mild solution $u(t)$ on J.

Proof. Let $\bar{\phi} : (-\infty, T) \to X$ be the extension of ϕ to $(-\infty, T]$ such that $\bar{\phi}(t) = \phi(0)$ on J. Consider the space $\mathfrak{B}''_h = \{u \in \mathfrak{B}'_h : u(0) = \phi(0)\}$ and $u(t) = \phi(t)$, for $t \in (-\infty, 0]$ endowed with the uniform convergence topology. Choose $r \geq \max\{\widetilde{M_S}(\|\phi(0)\|_{\mathfrak{B}_h} + M_2) + \widetilde{M_S}TM_1, \widetilde{M_S}M_3 + \widetilde{M_S}TM_1\}$ and consider space $\mathcal{B}_r = \{u \in \mathfrak{B}''_h : \|u\| \leq r\}$, then it is clear that $\mathcal{B}_r$ is a bounded, closed and convex subset in $\mathfrak{B}''_h$. Let us define the operator P and Q on $\mathcal{B}_r$ by $Pu(0) = \phi(0),\ Pu(t) = g_i(t, \bar{u}(t))$ for $t \in (t_i, s_i]$ and

$$Pu(t) = \begin{cases} S_\alpha(t)(\phi(0) - h(\bar{u}_{t_1}, \cdots, \bar{u}_{t_p})(0)), & t \in [0, t_1], \\ S_\alpha(t - s_i)g_i(s_i, \bar{u}(s_i)), & t \in [s_i, t_{i+1}], \end{cases}$$

$$Qu(t) = \begin{cases} \int_0^t S_\alpha(t-s)f(s, \bar{u}_{\rho(s,\bar{u}_s)}, \bar{u}(a_1(s)), \cdots, \bar{u}(a_m(s)))ds, & t \in [0, t_1], \\ \int_{s_i}^t S_\alpha(t-s)f(s, \bar{u}_{\rho(s,\bar{u}_s)}, \bar{u}(a_1(s)), \cdots, \bar{u}(a_m(s)))ds, & t \in [s_i, t_{i+1}], \end{cases}$$

where $\bar{u} : (-\infty, T] \to X$ is such that $\bar{u(0)} = \phi$ and $\bar{u} = u$ on J. Let $u, w \in \mathcal{B}_r$, then for $t \in [0, t_1]$ we have

$$\begin{aligned} \|Pu(t) + Qw(t)\|_X &\leq \|S_\alpha(t)\|_{L(X)}(\|\phi(0)\|_X + \|h(\bar{u}_{t_1}, \cdots, \bar{u}_{t_p})(0))\|_X \\ &\quad + \int_0^t \|S_\alpha(t-s)\|_{L(X)}\|f(s, \bar{w}_{\rho(s,\bar{w}_s)}, \bar{w}(a_1(s)), \cdots, \bar{w}(a_m(s)))\|_X ds \\ &\leq \widetilde{M_S}(\|\phi(0)\|_X + M_2) + \widetilde{M_S}TM_1. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we estimate as

$$\begin{aligned} \|Pu(t) + Qw(t)\|_X &\leq \|S_\alpha(t - s_i)\|_{L(X)}\|g_i(s_i, \bar{u}(s_i))\|_X \\ &\quad + \int_{s_i}^t \|S_\alpha(t-s)\|_{L(X)}\|f(s, \bar{w}_{\rho(s,\bar{w}_s)}, \bar{w}(a_1(s)), \cdots, \bar{w}(a_m(s)))\|_X ds \\ &\leq \widetilde{M_S}M_3 + \widetilde{M_S}TM_1. \end{aligned}$$

For all $t \in [0, T]$

$$\|Pu(t) + Qw(t)\|_X \leq \max\{\widetilde{M_S}(\|\phi(0)\|_X + M_2) + \widetilde{M_S}TM_1, \widetilde{M_S}M_3 + \widetilde{M_S}TM_1\} \leq r,$$

which implies that $Pu(t) + Qw(t) \in \mathcal{B}_r$.

Now, we shall show that P is contraction map. Let $u(t), u^*(t) \in \mathcal{B}_r$ for $t \in [0, t_1]$, we have

$$\begin{aligned} \|Pu(t) - Pu^*(t)\|_X &\leq \|S_\alpha(t)\|_{L(X)}(\|h(\bar{u}_{t_1}, \cdots, \bar{u}_{t_p})(0)) - h(\bar{u}^*_{t_1}, \cdots, \bar{u}^*_{t_p})(0))\|_X \\ &\leq \widetilde{M_S}\|K\|_\infty L_h\|u - u^*\|_{\mathfrak{B}''_h}. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we estimate as

$$\begin{aligned} \|Pu(t) - Pu^*(t)\|_X &\leq \|S_\alpha(t - s_i)\|_{L(X)}\|g_i(s_i, \bar{u}(s_i)) - g_i(s_i, \bar{u}^*(s_i))\|_X \\ &\leq \widetilde{M_S}L_{g_i}\|u - u^*\|_{\mathfrak{B}''_h}. \end{aligned}$$

For $t \in (t_j, s_j]$, we get

$$\|Pu(t) - Pu^*(t)\|_X \leq L_{g_j}\|u - u^*\|_{\mathfrak{B}''_h}, \qquad j = 1, 2, \cdots, N.$$

By above results, we obtain

$$\|Pu(t) - Pu^*(t)\|_X \leq \max\{\widetilde{M_S}\|K\|_\infty L_h, \widetilde{M_S}L_{g_i}\}\|u - u^*\|_{\mathfrak{B}''_h}.$$

Since $\max\{\widetilde{M_S}\|K\|_\infty L_h, \widetilde{M_S}L_{g_i}\} < 1$, which implies that P is a contraction map. Next, we shall show that Q is completely continuous on $\mathcal{B}_r$. So consider a sequence $u^n \to u$ in $\mathcal{B}_r$, for $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|Qu^n(t) - Qu(t)\|_X &\leq \int_{s_i}^t \|S_\alpha(t-s)\|_{L(X)}\|f(s, \bar{u}^n_{\rho(s,\bar{u}^n_s)}, \bar{u}^n(a_1(s)), \cdots, \bar{u}^n(a_m(s))) \\ &\quad - f(s, \bar{u}_{\rho(s,\bar{u}_s)}, \bar{u}(a_1(s)), \cdots, \bar{u}(a_m(s)))\|_X ds. \end{aligned}$$

Since function f is continuous, so $\|Qu^n(t) - Qu(t)\|_X \to 0$ as $n \to \infty$. Which implies that Q is continuous. It is easy to prove that Q maps bounded set into bounded set in $\mathcal{B}_r$. To do this we have for all $t \in [0, T]$

$$\|Qu(t)\|_X \leq T\widetilde{M_S}M_1 = C^*.$$

Finally, we show Q is a family of equi-continuous functions in $\mathcal{B}_r$. Let $l_1, l_2 \in [s_i, t_{i+1}]$, such that $s_i \leq l_1 < l_2 \leq t_{i+1}$. We have

$$\begin{aligned}
\|Q(u)(l_2) - Q(u)(l_1)\|_X &\leq \int_{s_i}^{l_1} \|S_\alpha(l_2 - s) - S_\alpha(l_1 - s)\|_{L(X)} \\
&\quad \times \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}, \bar{u}(a_1(s)), \cdots, \bar{u}(a_m(s)))\|_X ds \\
&\quad + \int_{l_1}^{l_2} \|S_\alpha(l_2 - s)\|_{L(X)} \\
&\quad \times \|f(s, \bar{u}_{\rho(s,\bar{u}_s)}, \bar{u}(a_1(s)), \cdots, \bar{u}(a_m(s)))\|_X ds \\
&\leq M_1 \int_{s_i}^{l_1} \|S_\alpha(l_2 - s) - S_\alpha(l_1 - s)\|_{L(X)} ds \\
&\quad + \widetilde{M_S} M_1 (l_2 - l_1).
\end{aligned}$$

Since $S_\alpha(t)$ is strongly continuous, so $\lim_{l_2 \to l_1} \|S_\alpha(l_2 - s) - S_\alpha(l_1 - s)\|_{L(X)} = 0$, which implies that $\|Q(u)(l_2) - Q(u)(l_1)\|_X \to 0$ as $l_2 \to l_1$. This proves that Q is a family of equi-continuous functions. So, we conclude that the operator Q is a completely continuous or compact operator by Arzela-Ascoli's theorem. Therefor by the theorem 3 there exist atleast one fixed point in $\mathcal{B}_r$. Hence we conclude that the system (10)-(12) has a mild solution $u(t)$ on J. This completes the proof of the theorem. ■

4 Example

Consider the following nonlocal impulsive fractional partial differential equation of the form

$$\begin{aligned}
\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) &= \frac{\partial^2}{\partial y^2} u(t,x) + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^\alpha \\
&\quad \times \frac{1}{16} \int_{-\infty}^{s} e^{2(\nu - s)} u(\nu - \rho_1(\nu)\rho_2(\|u\|)), u(a_1(\nu)), \cdots, u(a_m(\nu)), x) d\nu ds, \\
&\quad (t,x) \in \cup_{i=1}^{N} [s_i, t_{i+1}] \times [0, \pi], && (16) \\
u(t,0) &= u(t,\pi) = 0, \ \ t \geq 0, && (17) \\
u(t,x) &+ \sum_{i=1}^{p} \int_0^\pi K(x,\xi) u(t_i, \xi) d\xi = \phi(t,x), \ t \in (-\infty, 0], \ x \in [0,\pi], && (18) \\
u(t,x) &= G_i(t, u(t,x)), \ x \in [0,\pi], \ t \in (t_i, s_i], && (19)
\end{aligned}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha \in (0,1]$, $0 = t_0 = s_0 < t_1 \leq s_1 < \cdots < t_N \leq s_N < t_{N+1} = 1$ are fixed real numbers, $\phi \in \mathfrak{B}_h$, and p is a positive integer, $0 < t_0 < t_1, \cdots, < t_p < 1$. Let $X = L^2[0,\pi]$ and define the operator $A : D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) := \{w \in X : \ w, w'$ are absolutely continuous, $w'' \in X, w(0) = 0 = w(\pi)\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \ w \in D(A),$$

where $w_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \ n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ in X and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n) \omega_n, \ \text{for all } \omega \in X, \ \text{and every } t > 0.$$

The subordination principle of solution operator implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq 0}$, such that $\|S_\alpha(t)\|_{L(X)} \leq \widetilde{M_S}$ for $t \in [0,1]$.

Let $h(s) = e^{2s},\ s < 0$ then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$ and define

$$\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t, \phi) \in [0, 1] \times \mathfrak{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x),\ (\theta, x) \in (-\infty, 0] \times [0, \pi]$.

Set $u(t)(x) = u(t, x)$, and $\rho(t, \phi) = \rho_1(t)\rho_2(\|\phi(0)\|)$ we have

$$\begin{aligned}
&f(t, u(t - \rho_1(t)\rho_2(\|u\|)), u(a_1(t)), \cdots, u(a_m(t))) \\
&= \frac{1}{16}\int_{-\infty}^{0} e^{2(\nu)}\phi, u(a_1(\nu)), \cdots, u(a_m(\nu)))(x)d\nu, \\
&g_i(t, u)(x) = G_i(t, u(t, x)) = G_i(t, u)(x), \\
&h(u_{t_1}, \cdots, u_{t_p})(t) = \sum_{i=0}^{n} K_q u_{t_i}(t)
\end{aligned}$$

where $K_q z(x) = \int_0^\pi k(x, \xi)z(\xi)d\xi$ for $z \in L^2[0, \pi],\ x \in [0, \pi]$ then with these settings the equations (16)-(19) can be written in the abstract form of equation(10)-(12).

We assume that $\rho_i : [0, \infty) \to [0, \infty),\ i = 1, 2$, are continuous functions and taking $\widetilde{M_S} = \|K\| = K_b = 1$. The functions f, g_i and h are Lipschitz with Lipschitz constants L_f, Lg_i and L_h respectively with

$$\max\{L_h + L_f(1 + m), L_{g_i} + L_f(1 + m) : i = 1, \cdots, N\} < 1,$$

then there exists a unique mild solution u on $[0, 1]$ by the Theorem 4.

References

[1] S. G. Samko et al. Fractional integrals and derivatives theory and applications. *Gordon and Breach, Yverdon*. (1993).

[2] K. S. Miller and B. Ross. An introduction to the fractional falculus and differential equations. *John Wiley, New York*. (1993).

[3] A. A. Kilbas et al. Theory and applications of fractional differential equations. *North-Holland Mathematics Studies*. (2006).

[4] V. Lakshmikantham et al. Theory of fractional dynamic systems. *Cambridge Scientific Publishers*. (2009).

[5] I. Podlubny. Fractional Differential Equations. *Acadmic Press New York*. (1999).

[6] A. Chauhan and J. Dabas. Existence of mild solutions impulsive fractional order semilinear evolution equations with non local conditions. *Electronic. J. Diff. Equ.* 107(2011): 1–10.

[7] D. Bahuguna. Existence, uniqueness and regularity of solutions to semilinear nonlocal functional differential problems. *Nonlinear Anal.* 57(2004): 1021–1028.

[8] L. Byszewski and H. Acka. Existence of solutions of a semilinear functional differential evoluation nonlocal problems. *Nonlinear Anal.* 34(1998): 65–72.

[9] E. Hernandez and D. O'Regan, On a new class of abstract impulsive differential equations. *Pro. American Math. Society.* 141(2013): 1641–1649.

[10] M. Pierri et al. Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses. *Appl. Math. Comput.* 219(2013): 6743–6749.

[11] Y. Zhou and F. Jiao. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* 59(2010): 1063–1077

[12] J. Dabas et al. Existence of mild solution for impulsive fractional equation with infinity delay. *Int. J. Diff. Equ.* (2011).

[13] M. Benchohra and F. Berhoun. Impulsive fractional differential equations with state dependent delay. *Commun. Appl. Anal.* 14(2010): 213–224.

[14] M. Haase. The functional calculus for sectorial operators. *Operator theory Advan.* (2006).

[15] R. P. Agarwal and B. De Andrade. On fractional integro-differential equations with state-dependent delay. *Comput. Math. Appl.* 62(2011): 1143–1149.

[16] S. Abbas and M. Benchohra. Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay. *An Inter. J. Theory Appl.* (2010).

[17] M. M. Arjunan and V. Kavitha. Existence results for impulsive neutral functional differential equations with state-dependent delay. *Electron. J. Qual. Theory Diff. Equ.* 26(2009): 1–13.

[18] M. Benchohra et al. On fractional integro-differential inclusions with state-dependent delay in Banach spaces. *Appl. Anal.* 11(2011): 1–16.

[19] J. P. Carvalho et al. Existence results for a fractional equation with state-dependent delay. *Advan. Diff. Equ.* (2011).

[20] J. P. Carvalho dos Santos and M. M. Arjunan. Existence results for fractional neutral integro-differential equations with state-dependent delay. *Comput. Math. Appl.* 62(2011): 1275–1283.

[21] A. Chauhan et al. Integral boundary value problem for impulsive fractional functional differential equations with infinite delay. *Elect. J. Diff. Eq.* 229(2012): 1–13.

[22] J. Dabas and A. Chauhan. Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equations with infinity delay. *Math. Comput. Model.* 57 (2013): 754–763.

[23] M. A. Darwish and S. K. Ntouyas. Functional differential equations of fractional order with state-dependent delay. *Dynamic Syst. Appl.* 18(2009): 539–550.

[24] M. A. Darwish and S. K. Ntouyas. Semilinear functional differential equations of fractional order with state-dependent delay. *Electron. J. Diff. Equ.* (2009).

[25] V. Kavitha et al. Existence results for neutral functional fractional differential equations with state dependent-delay. *Malaya J. Mat.* 1(2012): 50–61.

[26] N. K. Tomar and J. Dabas. Controllability of impulsive fractional order semilinear evoluation equations with nonlocal conditions. *J. Non. Evol. Eq. Appl.* 5(2012): 57–67.

[27] J. Dabas and G. R. Gautam. Impulsive neutral fractional integro-differential equation with state dependent delay and integral boundary condition. *Elect. J. Diff. Equ.* 273(2013): 1–13.

[28] P. Kumara et al. On a new class of abstract impulsive functional differential equations of fractional order. *J. Nonlinear Sci. Appl.* 7(2014): 102–114.

[29] J. Wang and X. Li. Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses. *J. Appl. Math. Comput.* (2014).

[30] A. Anguraj and M. Lathamaheswari. Existence results for fractional differential equations with infinite delay and interval impulsive conditions. *Malaya J. Mat.* 1(2014): 16–23.