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Applications of Bell-polynomial Scheme in Constructing the Conversation Laws of the Nonlinear Evolution Equations Admitting Two-field Bilinear Forms

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Abstract: In this paper, the Bell-polynomial scheme is applied to obtain the infinite conservation laws for the nonlinear evolution equations (NLEEs) admitting two-field bilinear forms. Bell-polynomial scheme has been used to obtain certain bilinear Bäcklund transformations (BTs), Lax pairs and infinite conservation laws for the NLEEs with one-field bilinear forms. Based on the two-field bilinear forms and four-field bilinear BTs in the binary-Bell-polynominal form, the infinite conversation laws for a variable-coefficient modified Korteweg-de Vries equation arising in fluid and plasma physics as well as the Boussinesq-Burgers equations for the shallow water waves are construed systematically. Such procedure can be applied to other NLEEs admitting two-field bilinear forms.

Keywords: Bell polynomial; Infinite conservation law; Two-field Hirota system

1 Introduction

Integrability analysis is important for a nonlinear evolution equation (NLEE), which can be characterized by a number of properties such as the Painlevé text, Lax pair, infinite conservation laws, infinite symmetries and Hamiltonian structures [1]. Especially, conservation laws are essential in physical sciences, including the conservation of mass, energy, linear momentum and angular momentum [1, 2]. Derivation of the conserved quantities for a NLEE can be regarded as a key step to solve the initial-value problem through the inverse scattering method [2]. Moreover, conservation laws can be used to control the errors in the numerical integration for the NLEEs [3].

Conservation laws can be obtained by such methods as the Lax pair [4], Bäcklund transformation (BT) [4], formal solutions of the eigenfunctions [5, 6], Noether theorem [7], partial Lagrangians [8], Poisson brackets [9] and nonlocal conservation theorem [10]. Apart from them, Bell-polynomial scheme has been developed to deal with the infinite conservation laws for some NLEEs, for example, the non-isospectral and variable-coefficient Korteweg-de Vries (KdV) equation [11]. Connection between the Bell polynomials and integrability of a NLEE lies in that one can derive the bilinear BT in the $\mathscr{Y}$-polynomial (binary-Bell-polynomial) form systematically without the need of "exchange formulas" and construct the corresponding spectral formulation [12–14].

Although the method of constructing the infinite conservation laws has been given in Refs. [11, 15], it is mainly applicable to those NLEEs admitting one-field bilinear forms in terms of the $\mathscr{P}$-polynomials (special case of the $\mathscr{Y}$-polynomials). However, for those NLEEs admitting two-field bilinear forms, the Bell-polynomial scheme has not been

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used to construct the infinite conservation laws. In this paper, we will focus on such problem. In the procedure, the two-field bilinear form and four-field bilinear BT in terms of the $\mathscr{Y}$-polynomials will be involved.

The brief introduction of the Bell polynomials and the steps for constructing the infinite conservation laws via the Bell polynomials will be given. As an example of the single NLEEs, the non-isospectral and variable-coefficient modified KdV (mKdV) equation [16]

$$U_t + h_1(t)(U_{xxx} - 6U^2U_x) + 4h_2(t)U_x - h_3(t)(U + xU_x) = 0\,, \tag{1}$$

arising in the fields of fluid and plasma physics will be considered, where U is a real function of space variable x and time variable t, $h_j(t)$'s ($j = 1, 2, 3$) are all analytic functions. Moreover, we will consider the Boussinesq-Burgers equations [17]

$$\phi_t = -2\phi\phi_x + \frac{1}{2}\varphi_x\,, \tag{2a}$$

$$\varphi_t = \frac{1}{2}\varphi_{xxx} - 2(\phi\varphi)_x\,, \tag{2b}$$

as an example of the coupled NLEEs, where x and t are the normalized space and time, $\phi = \phi(x,t)$ and $\varphi = \varphi(x,t)$ denote respectively the horizontal velocity field and the height of the water surface above a horizontal bottom. Eqs. (2) can describe the propagation of the shallow water waves.

2 Bell-polynomial preliminary

Let $f = f(x_1, \ldots, x_n)$ be a C^∞ multi-variable function (x_j's are the variables), and then the multi-dimensional Bell polynomials can be defined as [12, 13]

$$Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) = e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}e^f\,, \tag{3}$$

where n_j's ($j = 1, \ldots, l$) are the nonzero integers, $f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l} f$ ($r_k = 0, \ldots, n_k, k = 1, \ldots, l$) and $Y_{n_1x_1,\ldots,n_lx_l}(f)$ denotes the multivariable polynomial with respect to $f_{r_1x_1,\ldots,r_lx_l}$.

Two-dimensional binary Bell polynomials ($\mathscr{Y}$-polynomial) are introduced as [13]

$$\mathscr{Y}_{nx,mt}(V,W) = Y_{nx,mt}(f)\Big|_{f_{nx,mt}=\begin{cases} V_{nx,mt}\,, & \text{if } n+m \text{ is odd}\,,\\ W_{nx,mt}\,, & \text{if } n+m \text{ is even}\,,\end{cases}} \tag{4}$$

where m and n are both integers, V and W are both the C^∞ functions of the variables x and t. Via the transformations

$$V = \ln(F/G)\,, \quad W = \ln(FG)\,, \tag{5}$$

with F and G being the functions of x and t, Expression (4) can be associated with the Hirota expression, that is,

$$\mathscr{Y}_{nx,mt}(V,W) = (FG)^{-1}D_x^nD_t^mF\cdot G\,, \tag{6}$$

where the Hirota bilinear operators are defined by [18]

$$D_x^nD_t^m(F\cdot G) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m F(x,t) \times G(x',t')\Big|_{x'=x,\,t'=t}\,.$$

Assume that one NLEE admits the two-field bilinear representation in the Bell-polynomial form

$$\mathscr{E}_1(V,W)=0\,,\quad \mathscr{E}_2(V,W)=0\,, \tag{7}$$

with $\mathscr{E}_j$'s ($j=1,2$) being the functions of V and W. Utilizing the relation

$$\mathscr{E}_j(V',W')-\mathscr{E}_j(V,W)=0\,,\quad j=1,2\,, \tag{8}$$

with V' and W' both satisfying Eqs. (7), one can obtain the sets of four-field bilinear BT in terms of the $\mathscr{Y}$-polynomials as below [19]:

$$\sum_j \big[a_j\mathscr{Y}_{k_jx,l_jt}(v_1,w_1)+b_j\exp(v_3-v_4)\mathscr{Y}_{m_jx,n_jt}(v_2,w_2)+c_j\exp(v_3-v_1)\mathscr{Y}_{p_jx,q_jt}(v_3,w_3)$$
$$+d_j\exp(v_3-v_2)\mathscr{Y}_{r_jx,s_jt}(v_4,w_4)\big]=0\,,\quad j=1,2,3,4\,, \tag{9}$$

where a_j's, b_j's, c_j's and d_j's are all constants, k_j's, l_j's, m_j's, n_j's, p_j's, q_j's, r_j's and s_j's ($j=1,2,3,4$) are all integers, and the mixing variables are introduced as

$$v_1=\ln\frac{G'}{G}\,,\quad v_2=\ln\frac{F'}{F}\,,\quad v_3=\ln\frac{F'}{G}\,,\quad v_4=\ln\frac{G'}{F}\,,$$
$$w_1=\ln GG'\,,\quad w_2=\ln FF'\,,\quad w_3=\ln GF'\,,\quad w_4=\ln FG'\,.$$

3 Construction of infinite conservation laws via Bell polynomials

3.1 Non-isospectral and variable-coefficient mKdV equation

In this section, we consider the non-isospectral and variable-coefficient mKdV equation [i.e.,Eq. (1)]. By the dimensionless transformation $U=V_x$ (or $U=-V_x$, the procedure can be performed in a similar way) with $V=V(x,t)$ as a dimensionless field, the bilinear form of Eq. (1) in terms of $\mathscr{Y}$-polynomials is given as below [20]

$$\mathscr{Y}_{2x}(V,W)=0\,, \tag{10a}$$

$$\mathscr{Y}_t(V)+h_1(t)\mathscr{Y}_{3x}(V,W)+[4h_2(t)-xh_3(t)]\mathscr{Y}_x(V)=0\,, \tag{10b}$$

where $W=W(x,t)$ is another dimensionless field auxiliary function. Subsequently, through the mixing variable procedure, the BT in the $\mathscr{Y}$-polynomial form can be obtained [20]

$$\mathscr{Y}_x(v_1)=V_x+\mu(t)\exp(v_2-v_1)\,, \tag{11a}$$

$$\mathscr{Y}_x(v_2)=-V_x+\lambda(t)\exp(v_1-v_2)\,, \tag{11b}$$

$$\mathscr{Y}_t(v_1)+h_1(t)\mathscr{Y}_{3x}(v_1,w_1)+[3\lambda(t)\mu(t)h_1(t)+4h_2(t)-xh_3(t)]\mathscr{Y}_x(v_1)=0\,, \tag{11c}$$

$$\mathscr{Y}_t(v_2)+h_1(t)\mathscr{Y}_{3x}(v_2,w_2)+[3\lambda(t)\mu(t)h_1(t)+4h_2(t)-xh_3(t)]\mathscr{Y}_x(v_2)=0\,, \tag{11d}$$

where $\lambda(t)$ and $\mu(t)$ are both arbitrary functions.

Now, we will construct the infinite conservation laws based on Bilinear Form (10) and BT (11). First, setting $\Gamma=\exp(v_1-v_2)$ (or $\Gamma=\exp(v_2-v_1)$, procedure can be performed in a similar way) and utilizing Eqs. (11a) and (11b), we obtain the Riccati-type equation from the derivative of Γ with respect to x,

$$\Gamma_x=2V_x\Gamma-\lambda(t)\Gamma^2+\mu(t)\,. \tag{12}$$

In order to solve Eq. (12) through the perturbation scheme, proper formal parameter needs to be chosen. Furthermore, the existence of recursion relations in the perturbation scheme relies on the terms Γ_x and Γ, which should be located at the different order of the formal parameter. Therefore, via the transformation $\Gamma = 1 + \widetilde{\Gamma}$ and the non-isospectral conditions [20]

$$\mu(t) = \lambda(t)\,, \quad \lambda'(t) = h_3(t)\lambda(t)\,, \tag{13}$$

we make a modification of Eq. (12) as the form

$$\widetilde{\Gamma}_x + 2(\varepsilon\, e^{\int h_3(t)dt} - V_x)\,\widetilde{\Gamma} + \varepsilon\, e^{\int h_3(t)dt}\,\widetilde{\Gamma}^2 - 2V_x = 0\,, \tag{14}$$

with ε as an arbitrary constant originates from $\lambda(t) = \varepsilon\, e^{\int h_3(t)dt}$. Next, substituting the expansion

$$\widetilde{\Gamma} = \sum_{n=1}^{\infty} f_n\, \varepsilon^{-n}\,, \tag{15}$$

with $f_n = f_n(x,t)$'s $(n = 1, 2, \ldots)$ as functions to be determined, into Eq. (14) and equating the coefficients of the same power of ε to be zero, we get the recursion relations

$$f_1 = V_x\,, \quad f_n = -\frac{1}{2}\, e^{-\int h_3(t)dt} f_{n-1,x} + V_x\, e^{-\int h_3(t)dt} f_{n-1} - \frac{1}{2}\sum_{k=1}^{n-1} f_k f_{n-k}\,, \quad n \geqslant 2\,. \tag{16}$$

On the other hand, solving Γ from Eq. (11b) and making the derivation of $\lambda(t)\,\Gamma$ with respect to t, we have

$$[\lambda(t)\,\Gamma]_t = (v_{2,x} + V_x)_t = (v_{2,t} + V_t)_x\,. \tag{17}$$

Via Eqs. (10), (11d) and the transformation $\Gamma = 1 + \widetilde{\Gamma}$, the divergence-type equation can be given as

$$\begin{aligned}\big[\lambda(t) + \lambda(t)\,\widetilde{\Gamma}\big]_t = \big\{&4\lambda(t)^2\, h_1(t) V_x - (1 + \widetilde{\Gamma})\,\lambda(t)\big[4\lambda(t)^2 h_1(t) + 4h_2(t) \\ &-x\, h_3(t) - 2h_1(t)V_x^2 + 2h_1(t)V_{2x}\big]\big\}_x\,.\end{aligned} \tag{18}$$

Finally, inserting Expression (15) and transformation $V_x = U$ into Eq. (18), and collecting the coefficients of each order of ε, we derive the infinite conservation laws of Eq. (1) as bellow:

$$\frac{\partial}{\partial t}\rho_n(x,t) + \frac{\partial}{\partial x}J_n(x,t) = 0\,, \quad n = 1, 2, \ldots\,, \tag{19}$$

with

$$\begin{aligned}&\rho_n(x,t) = e^{\int h_3(t)dt} f_n\,, \\ &J_n(x,t) = 4\, h_1(t)\, e^{3\int h_3(t)dt} f_{n+2} + e^{\int h_3(t)dt}\big[4h_2(t) - x h_3(t) - 2h_1(t)U^2 + 2h_1(t)U_x\big] f_n\,,\end{aligned}$$

where $\rho_n(x,t)$ and $J_n(x,t)$ are the conserved densities and conserved fluxes, respectively. The first three sets of conserved densities and fluxes are given as follows:

$$\rho_1(x,t) = U\,, \tag{20a}$$

$$J_1(x,t) = -(2U^3 - U_{2x})h_1(t) + 4\,U\,h_2(t) - U\,x\,h_3(t)\,, \tag{20b}$$

$$\rho_2(x,t) = \frac{1}{2}\, e^{-\int h_3(t)dt}(U^2 - U_x)\,, \tag{20c}$$

$$\begin{aligned}J_2(x,t) = -\frac{1}{2}\, e^{-\int h_3(t)dt}\big[&(3U^4 - 6U^2U_x + U_x^2 - 2UU_{2x})h_1(t) \\ &- 4(U^2 - U_x)h_2(t) + x(U^2 - U_x)h_3(t)\big]\,,\end{aligned} \tag{20d}$$

$$\rho_3(x,t) = \frac{1}{2}\, e^{-2\int h_3(t)dt}(U_{2x} - 2UU_x)\,, \tag{20e}$$

$$\begin{aligned}J_3(x,t) = \frac{1}{4}\, e^{-2\int h_3(t)dt}\big[&(12U^3U_x + U_{4x} - 6U^2U_{2x} - 12UU_x^2 - 2UU_{3x})h_1(t) \\ &- (8UU_x - 4U_{2x})h_2(t) + x(2UU_x - U_{2x})h_3(t)\big]\,.\end{aligned} \tag{20f}$$

The first conservation law is exactly equivalent to Eq. (1). Note that Conservation Laws (20) are the same as those given in Ref. [20]. However, compared to the studies in Ref. [20], this procedure of deriving the infinite conservation laws is more systematic and the expressions of conserved densities and fluxes are simpler.

3.2 Boussinesq-Burgers equations

In this part, we construct the infinite conservation laws of Eq. (2) via the Bell polynomials. Ref. [21] has given the bilinear form and BT in terms of $\mathscr{Y}$-polynomials. Via the transformations

$$\phi = c\,V_x\,, \quad \varphi = -2c^2 W_{2x}\,, \tag{21}$$

with $c = \pm\frac{1}{2}$, $V = V(x,t)$ and $W = W(x,t)$ being the dimensionless functions, Eqs. (2) can have the following bilinear equations in the $\mathscr{Y}$-polynomial form [21],

$$\mathscr{Y}_t(V) + c\,\mathscr{Y}_{2x}(V,W) = 0\,, \tag{22a}$$

$$4\,c\,\mathscr{Y}_{x,t}(V,W) + \mathscr{Y}_{3x}(V,W) = 0\,. \tag{22b}$$

Meanwhile, via the mixing variables, the BT in the $\mathscr{Y}$-polynomial form was obtained [21],

$$\mathscr{Y}_x(v_3) = \varrho\, e^{v_1 - v_2}\,, \tag{23a}$$

$$\mathscr{Y}_{2x}(v_3, w_3) - \eta \mathscr{Y}_x(v_3) + \varrho\, e^{v_1 - v_2} \mathscr{Y}_x(v_4) = 0\,, \tag{23b}$$

$$\mathscr{Y}_t(v_1) - c\,\mathscr{Y}_{2x}(v_1, w_1) = 0\,, \tag{23c}$$

$$\mathscr{Y}_t(v_2) - c\,\mathscr{Y}_{2x}(v_2, w_2) = 0\,, \tag{23d}$$

where ϱ and η are both arbitrary real constants.

Next, based on the obtained Bilinear Form (22) and BT (23), we will construct the infinite conservation laws of Eqs. (2). First of all, via $v_3 = v_2 + V$, $w_3 = v_2 + W$ and $v_4 = v_1 - V$, Eq. (23a) and (23b) are transformed into the form with respect to the mixing variables v_1 and v_2, i.e.,

$$v_{1,x} = \frac{\eta}{2} + \frac{1}{2\varrho} e^{v_2 - v_1} (V_{2x} - W_{2x})\,, \tag{24a}$$

$$v_{2,x} = -V_x + \varrho\, e^{v_1 - v_2}\,. \tag{24b}$$

Then, by setting $\Gamma = e^{v_1 - v_2}$ and via Eqs. (24), we get the Riccati-type equation,

$$\Gamma_x = \left(\frac{\eta}{2} + V_x\right)\Gamma - \varrho\,\Gamma^2 + \frac{1}{2\varrho}\,(V_{2x} - W_{2x})\,, \tag{25}$$

from which we find that η should be chosen as the formal parameter. Therefore, substituting the expansion of Γ with respect to η, i.e.,

$$\Gamma = \sum_{n=1}^{\infty} g_n \eta^{-n}\,, \tag{26}$$

with g_n's ($n = 1, 2, \ldots$) as functions of x and t to be determined, into Eq. (25) and collecting the same power of η, we obtain the following recursion relations,

$$g_1 = \frac{1}{\varrho}\,(W_{2x} - V_{2x})\,, \quad g_{n+1} = 2\left(g_{n,x} - V_x g_n + \varrho \sum_{l=1}^{n-1} g_l g_{n-l}\right), \quad n \geqslant 1\,. \tag{27}$$

Next, solving Γ from Eq. (24b) and making the derivation about t leads to

$$\Gamma_t = \frac{1}{\varrho}(v_{2,x} + V_x)_t = \frac{1}{\varrho}(v_{2,t} + V_t)_x\,. \tag{28}$$

Replacing $v_{2,t}$ and V_t in Eq. (28) via Eqs. (22a), (23d) and the expression $w_2 = W + V + v_2$, we get the divergence-type equation

$$\Gamma_t = \left\{ \frac{c}{2}\left[(\eta - 2V_x)\Gamma + \frac{1}{\varrho}\,(V_{2x} - W_{2x}) \right] \right\}_x . \tag{29}$$

Finally, with the substitution of Expression (26) along with transformations $V_x = \phi/c$ and $W_{2x} = -\varphi/(2c^2)$ into Eq. (29), infinite conservation laws of Eq. (2) can be given as

$$\frac{\partial}{\partial t}\rho_n(x,t) + \frac{\partial}{\partial x}J_n(x,t) = 0\,, \quad n = 1, 2, \dots\,, \tag{30}$$

with

$$\rho_n = g_n\,, \quad J_n = -\frac{c}{2}\,g_{n+1} + \phi\, g_n\,,$$

where $\rho_n = \rho_n(x,t)$ and $J_n = J_n(x,t)$ are the conserved densities and conserved fluxes, respectively. The first three conservation laws can be given as follows:

$$\begin{aligned}
&\rho_1 = -\frac{1}{2\,c^2\varrho}(\varphi + 2\,c\,\phi_x)\,, \quad J_1 = -\frac{1}{c^2\varrho}\Big(\phi\varphi + 2\,c\,\phi\phi_x - c^2\phi_{2x} - \frac{c}{2}\varphi_x\Big)\,,\\
&\rho_2 = \frac{1}{c^3\varrho}(\phi\varphi + 2c\phi\phi_x - 2c^2\phi_{2x} - c\varphi_x)\,,\\
&J_2 = \frac{1}{4c^3\varrho}(8\phi^2\varphi - \varphi^2 + 16c\phi^2\phi_x - 8c\varphi\phi_x - 12c^2\phi_x^2 + 8c^3\phi_{3x} - 24c^2\phi\phi_{2x} - 12c\phi\varphi_x + 4c^2\varphi_{2x})\,,\\
&\rho_3 = \frac{1}{2c^4\varrho}(-4\phi^2\varphi + \varphi^2 - 8c\phi^2\phi_x + 8c\varphi\phi_x + 12c^2\phi_x^2 - 8c^3\phi_{3x} + 16c^2\phi\phi_{2x} + 8c\phi\varphi_x - 4c^2\varphi_{2x})\,,\\
&J_3 = \frac{1}{c^4\varrho}\big\{2\big[-2\phi^3\varphi + \phi\varphi^2 + 12c^2\phi\phi_x^2 - 8c^3\phi\phi_{3x} + 2c^4\phi_{4x} + 10c^2\phi^2\phi_{2x} - 3c^2\varphi\phi_{2x} + 5c\phi^2\varphi_x\\
&\qquad -c\varphi\varphi_x - c\phi_x(4\phi^3 - 8\phi\varphi + 12c^2\phi_{2x} + 5c\varphi_x) + c^3\varphi_{3x} - 4c^2\phi\varphi_{2x}\big]\big\}\,.
\end{aligned}$$

The first conservation law can be exactly decomposed into Eq. (2) with $c = \pm 1/2$. It should be pointed out that this procedure can also be performed on other coupled NLEEs admitting two-field bilinear form such as the non-isospectral AKNS system [22].

3.3 Procedure of constructing the infinite conservation laws

Through the above analysis, the procedure of constructing the infinite conservation laws via the Bell polynomials can be concluded as the following four steps:

Step 1: From the space-derivative part of the four-field bilinear BT in terms of the $\mathscr{Y}$-polynomials, adopt the exponential function about two odd-order mixing variables (e.g., $e^{v_j - v_l}$, $j, l = 1, 2, 3, 4$ and $j \neq l$) as the Γ function. The derivative of Γ with respect to the space variable x leads to the Riccati-type equation.

Step 2: Substitute the series expansion of Γ with respect to a formal parameter into the Riccati-type equation and equate the coefficients of the same power of the parameter to zero to obtain the recurrence relations. Note that the choice of formal parameter needs to ensure the existence of power difference between the functions Γ_x and Γ.

Step 3: Solve the space-derivative part of the four-field bilinear BT in terms of $\mathscr{Y}$-polynomials to derive another expression of Γ. Calculate the derivative on both sides of this expression with respect to the time variable t and switch the derivative to get the divergence-type equation.

Step 4: Insert Γ into the divergence-type equation and collect the coefficients of the same power of the formal parameter to derive the infinite conservation laws.

Such procedure can be applied to the single and coupled NLEEs admitting two-field bilinear forms.

4 Conclusions

In this paper, we have extended the Bell-polynomial scheme to the construction of infinite conservation laws for the NLEEs admitting two-field bilinear forms. As an example of the single NLEEs, the non-isospectral and variable-coefficient mKdV equation arising in fluid and plasma physics [i.e., Eq. (1)] has been considered and infinite conservation laws in Expressions (19) have been derived based on the two-field Hirota method [as seen in Eqs. (10)] and four-field bilinear BT [as seen in Eqs. (11)] in the $\mathscr{Y}$-polynominal form. It is found that this procedure can be carried out under the variable-coefficient conditions. Moreover, the Boussinesq-Burgers equations for the shallow water waves [i.e., Eq. (2)] as an example of the coupled NLEEs have been analyzed and infinite conservation laws [as seen in Eqs. (30)] have been obtained in a similar way. Finally, four general steps in this method have been concluded, which can be applicable to the single and coupled NLEEs admitting two-field bilinear forms.

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