ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.21(2016) No.2, pp.106-112

Bursting Synchronization of Two Fast-slow Oscillators with Mismatch in Parameters

Suzhen Zhang, Huilin Lu, Xiaoke Chen, Zhengdi Zhang *
Faculty of Science, Jiangsu University, Zhenjiang 212013, China
(Received 10 October 2015 , accepted 11 January 2016)

Abstract: The bursting synchronization between two non-autonomous systems with mismatched parameters is investigated in this paper. A fast-slow model is introduced based on an extended BVP oscillator with periodical excitation, which may present bursting oscillations when system parameters sets are taken properly. The non-autonomous equations are transformed into autonomous equations, and accordingly the synchronization of two systems with mismatch in parameters is transformed to synchronization of two identical systems with different initial values. Some sufficient conditions for the stabilization and synchronization are derived by using theories in Lyapunov stability and cascade-connected system. Numerical simulations are then used to confirm the effectiveness of the proposed schemes.

Keywords: Bursting synchronization; fast-slow oscillator; Cascade-connected system; Lyapunov function

1 Introduction

Synchronization control between dynamic systems has been extensively studied in many research fields over the last few decades, such as biology, ecology, sociology, power grids, climatology, etc.[1-5]. Some important methods have been proposed to investigate synchronization between different systems and various kinds of synchronizations have been explored, such as lag synchronization[6], generalized synchronization[7], phase synchronization[8], anti-phase synchronization[9] and identical synchronization[10]. Up to now, most of the work is confined to vector field with single time scalewhile many dynamical systems in neuronal networks, chemical reactor, secure communication, biomedical may involve different time scales, which often behave in periodic bursting oscillations characterized by a combination of relatively large amplitude and nearly harmonic small amplitude oscillations, known as spiking state and quiescent state respectively[11]. Bursting phenomena can be observed when the variables alternate between these two states. Most of the reports about dynamic system with different time scales are focused on dynamical characteristics as well as bifurcation mechanism of single multi-scale systems[12], such as cusp type bursting in photosensitive B-Z reaction system[13], tea-cup attractor in ecological model [14], two types of fast-slow bursters and related bifurcation in the modified Morris–Lecar neuron[15] and fold/Hopf bursting, fold/Hopf/homoclinic bursting and bifurcation mechanism of periodically driven oscillators [16]. Clinically, the connection between bursting and synchronization is extremely important, since synchronization in large neural populations is widely viewed as a hallmark of seizers. Various areas of the basal ganglia have been found to exhibit bursting synchronization related to Parkinson's disease and resting tremor[17]. By observing bursting synchronization in cell cultures of cortical neurons, it is found that periodic synchronized bursting plays a critical role in the development of synaptic connections[18-20].

In this paper, we study the bursting synchronization of two coupled identical non-autonomous systems with mismatch in parameters. Based on an extended BVP oscillator, a generalized model with two time scales has been established via taking suitable values of the parameters. Firstly, the scheme to realize synchronization of the system with variable parameter is given. Numerical simulations are then used to verify the effectiveness of the proposed method.

*Corresponding author. *E-mail address*: dyzhang72@126.com

IJNS.2016.04.15/906

2 Mathematical model

The extended BVP oscillator is obtained by adding a capacitor to the original BVP oscillator[21], constructed by two same capacitors C, an inductor L, a linear resistor r and a nonlinear resistor $g(v_1)$. It can also be introduced as a modified Chua's circuit by swapping locations of $L$and r in the Chua's system[22]. A three-dimensional system[23] can be used to describe the dynamics of the oscillation. In order to appear bursting phenomenon, we incorporate an alternating voltage to external periodic excitation u in the circuit, shown in Fig.1. The corresponding circuit equations can be described by:

$$C\frac{dv_1}{dt} = -i - g(v_1), C\frac{dv_2}{dt} = i - \frac{v_2}{r} + \frac{u}{r}\cos wt, L\frac{di}{dt} = v_1 - v_2. \tag{1}$$

Where the $v - i$characteristic of the nonlinear resistor abide by the following function:

$$g(v_1) = -av_1 - \frac{1}{2}\left(|1 + bv_1| - |1 - bv_1|\right).$$

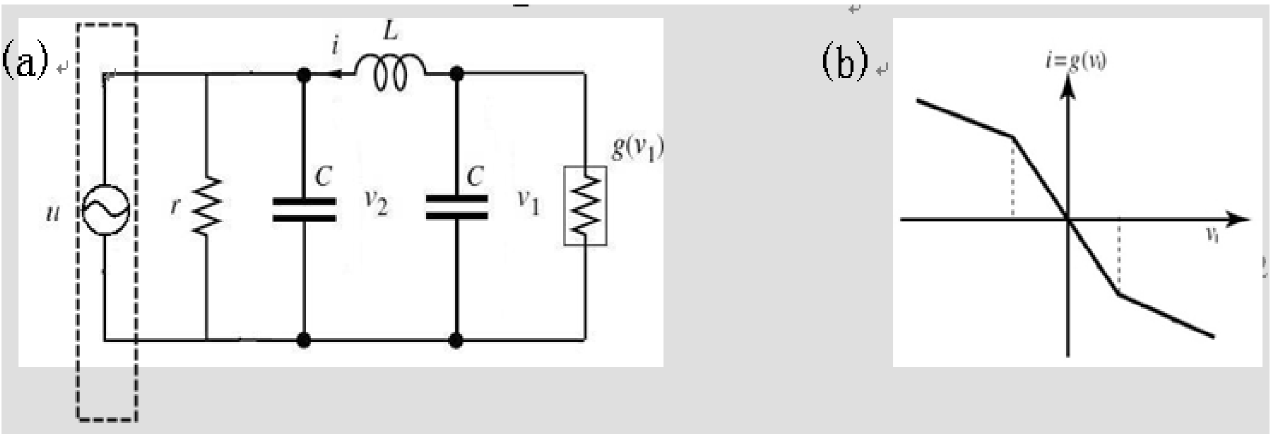

Figure 1: (a) The circuit diagram of the non-smooth BVP oscillator; (b) The $v - i$ characteristics of the nonlinear resistor

The dimensionless equation of the extended BVP system (1) is described as follows:

$$\dot{x} = -z + Ax + \frac{1}{2}\left(|1 + Bx| - |1 - Bx|\right), \dot{y} = z - \delta y + U\cos(\Omega\tau), \dot{z} = x - y, \tag{2}$$

where

$$U = \frac{u}{r}\sqrt{L/C}\Omega = w\sqrt{LC}\cdot = {}^{d}/_{d\tau}, \quad \tau = \frac{1}{\sqrt{LC}}t, \quad A = a\sqrt{L/C}, \quad B = bc\sqrt{L/C},$$

$$\delta = \frac{1}{r}\sqrt{L/C}, \quad x = \frac{v_1}{b}\sqrt{C/L}, \quad y = \frac{v_2}{b}\sqrt{C/L}z = i/b.$$

When there is order gap between the natural frequency and the excited frequency, a two-scale, namely, a fast-slow coupled non-smooth generalized BVP circuit system has been established. The dynamics with multiple time scales may often behave in periodic movements characterized by the combination of large and small amplitudes of oscillations, which will produce multi-modal coupling oscillatory behavior. Obviously, the system (2) is non-autonomous since the time τ is explicitly contained in the system.

3 Bursting synchronization of two systems with parameter mismatches

In this section, bursting synchronization will be constructed between two non-autonomous extended BVP oscillators with different parameters. The driving system and the response system are supposed to possess different frequency Ω and amplitude U. Generally speaking, parameter mismatches are considered to have a detrimental effect on the synchronization quality between coupled identical systems: in the case of small parameter mismatches the synchronization error does not decay to zero or even a nonzero mean. Larger values of parameter mismatches can even result in the loss of synchronization. Here, we would firstly use unidirectional coupled synchronization method and the stability method of the cascade-connected system from the nonlinear differential geometric control theory to reduce the dimension, which transforms the non-autonomous equations into autonomous equations. Thereby the synchronization of two non-autonomous systems with parameter mismatches can be regarded as the different initial conditions in the autonomous systems.

3.1 Transform non-autonomous system to autonomous system

We firstly convert the non-autonomous equation into an autonomous equation, and the system parameters Ω and U will be transferred to values of two new state variables accordingly. It can be easily verified the time-dependent factor $U\cos(\Omega\tau)$ of the system (2) is a solution of the equation:

$$\ddot{x}_4 = -\Omega^2 x_4. \tag{3}$$

Then we can introduce new first-order bivariate equations written as

$$\dot{x}_4 = -x_5, \quad \dot{x}_5 = \Omega^2 x_4, \tag{4}$$

which represent the cosine factor in the system (2) whose amplitude U is determined by the initial values of variables x_4 and x_5.

Let $x_1 = x$,$x_2 = y, x_3 = z$, system (2) can be translated to autonomous first-order differential equations written as following:

$$\dot{x}_1 = -x_3 + Ax_1 + g(x_1), \quad \dot{x}_2 = x_3 - \delta x_2 + x_4, \dot{x}_3 = x_1 - x_2, \dot{x}_4 = -x_5, \dot{x}_5 = \Omega^2 x_4. \tag{5}$$

Here, bivariate unidirectional coupled synchronization method will be used. We consider linear coupling terms to the first and the fourth formula respectively, the corresponding response system is expressed as following:

$$\dot{y}_1 = -y_3 + Ay_1 + g(y_1) + k_1(x_1 - y_1), \quad \dot{y}_2 = y_3 - \delta y_2 + y_4, \dot{y}_3 = y_1 - y_2,$$

$$\dot{y}_4 = -y_5 + k_4(x_4 - y_4), \dot{y}_5 = \Omega^2 y_4, \tag{6}$$

where k_1 and k_4 are the coupling coefficients. It seems apparently the drive system (5) and the response system (6) do not contain excitation voltages, they actually have already been implied in variables x_4 and y_4, respectively. So we can transfer different parameters of the excitation voltage into different initial values in two systems because the different initial values of x_4, x_5, y_4 and y_5 will result in different amplitude of the excitation voltage. Define the error variables as $e_1 = x_1 - y_1, \quad e_2 = x_2 - y_2, \quad e_3 = x_3 - y_3, \quad e_4 = x_4 - y_4, \quad e_5 = x_5 - y_5$, the corresponding error system can be expressed by:

$$\dot{e}_1 = -e_3 + Ae_1 + (g(x_1) - g(y_1)) - k_1 e_1, \quad \dot{e}_2 = e_3 - \delta e_2 + e_4, \quad \dot{e}_3 = e_1 - e_2,$$

$$\dot{e}_4 = -e_5 - k_4 e_4, \dot{e}_5 = \Omega^2 e_4. \tag{7}$$

Since $g(x_1)$ and $g(y_1)$ are piecewise linear, we obtain the following equation:

$$g(x_1) - g(y_1) = c(x_1 - y_1) = ce_1. \tag{8}$$

Substitute (8) into (7) leads to

$$\dot{e}_1 = (A + c - k_1)e_1 - e_3, \dot{e}_2 = e_3 - \delta e_2 + e_4, \dot{e}_3 = e_1 - e_2, \dot{e}_4 = -e_5 - k_4 e_4, \dot{e}_5 = \Omega^2 e_4. \tag{9}$$

Therefore, the bursting synchronization of the two systems is equivalent to the stability of error system (9) at the origin. In the following, the stability theory of cascade-connected system is used to analyze this issue.

3.2 Stability analysis of the error system

We consider the system written as:

$$\dot{x} = f(x, z), \quad \dot{z} = g(z). \tag{10}$$

When $x \in R^n, \quad z \in R^m, f(0,0) = 0, g(0) = 0$, while $f(x,z)$and $g(z)$ are locally Lipschitz in $R^n \times R^m$, (10) is called cascade-connected system. In terms of stability, it contains the following two important properties[24]:

Lemma 1: For the system (10), if $x = f(x,0)$ and $z = g(z)$ are locally asymptotically stable when $x = 0$ and $z = 0$, then system (10) is locally asymptotically stable when $(x,z) = (0,0)$.

Lemma 2: For the system (10), while$x = f(x,0)$ and $z = g(z)$ are globally asymptotically stable when $x = 0$ and $z = 0$, and all solutions of the system (10) are bounded, then system (10) is globally asymptotically stable when $(x,z) = (0,0)$.

Obviously, the error system (9) is a cascade-connected system. we first discuss the subsystem

$$\dot{e}_4 = -e_5 - k_4 e_4, \quad \dot{e}_5 = \Omega^2 e_4. \tag{11}$$

Theorem1. When $k_4 > 0$, the zero solution of system (11) is globally asymptotically stable.

Proof. Choose the Lyapunov function as

$$V_1(e_4, e_5) = \frac{1}{2} w^2 e_4^2 + \frac{1}{2} e_5^2, \tag{12}$$

we can obtain

$$\dot{V}_1(e_4, e_5) = w^2 e_4 \dot{e}_4 + e_5 \dot{e}_5 = w^2 e_4(-e_5 - k_4 e_4) + e_5 w^2 e_4 = -w^2 k_4 e_4^2. \tag{13}$$

Since $V_1(e_4, e_5)$ is positive definite, and $\dot{V}_1(e_4, e_5)$ is negative definite when $k_4 > 0$, According to the stability theorem of Lyapunov, we can come to a conclusion that the zero solution of subsystem (11) is globally asymptotically stable.

Secondly, we consider the other subsystem

$$\dot{e}_1 = (A + c - k_1)e_1 - e_3, \dot{e}_2 = e_3 - \delta e_2 + e_4, \dot{e}_3 = e_1 - e_2. \tag{14}$$

According to Lemma 2, we can know that above subsystem requires $e_4 = 0$. Therefore, eq.(14) can be written as

$$\dot{e}_1 = (A + c - k_1)e_1 - e_3, \quad \dot{e}_2 = e_3 - \delta e_2, \quad \dot{e}_3 = e_1 - e_2. \tag{15}$$

Theorem 2. When$A + c - k_1 < 0, \delta > 0$, the zero solution of system (15) is globally asymptotically stable.

Proof. Choose the Lyapunov function as

$$V_2(e_1, e_2, e_3) = e_1^2 + e_2^2 + e_3^2. \tag{16}$$

We have

$$\begin{aligned} \dot{V}_2(e_1, e_2, e_3) &= 2e_1\dot{e}_1 + 2e_2\dot{e}_2 + 2e_3\dot{e}_3 = 2e_1\left[(A + c - k_1)e_1 - e_3\right] + 2e_2(-\delta e_2 + e_3) + 2e_3(e_1 - e_2) \\ &= 2(A + c - k_1)e_1^2 - 2\delta e_2^2. \end{aligned} \tag{17}$$

When $A + c - k_1 < 0$ and $\delta > 0$, we can conclude that the zero solution of system (15) is globally asymptotically stable.

Theorem 3. When$A + c - k_1 < 0,, \delta > 0, k_4 > 0$,the error system (9) is globally asymptotically stable at the origin.

Proof. Due to the change rate of the system (9) per unit volume is

$$V = \frac{\partial \dot{e}_1}{\partial e_1} + \frac{\partial \dot{e}_2}{\partial e_2} + \frac{\partial \dot{e}_3}{\partial e_3} + \frac{\partial \dot{e}_4}{\partial e_4} + \frac{\partial \dot{e}_5}{\partial e_5} = (A + c - k_1) - \delta - k_4. \tag{18}$$

Under the condition of that the subsystem (11) is globally asymptotically stable when $k_4 > 0$ and the subsystem (15) is globally asymptotically stable when$A + c - k_1 < 0, \quad \delta > 0$, we conclude that $\Delta V < 0$, that is to say, the volume of the phase needs to shrink. Thus the trajectories starting from the different initial state must shrink to a limited part in the phase space finally. Then all the solutions of system (9) are bounded. According to Lemma 2, when the coupling coefficient k_1 and k_4 satisfy that

$$A + c - k_1 < 0, \delta > 0, k_4 > 0, \tag{19}$$

the error system (9) is globally asymptotically stable at the origin, driving system(5) and response system (6) will achieve synchronization.

4 Numerical simulation

With the change of time, the difference of the corresponding variables become a constant in the original system (5) and the synchronization system (6), which is $x_1 - y_1$, $x_2 - y_2$ and $x_3 - y_3$ finally become a constant. Then bursting synchronization gets verified. The system parameters are selected as A=0.8, B=1.0, $\delta = 1.2$ and $\Omega = 0.01$. When the initial values are chosen as $x_1(0) = 1.0$, $x_2(0) = 1.0$, $x_3(0) = 0.1$, $x_4(0) = 0.6$ and $x_5(0) = 0$(the amplitude of the excitation voltage $U = 0.6$can be obtained from this group of initial values), periodic bursting can be observed in the system (5), the details of which is presented in Fig. 3.1. Bursting oscillation of the system may be called symmetric focus/focus type bursting. Fold-type non-smooth bifurcation connects the quiescent state and spiking state. According to (19), the coupling coefficient of the synchronization system (6) can be chosen as $k_1 = 88$ and $k_2 = 2$. The initial values of the synchronization system (6) are chosen as $y_1(0) = 9.0$,$y_2(0) = 14.0$,$y_3(0) = 30$, $y_4(0) = 2.0$ and $y_5(0) = 2.0$(the amplitude of the excitation voltage $U = 2.0$ can be obtained from this group of initial values).

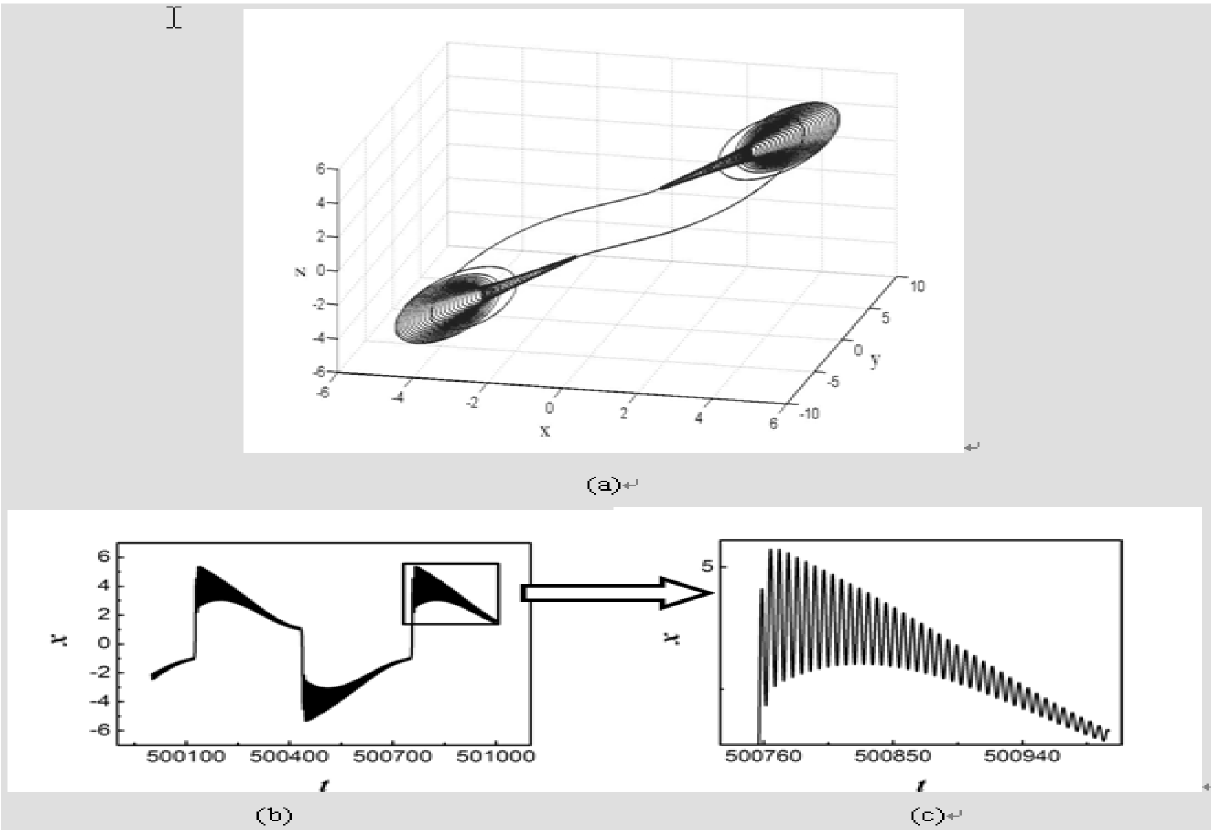

Figure 2: Periodic bursting solution (.Ω=0.01): (a) The phase portrait; (b) The time series;(c) Enlarged portrait of (b)

Now we use numerical simulations to verify the theoretical results presented above. The error signals e_i $(i = 1, 2, 3)$ varying with time t are shown in Fig 3.2(a). It shows the signals have become straight lines which passes through the origin and parallels to a time-axis after a very short time, which indicates that all state variables of the system (5) and the system (6) have rapidly reachedglobal synchronization. If other different initial values are taken, the same effect can also be obtained. For an example, we chose $y_1(0) = 4.0$, $y_2(0) = 2.0$, $y_3(0) = 40$, $y_4(0) = 4.0$ and $y_5(0) = 2.0$, the perfect synchronization effect can be given, the results of which are shown in Fig 3.2(b).

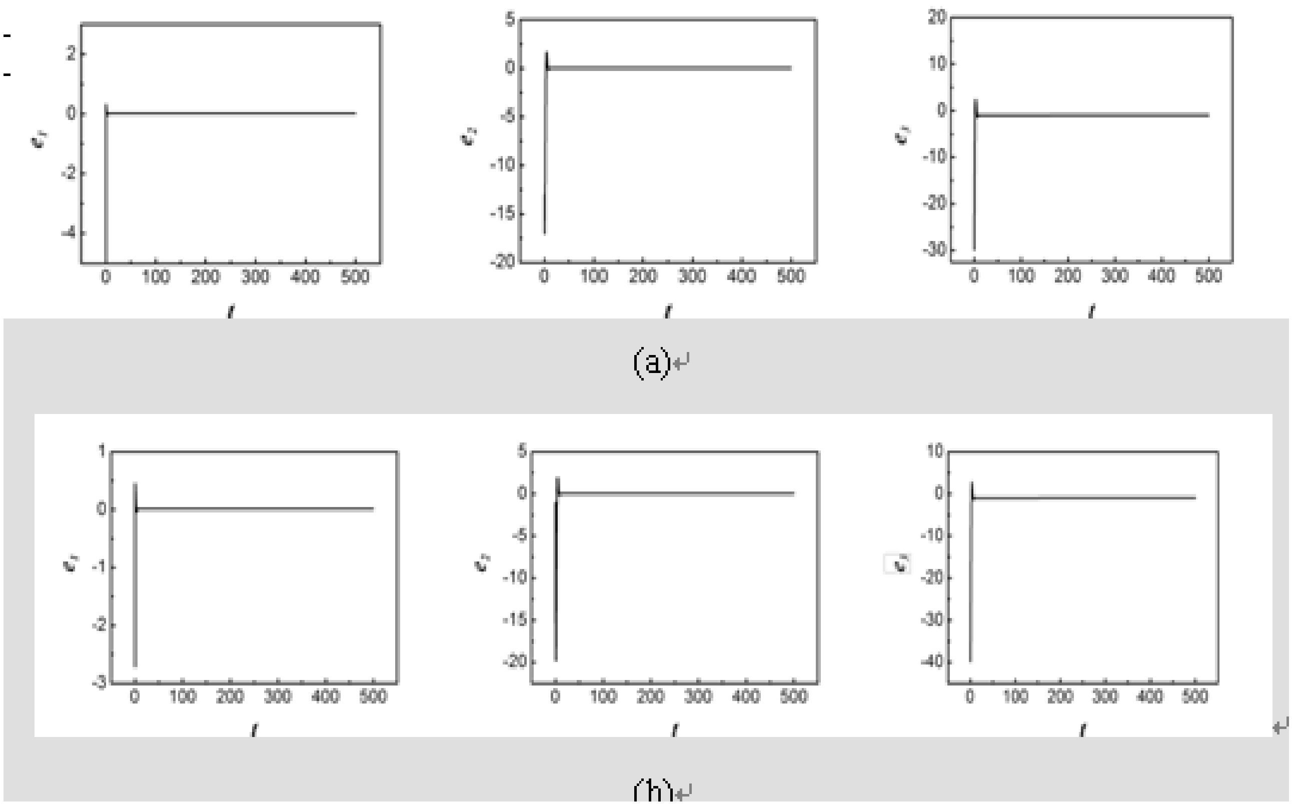

Figure 3: The time series for error.

5 Conclusions

We have investigated the bursting synchronization of two the non-autonomous BVP oscillators. The conditions of coupling coefficients have been obtained by the stability theorem of Lyapunov as well as the nonlinear differential geometric control theory. Two coupling terms can be designed in any two formulas of the system (6). In particular, as the coupling increases, two coupled bursters transit from non-synchronization to complete synchronization. Thus, the synchronization with parameter mismatches can be realized. Simulation shows that the error signal quickly close to zero, which verifies fast synchronization between two systems with different initial values.

Acknowledgments

The authors are supported by the National Natural Science Foundation of China (Grant No.11272135, 11472116), the 2014 Qinlan Project of Jiangsu Province, the Science Research Innovation Foundation of Jiangsu Province(Grant No.1291480004).

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