ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.21(2016) No.3, pp.145-150

Blow-up Profiles To a Nonlinear Degenerate Parabolic System with Nonlocal Boundary Condition

Guangsheng Zhong[1,2,*]
[1] Nonlinear Scientific Research Center, Jiangsu University, Jiangsu, 212013,P.R.China
[2] School of Science,Nantong University , Jiangsu, 226007, P.R.China
(Received 30 November 2015, accepted 28 February 2016)

Abstract: This paper deals with a nonlocal degenerate parabolic system with weighted nonlocal boundary condition. We give out the criteria for finite time blow-up or global existence, which shows the weight functions play the substantial roles in determining whether the solutions will blow-up or not.

Keywords: parabolic system; nonlocal boundary condition; global existence; blow up

1 Introduction

In this paper, we consider the following degenerate parabolic system

$$u_t = \Delta u^m + av^{p_1} \|v\|_{\alpha}^{p_2}, v_t = \Delta v^n + bu^{q_1} \|u\|_{\beta}^{q_2}, \qquad x \in \Omega, t > 0, \tag{1.1}$$

with nonlocal boundary conditon

$$u(x,t) = \int_{\Omega} f(x,y)u(y,t)dy, \;\; v(x,t) = \int_{\Omega} g(x,y)v(y,t)dy, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $R^N(N \geq 1)$ with smooth boundary $\partial\Omega$,$m, n > 1$,$\alpha, \beta \geq 1$, $a, b > 0$,$p_i, q_i \geq 0$,$(i = 1, 2)$ and $(p_1 + p_2)(q_1 + q_2) > 0$, $\|.\|_{\alpha}^{\alpha} = \int_{\Omega} |.|^{\alpha} dx$. While $f(x, y), g(x, y)$ are nonnegative and continuous on $\partial\Omega \times \overline{\Omega}$, $u_0(x), v_0(x) \in C^{2+\theta}(\overline{\Omega})$ with $0 < \theta < 1$, $u_0(x), v_0(x) \geq 0$, and satisfy the compatibility conditions $u_0(x) = \int_{\Omega} f(x, y)u_0(y)dy$, $v_0(x) = \int_{\Omega} g(x, y)v_0(y)dy$, $x \in \partial\Omega$, respectively.

There have been many articles dealing with properties of solutions to degenerate parabolic equations or system with homogeneous Dirichlet boundary conditions (see [1, 5, 7,9] and references therein). In [3,4], Galaktionov et al. considered the system

$$u_t = \Delta u^{v+1} + v^p, \quad v_t = \Delta v^{\mu+1} + u^q, \qquad x \in \Omega, t > 0, \tag{1.4}$$

subject to homogeneous Dirichlet boundary condition. They proved that every solution of (1.4) is global if $pq < (1 + v)(1 + \mu)$, and there are solutions blow up in finite time for sufficiently large initial date if $pq > (1 + v)(1 + \mu)$, which show that $p_c = pq - (1 + v)(1 + \mu)$ is the critical exponent of (1.4).

Furthermore, Deng et al. in [8] considered the system

$$u_t = \Delta u^m + \|v\|_{\alpha}^{p}, \quad v_t = \Delta v^n + \|u\|_{\beta}^{q}, \qquad x \in \Omega, t > 0, \tag{1.5}$$

subject to homogeneous Dirichlet boundary condition. They proved the critical exponent is $p_c = pq - mn$.

*Corresponding author. *E-mail address*: zhgs917@163.com

IJNS.2016.06.15/910

However, parabolic equations with nonlocal boundary conditions have been studied as well (see [10]-[13]). For example, Kong et al. in [10], by using some ideas of Souplet [7], obtained the blow-up conditions for the following system

$$u_t = \Delta u + u^m \int_\Omega v^n(x,t)dx, \quad v_t = \Delta v + v^q \int_\Omega u^p(x,t)dx, \qquad x \in \partial\Omega, t > 0 \,, \tag{1.6}$$

subject to nonlocal boundary (1.2). The typical characterization of (1.6) is the completely coupled of the nonlocal sources, which leads to the analysis of the simultaneous blow-up.

Motivated by [3],[4] and [8]-[10], we discuss how the weight functions in the boundary conditions affect the global and blow-up properties for problem (1.1)-(1.3). We will show that the weight functions $f(x,y), g(x,y)$ play substantial roles in determining blow-up or not of solutions.

This paper is organized as follows. In the next section, we give the comparison principle of the solution of problem (1.1)-(1.3). In Section 3, we will consider the global existence and blow-up of solution.

2 Comparison principle

Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$ for $0 < T < \infty$. As it is now well known that degenerate equation need not possess classical solutions, we begin by giving a precise definition of a weak solution of (1.1)-(1.3).

Definition 2.1 *A vector function $(w(x,t), z(x,t))$ defined on $\overline{Q_T}$, for some $T > 0$, is called a sub (or super) solution of problem (1.1)-(1.3) on $\overline{Q_T}$, if $w(x,t), z(x,t) \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ and satisfy*

$$\begin{array}{ll} w_t \le (\ge)\Delta w^m + a z^{p_1} \|z\|_\alpha^{p_2}\,, & x \in \Omega, t > 0, \\ z_t \le (\ge)\Delta z^n + b w^{q_1} \|w\|_\beta^{q_2}\,, & x \in \Omega, t > 0, \\ (w(x,t), z(x,t)) \le (\ge) \left(\int_\Omega f(x,y)w(y,t)dy, \int_\Omega g(x,y)z(y,t)dy\right), & x \in \partial\Omega, t > 0, \\ (w(x,0), z(x,0)) \le (\ge)(u_0(x), v_0(x)), & x \in \Omega. \end{array} \tag{2.1}$$

A weak solution of problem (1.1)-(1.3) is a vector function which is both a subsolution and a supersolution of problem (1.1)-(1.3). Regarding the subsolution and the supersolution, we have the following comparison lemma which plays an important role in our proof.

Lemma 2.1 *(Comparison principle). Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be a nonnegative subsolution and a nonnegative supersolution of problem (1.1)-(1.3), respectively. Then $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$ in $\overline{Q_T}$, if $(\underline{u}(x,0), \underline{v}(x,0)) \le (\overline{u}(x,0), \overline{v}(x,0))$, and*

$$\underline{u}, \underline{v} \ge \delta > 0 \quad \text{or} \quad \overline{u}, \overline{v} \ge \delta > 0. \tag{2.2}$$

Proof. Taking $\psi(x,t) \in \Psi \equiv \{\psi \in C(\overline{Q_T}); \psi_t, \Delta\psi \in C(Q_T) \cap L^2(Q_T); \psi \ge 0, \psi(x,t)\big|_{\partial\Omega \times (0,T)} = 0\}$ and multiplying the first and second inequalities in (2.1) by $\psi(x,t)$, then integrate them on $\Omega \times (0,t)$ for any $0 < t < T$, we obtain

$$\begin{array}{c} \int_\Omega \underline{u}(x,t)\psi(x,t)dx \le \int_\Omega \underline{u}(x,0)\psi(x,0)dx + \int_0^t \int_\Omega (\underline{u}\psi_\tau + \underline{u}^m \Delta\psi + a\underline{v}^{p_1} \|\underline{v}\|_\alpha^{p_2} \psi)dxd\tau \\ - \int_0^t \int_{\partial\Omega} \frac{\partial\psi}{\partial\eta} \left(\int_\Omega f(x,y)\underline{u}(y,\tau)\right)^m dSd\tau, \end{array} , \tag{2.3}$$

$$\begin{array}{c} \int_\Omega \underline{v}(x,t)\psi(x,t)dx \le \int_\Omega \underline{v}(x,0)\psi(x,0)dx + \int_0^t \int_\Omega (\underline{v}\psi_\tau + \underline{v}^n \Delta\psi + b\underline{u}^{q_1} \|\underline{u}\|_\beta^{q_2} \psi)dxd\tau \\ - \int_0^t \int_{\partial\Omega} \frac{\partial\psi}{\partial\eta} \left(\int_\Omega g(x,y)\underline{v}(y,\tau)\right)^n dSd\tau. \end{array} , \tag{2.4}$$

where η is the unit outward normal to the lateral boundary of Q_T . On the other hand, the supersolution $(\overline{u}, \overline{v})$ satisfies the reverse inequality

$$\begin{array}{c} \int_\Omega \overline{u}(x,t)\psi(x,t)dx \ge \int_\Omega \overline{u}(x,0)\psi(x,0)dx + \int_0^t \int_\Omega (\overline{u}\psi_\tau + \overline{u}^m \Delta\psi + a\overline{v}^{p_1} \|\overline{v}\|_\alpha^{p_2} \psi)dxd\tau \\ - \int_0^t \int_{\partial\Omega} \frac{\partial\psi}{\partial\eta} \left(\int_\Omega f(x,y)\overline{u}(y,\tau)\right)^m dSd\tau, \end{array} , \tag{2.5}$$

$$\begin{array}{c} \int_\Omega \overline{v}(x,t)\psi(x,t)dx \ge \int_\Omega \overline{v}(x,0)\psi(x,0)dx + \int_0^t \int_\Omega (\overline{v}\psi_\tau + \overline{v}^n \Delta\psi + b\overline{u}^{q_1} \|\overline{u}\|_\beta^{q} \psi)dxd\tau \\ - \int_0^t \int_{\partial\Omega} \frac{\partial\psi}{\partial\eta} \left(\int_\Omega g(x,y)\overline{v}(y,\tau)\right)^n dSd\tau. \end{array} , \tag{2.6}$$

By (2.3) and (2.5), we have

$$\begin{aligned}
&\int_\Omega [\underline{u}(x,t)-\overline{u}(x,t)]\psi(x,t)dx \\
&\quad\le \int_\Omega [\underline{u}(x,0)-\overline{u}(x,0)]\psi(x,0)dx \ + \int_0^t\int_\Omega [\psi_\tau + E_1(x,\tau)\Delta\psi](\underline{u}-\overline{u})dxd\tau \\
&\quad\ + \int_0^t\int_\Omega \left(a\,\|\underline{v}\|_\alpha^{p_2}\,\psi E_2(x,\tau)(\underline{v}-\overline{v}) + a\overline{v}^{p_1}\psi E_4(\tau)\int_\Omega E_3(x,\tau)(\underline{v}-\overline{v})dx\right)dxd\tau \\
&\quad\ - \int_0^t\int_{\partial\Omega} \frac{\partial\psi}{\partial\eta} m\xi^{m-1}\left(\int_\Omega f(x,y)(\underline{u}(y,\tau)-\overline{u}(y,\tau))dy\right)dSd\tau,
\end{aligned}$$

where

$E_1(x,\tau) = \int_0^1 m(\theta\underline{u} + (1-\theta)\overline{u})^{m-1}d\theta, \quad E_2(x,\tau) = \int_0^1 p_1(\theta\underline{v}+(1-\theta)\overline{v})^{p_1-1}d\theta,$

$E_3(x,\tau) = \int_0^1 \alpha(\theta\underline{v} + (1-\theta)\overline{v})^{\alpha-1}d\theta, \quad E_4(\tau) = \int_0^1 \frac{p_2}{\alpha}(\theta\,\|\underline{v}\|_\alpha^{p_2} + (1-\theta)\,\|\overline{v}\|_\alpha^{p_2})^{\frac{p_2}{\alpha}-1}d\theta,$

ξ is a function between $\int_\Omega f(x,y)\underline{u}(y,t)dy$ and $\int_\Omega f(x,y)\overline{u}(y,t)dy$.

Since $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are bounded Q_T, it follows from $m>1, \alpha\ge 1$ that $E_1(x,\tau),\ E_3(x,\tau)$ are bounded nonnegative functions. Similarly, $E_2(x,\tau), E_4(\tau)$ are bounded if $p_1, \frac{p_2}{\alpha} \ge 1$. Now, if $p_1, \frac{p_2}{\alpha} < 1$, we have $E_2(x,\tau) \le \delta^{p_1-1}$, $E_4(\tau) \le (\delta^{p_2}|\Omega|^{\frac{p_2}{\alpha}})^{\frac{p_2}{\alpha}-1}$, by the assumptions (2.2). In addition, $\partial\psi/\partial\eta \le 0$ on $\partial\Omega$. Thus, we can choose the appropriate test function ψ as in [2, p.118-123] to obtain

$$\begin{aligned}
&\int_\Omega [\underline{u}(x,t)-\overline{u}(x,t)]_+dx \\
&\quad\le C_1\int_\Omega [\underline{u}(x,0)-\overline{u}(x,0)]_+dx \ + C_2\int_0^t\int_\Omega [\underline{v}-\overline{v}]_+dxd\tau \\
&\quad\le C_2\int_0^t\int_\Omega [\underline{v}-\overline{v}]_+dxd\tau.
\end{aligned} \tag{2.7}$$

where $w_+ \equiv \max\{w,0\}$ and using $\underline{u}(x,0)-\overline{u}(x,0)\le 0$. Similarly, we can prove

$$\int_\Omega [\underline{v}(x,t)-\overline{v}(x,t)]_+dx \ \le C_3\int_0^t\int_\Omega [\underline{u}-\overline{u}]_+dxd\tau. \tag{2.8}$$

Now, (2.7),(2.8) combined with the Gronwall's lemma show that $(\underline{u},\underline{v}) \le (\overline{u},\overline{v})$, Thus, this proof is completed. ∎

Local in time existence of positive classical solutions of problem (1.1)-(1.3) can be obtained by using fixed point theorem (see [6]), the representation formula and the contraction mapping principle as in [11]. By the above comparison principle, we get the uniqueness of solution to the problem. The proof is more or less standard, so is omitted here.

Remark 1 *If $p_1\ge 1,\ p_2\ge 1,\ p_2\ge\alpha,\ q_2\ge\beta$, then the condition of (2.2) is not necessary in Lemma 2.1. In this case, it is also easy to see that the solution of problem (1.1)-(1.3) is unique.*

3 Global existence and finite time blow-up

We use the upper and lower solutions method to deduce conditions on the global existence or blow-up in finite time for problem (1.1)-(1.3), and it can be observed that the weight functions $f(x,y),\ g(x,y)$ play an important role in the discussions.

Theorem 3.1 *Suppose that $\int_\Omega f(x,y)dy\ge 1, \int_\Omega g(x,y)dy \ge 1$ for any $x\in\partial\Omega$. If $(p_1+p_2)(q_1+q_2)>1$, then any solution of problem (1.1)-(1.3) with positive initial data blows up in finite time.*

Proof. Let $(F(t),H(t))$ be the unique solution of the following ordinary differential equations

$$\begin{array}{ll}
F'(t) = a|\Omega|^{\frac{p_2}{\alpha}}H^{p_1+p_2}(t), & H'(t) = b|\Omega|^{\frac{q_2}{\beta}}F^{q_1+q_2}(t), \quad t>0, \\
F(0) = \frac{1}{2}\min\limits_{\overline{\Omega}} u_0(x) > 0, & H(0) = \frac{1}{2}\min\limits_{\overline{\Omega}} v_0(x) > 0,
\end{array} \quad . \tag{3.1}$$

Under the condition $(p_1+p_2)(q_1+q_2)>1$, we know that $(F(t),H(t))$ blows up in the finite time T^* . Set $\underline{u}(x,t) = F(t),\ \underline{v}(x,t) = H(t)$, then we obtain

$$\begin{array}{lll}
\underline{u}_t = \Delta\underline{u}^m + a\underline{v}^{p_1}\,\|\underline{v}\|_\alpha^{p_2}\,, & \underline{v}_t = \Delta\underline{v}^n + b\underline{u}^{q_1}\,\|\underline{u}\|_\beta^{q_2}\,, & x\in\Omega, t>0, \\
\underline{u}(x,t) \le \int_\Omega f(x,y)\underline{u}(y,t)dy, & \underline{v}(x,t)\le\int_\Omega g(x,y)\underline{v}(y,t)dy, & x\in\partial\Omega, t>0, \\
\underline{u}(x,0)\le u_0(x),\ \underline{v}(x,0)\le v_0(x), & & x\in\Omega,
\end{array} \quad . \tag{3.2}$$

By Lemma 2.1, we get $(\underline{u}(x,t),\underline{v}(x,t)) \le (u(x,t),v(x,t))$. So $(u(x,t),v(x,t))$ blows up in a finite time for any positive initial data. ∎

Theorem 3.2 *Suppose that* $\int_\Omega f(x,y)dy < 1, \int_\Omega g(x,y)dy < 1$ *for any* $x \in \partial\Omega$.
(1) If $[(p_1+p_2)(q_1+q_2) < mn$*, then every nonnegative solution of problem (1.1)-(1.3) is global;*
(2) If $(p_1+p_2)(q_1+q_2) = mn$ *, then every nonnegative solution of problem (1.1)-(1.3) is global provided that* a *and* b *are sufficiently small;*
(3)If $(p_1+p_2)(q_1+q_2) > mn$*, then the nonnegative solution of problem (1.1)-(1.3) exists globally for sufficiently small initial data and blows up in finite time for sufficiently large initial data.*

Proof. Let $\varphi_1(x),\ \varphi_2(x)$ be the positive solution of the following linear elliptic problem, respectively

$$-\Delta\varphi_1(x) = \varepsilon_1, \quad x \in \Omega, \quad \varphi_1(x) = \int_\Omega f(x,y)dy, \ \ x \in \partial\Omega, \tag{3.3}$$

and

$$-\Delta\varphi_2(x) = \varepsilon_2, \quad x \in \Omega, \quad \varphi_2(x) = \int_\Omega g(x,y)dy, \ \ x \in \partial\Omega, \tag{3.4}$$

where $\varepsilon_1,\ \varepsilon_2$ are positive constants such that $0 < \varphi_i(x) < 1 \ \ (i = 1,2)$ (since $\int_\Omega f(x,y)dy < 1, \int_\Omega g(x,y)dy < 1$, there exist such ε_i). Set $\overline{K_1} = \max_{x\in\overline{\Omega}} \varphi_1(x), \ \ \overline{K_2} = \max_{x\in\overline{\Omega}} \varphi_2(x)$. We define the functions $\overline{u}, \overline{v}$ as following:

$$\overline{u}(x,t) = (M\varphi_1(x))^{l_1}, \quad \overline{v}(x,t) = (M\varphi_2(x))^{l_2}, \tag{3.5}$$

where $0 < l_1,\ l_2 < 1$ satisfy $ml_1,\ n\,l_2 < 1$ and M is a positive constant to be determined later. Then, we have for $x \in \Omega, t > 0$,

$$\begin{aligned} &\overline{u}_t - \Delta\overline{u}^m - a\overline{v}^{p_1}\, \|\overline{v}\|_\alpha^{p_2} \\ &= -ml_1(ml_1-1)\varphi_1^{ml_1-2}|\nabla\varphi_1|^2 M^{ml_1} + ml_1\varphi_1^{ml_1-1}M^{ml_1}\varepsilon_1 - a(M\varphi_2)^{p_1 l_2}(\textstyle\int_\Omega (M\varphi_2)^{\alpha l_2}dx)^{\frac{p_2}{\alpha}} \ , \\ &\geq ml_1\overline{K_1}^{ml_1-1}M^{ml_1}\varepsilon_1 - a|\Omega|^{p_2/\alpha}(M\overline{K_2})^{(p_1+p_2)l_2}, \end{aligned} \tag{3.6}$$

$$\overline{v}_t - \Delta\overline{v}^n - b\overline{u}^{q_1}\, \|\overline{u}\|_\beta^{q_2} \geq nl_2\overline{K_2}^{nl_2-1}M^{nl_2}\,\varepsilon_2 - b|\Omega|^{q_2/\beta}(M\overline{K_1})^{(q_1+q_2)l_1}, \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \overline{u}(x,t)|_{\partial\Omega} &= M^{l_1}(\textstyle\int_\Omega f(x,y)dy)^{l_1} \geq M^{l_1}\int_\Omega f(x,y)dy \\ &\geq M^{l_1}\textstyle\int_\Omega f(x,y)\varphi_1^{l_1}(y)dy = \int_\Omega f(x,y)\overline{u}(y,t)dy, \end{aligned} \ , \tag{3.8}$$

In a similar way, we have

$$\overline{v}(x,t)|_{\partial\Omega} \geq \int_\Omega g(x,y)\overline{v}(y,t)dy, \tag{3.9}$$

Here, we use the assumptions $0 < \varphi_1(x), \varphi_2(x) < 1$ and $\int_\Omega f(x,y)dy < 1, \int_\Omega g(x,y)dy < 1$. Denote

$$M_1 = \Big(a(ml_1\varepsilon_1)^{-1}|\Omega|^{p_2/\alpha}\overline{K_1}^{1-ml_1}\overline{K_2}^{(p_1+p_2)l_2}\Big)^{1/(ml_1-(p_1+p_2)l_2)},$$

$$M_2 = \Big(b(nl_2\varepsilon_2)^{-1}|\Omega|^{q_2/\beta}\overline{K_1}^{(q_1+q_2)l_1}\overline{K_2}^{1-nl_2}\Big)^{1/(nl_2-(q_1+q_2)l_1)}.$$

(1) If $(p_1+p_2)(q_1+q_2) < mn$, then there exist positive constants l_1, l_2 such that $(p_1+p_2)/m < l_1/l_2 < n/(q_1+q_2)$, and $ml_1,\ n\,l_2 < 1$. Hence

$$ml_1 > (p_1+p_2)l_2, \ \ nl_2 > (q_1+q_2)l_1.$$

Thus, we can choose sufficiently large M such that $M > \max\{M_1, M_2\}$ and

$$(M\varphi_1(x))^{l_1} \geq u_0(x), \ \ (M\varphi_2(x))^{l_2} \geq v_0(x). \tag{3.10}$$

From (3.6)-(3.10), we see that $(\overline{u}, \overline{v})$ is a positive supersolution of problem (1.1)-(1.3). Hence $(u,v) \leq (\overline{u}, \overline{v})$ by comparison principle, which implies (u,v) exists globally.

(2) If $(p_1+p_2)(q_1+q_2)=mn$, then there exist positive constants l_1,l_2 such that $ml_1=(p_1+p_2)l_2,\ \ nl_2=(q_1+q_2)l_1$ and $ml_1,\ n\,l_2<1$. According to (3.6)-(3.7), we can choose a,b to satisfy

$$a<ml_1\varepsilon_1|\Omega|^{-p_2/\alpha}\overline{K_1}^{-1}(\overline{K_1}/\overline{K_2})^{ml_1},\qquad b<nl_2\varepsilon_2|\Omega|^{-q_2/\beta}\overline{K_2}^{-1}(\overline{K_2}/\overline{K_1})^{nl_2}.$$

Then

$$\overline{u}_t-\Delta\overline{u}^m-a\overline{v}^{p_1}\left\|\overline{v}\right\|_\alpha^{p_2}\geq 0, \tag{3.11}$$

$$\overline{v}_t-\Delta\overline{v}^n-b\overline{u}^{q_1}\left\|\overline{u}\right\|_\beta^{q_2}\geq 0. \tag{3.12}$$

Furthermore, choose large enough M to satisfy (3.10). Then, it follows from (3.8)-(3.12) that $(\overline{u},\overline{v})$ is a positive supersolution of problem (1.1)-(1.3). Therefore, (u,v) exists globally.
(3) If $(p_1+p_2)(q_1+q_2)>mn$, then there exist positive constants l_1,l_2 such that $ml_1,\ n\,l_2<1$ and

$$ml_1<(p_1+p_2)l_2,\ \ nl_2<(q_1+q_2)l_1. \tag{3.13}$$

So, we can choose M sufficiently small such that $M<\min\{M_1,M_2\}$. Assume that $u_0(x)$ and $v_0(x)$ are sufficiently small to satisfy (3.10), and it follows that $(\overline{u},\overline{v})$ defined by (3.5) is a positive supersolution of problem (1.1)-(1.3). Hence, (u,v) exists globally.

Next, we will prove the bow-up result. Let $\phi(x)$ be a nontrivial nonnegative continuous function, which vanishes on $\partial\Omega$. Without loss of generality, we may assume that $0\in\Omega$ and $\phi(0)>0$. Set

$$\underline{u}(x,t)=\frac{1}{(T-t)^{l_1}}\omega^{1/m}\left(\frac{|x|}{(T-t)^\sigma}\right),\ \ \underline{v}(x,t)=\frac{1}{(T-t)^{l_2}}\omega^{1/n}\left(\frac{|x|}{(T-t)^\sigma}\right),$$

with

$$\omega(r)=\frac{R^3}{12}-\frac{R}{4}r^2+\frac{1}{6}r^3,r=\frac{|x|}{(T-t)^\sigma},0\leq r\leq R, \tag{3.14}$$

where $l_1,l_2,\sigma>0$ and $0<T<1$ are constants to be chosen later. Clearly, $0\leq\omega(r)\leq R^3/12$ and $\omega(r)$ is nonincreasing, since $\omega'(r)=r(r-R)/2\leq 0$. We let T be small enough so that

$$\mathrm{supp}\underline{u}(x,t)=\mathrm{supp}\underline{v}(x,t)=\overline{B(0,R(T-t)^\sigma)}\subset\overline{B(0,RT^\sigma)}\subset\Omega. \tag{3.15}$$

Obviously, $(\underline{u},\underline{v})$ becomes unbounded as $t\to T^-$, at the point $x=0$. Calculating directly, we obtain that

$$\begin{aligned}&\underline{u}_t-\Delta\underline{u}^m-a\underline{v}^{p_1}\left\|\underline{v}\right\|_\alpha^{p_2}\\&=\tfrac{ml_1\omega^{1/m}(r)+\sigma r\omega'(r)\omega^{(1-m)/m}}{m(T-t)^{l_1+1}}+\tfrac{NR-(N+1)r}{2(T-t)^{ml_1+2\sigma}}-\tfrac{a\omega^{p_1/n}(r)}{(T-t)^{l_2(p_1+p_2)}}\left(\int_{B(0,R(T-t)^\sigma)}|\omega(\tfrac{|x|}{(T-t)^\sigma})|^{\alpha/n}dx\right)^{p_2/\alpha},\end{aligned}$$

$$\begin{aligned}&\underline{v}_t-\Delta\underline{v}^n-b\underline{u}^{q_1}\left\|\underline{u}\right\|_\beta^{q_2}\\&=\tfrac{nl_2\omega^{1/n}(r)+\sigma r\omega'(r)\omega^{(1-n)/n}}{n(T-t)^{l_2+1}}+\tfrac{NR-(N+1)r}{2(T-t)^{nl_2+2\sigma}}-\tfrac{b\omega^{q_1/m}(r)}{(T-t)^{l_1(q_1+q_2)}}\left(\int_{B(0,R(T-t)^\sigma)}|\omega(\tfrac{|x|}{(T-t)^\sigma})|^{\beta/m}dx\right)^{q_2/\beta},\end{aligned}$$

Case1. If $0\leq r\leq NR/(N+1)$, we have $\omega(r)\geq R^3(3N+1)\Big/12(N+1)^3$, then

$$\begin{aligned}&\underline{u}_t-\Delta\underline{u}^m-a\underline{v}^{p_1}\left\|\underline{v}\right\|_\alpha^{p_2}\\&\leq\tfrac{l_1(R^3/12)^{1/m}}{(T-t)^{l_1+1}}+\tfrac{NR-(N+1)r}{2(T-t)^{ml_1+2\sigma}}-\tfrac{aM_1}{(T-t)^{l_2(p_1+p_2)-(N\sigma p_2/\alpha)}}\left(\tfrac{R^3(3N+1)}{12(N+1)^3}\right)^{p_1/n},\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\underline{v}_t-\Delta\underline{v}^n-b\underline{u}^{q_1}\left\|\underline{u}\right\|_\beta^{q_2}\\&\leq\tfrac{l_2(R^3/12)^{1/n}}{(T-t)^{l_2+1}}+\tfrac{NR-(N+1)r}{(T-t)^{nl_2+2\sigma}}-\tfrac{bM_2}{(T-t)^{l_1(q_1+q_2)-(N\sigma q_2/\beta)}}\left(\tfrac{R^3(3N+1)}{12(N+1)^3}\right)^{q_1/m},\end{aligned} \tag{3.17}$$

where $M_1=\left(\int_{B(0,R)}\omega_+^{\alpha/n}(|\xi|)d\xi\right)^{p_2/\alpha},\ \ M_2=\left(\int_{B(0,R)}\omega_+^{\beta/m}(|\xi|)d\xi\right)^{q_2/\beta}$.
Case2. If $NR/(N+1)<r\leq R$, then

$$\underline{u}_t-\Delta\underline{u}^m-a\underline{v}^{p_1}\left\|\underline{v}\right\|_\alpha^{p_2}\leq\frac{l_1(R^3/12)^{1/m}}{(T-t)^{l_1+1}}-\frac{(N+1)r-NR}{2(T-t)^{ml_1+2\sigma}}, \tag{3.18}$$

$$\underline{v}_t - \Delta\underline{v}^n - b\underline{u}^{q_1} \|\underline{u}\|_\beta^{q_2} \le \frac{l_2(R^3/12)^{1/n}}{(T-t)^{l_2+1}} - \frac{(N+1)r - NR}{(T-t)^{nl_2+2\sigma}}. \tag{3.19}$$

When l_1, l_2 are large enough to satisfy (3.13) and $l_1(m-1) > 1,\ l_2(n-1) > 1$, we can choose positive constant σ be sufficiently small such that

$$\sigma < \min\{\frac{l_2(p_1+p_2) - ml_1}{2+(Np_2/\alpha)}, \frac{l_1(q_1+q_2) - nl_2}{2+(Nq_2/\beta)}\},$$

Thus, we have

$$l_2(p_1+p_2) - (N\sigma p_2/\alpha) > ml_1 + 2\sigma > l_1 + 1, \quad l_1(q_1+q_2) - (N\sigma q_2/\beta) > nl_2 + 2\sigma > l_2 + 1.$$

Hence, for sufficiently small $T > 0$, (3.16)-(3.19) imply that

$$\underline{u}_t - \Delta\underline{u}^m - a\underline{v}^{p_1} \|\underline{v}\|_\alpha^{p_2} \le 0, \quad \underline{v}_t - \Delta\underline{v}^n - b\underline{u}^{q_1} \|\underline{u}\|_\beta^{q_2} \le 0, \quad (x,t) \in Q_T.$$

Since $\phi(0) > 0$ and $\phi(x)$ is continuous, there exist two positive constants ρ and ε such that $\phi(x) \ge \varepsilon$, for all $x \in B(0,\rho) \subset \Omega$. Choose T small enough to ensure $B(0, RT^\sigma) \subset B(0,\rho)$, hence $\underline{u} \le 0, \underline{v} \le 0$ on S_T. Under the assumption that $\int_\Omega f(x,y)dy < 1, \int_\Omega g(x,y)dy < 1$ for any $x \in \partial\Omega$, we have $\underline{u}(x,t) \le \int_\Omega f(x,y)\underline{u}(y,t)dy,\ \ \underline{v}(x,t) \le \int_\Omega g(x,y)\underline{v}(y,t)dy$ on S_T. Furthermore, choose $u_0(x),\ v_0(x)$ so large that $u_0(x) \ge \underline{u}(x,0), v_0(x) \ge \underline{v}(x,0)$ .By comparison principle, we have $(\underline{u}, \underline{v}) \le (u,v)$. It shows that solution (u,v) blows up in finite time. ∎

Acknowledgments

The author would like to thank the referees for their valuable suggestions and comments on this manuscript.

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