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Homotopy Perturbation Sumudu Transform Method with He's Polynomial for Solutions of Some Fractional Nonlinear Partial Differential Equations

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Abstract: In this paper, the homotopy perturbation Sumudu transform method (HPSTM) is applied to solve fractional nonlinear partial differential equations. The HPSTM is a combined form of Sumudu transform and Homotopy perturbation method (HPM). The nonlinear terms can be easily handled by the use of He's polynomials. The results shows that the HPSTM is very efficient, simple and avoids the round-off errors. Further the time-fractional K (2,2), Sawada-Kotera and KdV equation of fifth order, are solved as test examples to illustrate the present scheme.

Keywords: fractional derivative; Sawada-Kotera Equation; K(2,2) and KdV equations; homotopy perturbation Sumudu transform method; He's polynomials

1 Introduction

Most of the problems in physics, chemistry and biology can be converted into the form of nonlinear partial differential equations. Therefore many methods to solve these problems are increasing attention in recent years [1–3]. It is very difficult to solve nonlinear problems and it is often more difficult to find an analytic solution. Various methods were proposed to find approximate solutions of nonlinear equations [4–13]. He developed the homotopy perturbation method (HPM) by combining the homotopy in topology and classical perturbation techniques, which has been applied to solve many linear and nonlinear differential equations [14–29]. In the recent years, the homotopy perturbation method is combined with Laplace transformation method and the variational iteration method to produce a highly effective technique for handling nonlinear terms is known as homotopy perturbation transform method (HPTM). The use of He's polynomials in the nonlinear terms was first introduced by Ghorbani [30, 31]. Later on many researcher use homotopy perturbation transform method for different type of linear and nonlinear differential equations [32–38].Recently, another such a combination in which the sumudu transformation method and homotopy perturbation method are applied to solve nonlinear problems is known as homotopy perturbation Sumudu transform method (HPSTM) [39, 40]. Many researcher use HPSTM to obtain the analytical exact and approximate solutions for linear and nonlinear partial differential equations [41–46].

In this paper, HPSTM is applied to find the solution of some nonlinear partial differential equations (the Sawada-Kotera equation, KdV equation of fifth order and K (2,2) equations all of time-fractional type). This method provides the solution in rapid convergent series which leads the solution in a closed form. Comparison between HPSTM and HPM are illustrated which shows that HPSTM is highly efficient for solving nonlinear equations.

2 Homotopy perturbation Sumudu transform method (HPSTM)

To illustrate the basic idea of this method, we consider a gerneral fractional nonlinear partial differential equation with initial conditions of the form

$$D_t^{\alpha} u(x,t) + Lu(x,t) + Nu(x,t) = g(x,t), \tag{1}$$

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Subjected to the initial condition

$$D_0^s u(x,0) = g_s, (s = 0,1,2,3,\cdots\cdots, m-1), D_0^m u(x,0) = 0, m = [\alpha], \tag{2}$$

where $m-1 < \alpha \leq m$, D_t^α is the Caputo fractional derivative of the function $u(x,t)$, $g(x,t)$ is the source term, L linear differential and N is the nonlinear differential operator respectively. Applying the Sumudu transform on both sides of above equation

$$S[D_t^\alpha u(x,t) + Lu(x,t) + Nu(x,t)] = S[g(x,t)] \tag{3}$$

Using the linear proprty of Sumudu transorm

$$S[D_t^\alpha u(x,t)] + S[Lu(x,t)] + S[Nu(x,t)] = S[g(x,t)] \tag{4}$$

Using the differentiation property of the Sumudu transform, and using the initial conditions, we get

$$S[u(x,t)] = \sum_{k=0}^{m-1} u^{-\alpha+k} u^{(k)}(x,0) + u^\alpha S[g(x,t)] - u^\alpha S[Lu(x,t) + Nu(x,t)] \tag{5}$$

operating the inverse Sumudu transform on both sidesoperating with the Sumudu inverse on both sides

$$u(x,t) = H(x,t) - S^{-1}\left[u^\alpha S[Lu(x,t) + Nu(x,t) - g(x,t)]\right], \tag{6}$$

where $H(x,t)$ represents the term arising from the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{7}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \tag{8}$$

for some He's polynomials H_n that are given by

$$H_n(u_0,...,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}(p^i u_i)\right)\right]_{p=0}, n = 0,1,2,3... \tag{9}$$

Substituting Equation (6) and (7)in equation (5), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = H(x,t) - p\left(S^{-1}\left[u^\alpha S\left[L\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u) - g(x,t)\right]\right]\right), \tag{10}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained

$$\begin{aligned} p^0 &: u_0(x,t) = -H(x,t), \\ p^1 &: u_1(x,t) = -S^{-1}\left[u^\alpha S[Lu_0(x,t) + H_0(u) - g(x,t)]\right], \\ p^2 &: u_2(x,t) = -S^{-1}\left[u^\alpha S[Lu_1(x,t) + H_1(u) - g(x,t)]\right], \\ p^3 &: u_3(x,t) = -S^{-1}\left[u^\alpha S[Lu_2(x,t) + H_2(u) - g(x,t)]\right], \\ &\cdots \end{aligned} \tag{11}$$

Setting $p = 1$ results the approximate solution of equation (1)

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)\ldots \tag{12}$$

3 Applications

In order to elucidate the solution procedure of the homotopy perturbation Sumudu transform method, we consider the following three examples:

Example 1 *Consider the fractional $K(2,2)$ equation, where $0 < \alpha \leq 1$.*

$$D_t^\alpha u + 2uu_x + 2uuu_{xxx} + 6u_x u_{xx} = 0 \tag{13}$$

Subject to the initial condition

$$u(x,0) = x, \tag{14}$$

By applying the aforesaid method subject to the initial condition, we get

$$S[D_t^\alpha u] = -S\left[2u\frac{\partial u}{\partial x} + 2u\frac{\partial^3 u}{\partial x^3} + 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right] \tag{15}$$

Applying the inverse Sumudu transform

$$u(x,t) = xt - S^{-1}\left[u^\alpha S[2u\frac{\partial u}{\partial x} + 2u\frac{\partial^3 u}{\partial x^3} + 6\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}]\right] \tag{16}$$

By applying the homotopy perturbation method, using

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t),$$

where the homotopy parameter p is considered as small parameter $p\epsilon[0,1]$, the nonlinear term decompose by He's polynomial, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = xt - S^{-1}\left[u^\alpha S[\sum_{n=0}^{\infty} p^n H_n(u)]\right], \tag{17}$$

where $H_n(u)$ are He's polynomials that represent the nonlinear terms. The first few components of He's polynomials, for example, are given by

$$H_0^1(u) = 2u_0u_{0x} + 2u_0u_{0xxxx} + 6u_{0x}u_{0xx}$$
$$H_1^1(u) = 2(u_0u_{1x} + u_1u_{0x}) + 2(u_0u_{1xxx} + u_1u_{0xxx}) + 6(u_{0x}u_{1xx} + u_{1x}u_{0xx})$$
$$H_2^1(u) = 2(u_0u_{2x} + u_1u_{1x} + u_2u_{0x}) + 2(u_0u_{2xxx} + u_1u_{1xxx} + u_2u_{0xxx}) + 6(u_{0x}u_{2xx} + u_{1x}u_{1xx} + u_{2x}u_{0xx})$$
$$\cdots$$

Comparing the coefficient of like powers of p, we have

$$p^0 : u_0(x,t) = x$$
$$p^1 : u_1(x,t) = \frac{-2x}{\Gamma(1+\alpha)}t^\alpha$$
$$p^2 : u_2(x,t) = 2^3\frac{x}{\Gamma(1+2\alpha)}t^{2\alpha}$$
$$p^3 : u_3(x,t) = -\left(\frac{2^5}{\Gamma(1+3\alpha)} + \frac{2^3\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)xt^{3\alpha}$$
$$\cdots$$

Therefor, the series solution is

$$u(x,t) = x - \frac{2x}{\Gamma(1+\alpha)}t^\alpha + 2^3\frac{x}{\Gamma(1+2\alpha)}t^{2\alpha} - \left(\frac{2^5}{\Gamma(1+3\alpha)} + \frac{2^3\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)xt^{3\alpha} + \dots \tag{18}$$

Setting $p = 1$ results the approximate solution as $\alpha = 1$:

$$u(x,t) = x - 2xt + 4xt^2 - 8xt^3 + \dots \tag{19}$$

In closed form $u(x,t) = \frac{x}{1+2t}$.

Example 2 *Consider the time fractional Sawada Kotera equation*

$$D_t^\beta u + 45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx} = 0; t > 0, 0 < \beta \le 1 \tag{20}$$

with initial condition

$$u(x,0) = 2k^2 \operatorname{sech}^2(kx) \tag{21}$$

The exact solution for $\beta = 1$ is

$$u(x,t) = 2k^2 \operatorname{sech}^2(k(x - 16kt)) \tag{22}$$

By applying the aforesaid method subject to the initial condition, we get

$$S[D_t^\beta u] = -S[45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx}] \tag{23}$$

Applying the inverse Sumudu transform

$$u(x,t) = u(x,0) - S^{-1}\left[u^\beta S[45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx}]\right] \tag{24}$$

By applying the homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - pS^{-1}\left[u^\beta S\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxxxx} + \left(\sum_{n=0}^{\infty} p^n H_n(u)\right)\right]\right] \tag{25}$$

where

$$\sum_{n=0}^{\infty} p^n H_n(u) = 45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx}$$

where $H_n(u)$ are He's polynomials that represent the nonlinear terms. The first few components of He's polynomials, for example, are given by

$$H_0(u) = 45u_0^2 u_{0x} + 15u_{0x} u_{0xx} + 15u_0 u_{0xxx}$$

$$H_1(u) = 45(2u_0 u_1 u_{0x} + u_0^2 u_{1x}) + 15(u_{1x} u_{0xx} + u_{0x} u_{1xx}) + 15(u_1 u_{0xxx} + u_{1xxx} u_0)$$

$$\begin{aligned} H_2(u) = {} & 45[(2u_1 u_2 + 2u_0 u_3)u_{0x} + (u_1^2 + 2u_0 u_2)u_{1x} + 2u_0 u_1 u_{2x} + u_0^2 u_{3x}] \\ & + 15[u_0 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx}] + 15[u_{0x} u_{2xx} + u_{1x} u_{1xx} + u_{2x} u_{0xx}] \end{aligned} \tag{26}$$

$$\cdots$$

Comparing the coefficient of like powers of p, we have

$$p^0 : u_0(x,t) = 2k^2 \operatorname{sech}^2(kx)$$

$$p^1 : u_1(x,t) = 64k^7 \operatorname{sech}^2(kx) \tanh(kx) \frac{t^\beta}{\Gamma(1+\beta)}$$

$$p^2 : u_2(x,t) = -512 \operatorname{sech}^2(kx)(3\operatorname{sech}^2(kx) - 2) \frac{t^{2\beta}}{\Gamma(1+2\beta)}$$

$$\cdots$$

Therefore, the series solution is

$$u(x,t) = 2k^2 \operatorname{sech}^2(kx) + 64k^7 \operatorname{sech}^2(kx)\tanh(kx)\frac{t^\beta}{\Gamma(1+\beta)} - 512\operatorname{sech}^2(kx)(3\operatorname{sech}^2(kx) - 2)\frac{t^{2\beta}}{\Gamma(1+2\beta)} \tag{27}$$

Example 3 *Consider the time fractional KdV equation.*

$$D_t^\beta u + 2u^2 u_x + 6u_x u_{xx} + 3uu_{xxx} + u_{xxxxx} = 0; t > 0, 0 < \beta \leq 1 \tag{28}$$

with initial condition

$$u(x, 0) = 10k^2(3\operatorname{sech}^2(kx) - 1) \tag{29}$$

By applying the aforesaid method subject to the initial condition, we get

$$S[D_t^\beta u] = -S[2u^2 u_x + 6u_x u_{xx} + 3uu_{xxx} + u_{xxxxx}] \tag{30}$$

Applying the inverse Sumudu transform

$$u(x,t) = u(x,0) - S^{-1}\left[u^\beta S[2u^2 u_x + 6u_x u_{xx} + 3uu_{xxx} + u_{xxxxx}]\right] \tag{31}$$

By applying the homotopy perturbation method

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) - pS^{-1}\left[u^\beta S\left[\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxxxx} + \left(\sum_{n=0}^{\infty} p^n H_n(u)\right)\right]\right] \tag{32}$$

where

$$\sum_{n=0}^{\infty} p^n H_n(u) = 2u^2 u_x + 6u_x u_{xx} + 3uu_{xxx}$$

where $H_n(u)$ are He's polynomials that represent the nonlinear terms. The first few components of He's polynomials, for example, are given by

$$H_0(u) = 2u_0^2 u_{0x} + 6u_{0x}u_{0xx} + 3u_0 u_{0xxx}$$

$$H_1(u) = 2(2u_0 u_1 u_{0x} + u_0^2 u_{1x}) + 6(u_{1x}u_{0xx} + u_{0x}u_{1xx}) + 3(u_1 u_{0xxx} + u_{1xxx}u_0)$$

$$\begin{aligned} H_2(u) = {} & 2[(2u_1 u_2 + 2u_0 u_3)u_{0x} + (u_1^2 + 2u_0 u_2)u_{1x} + 2u_0 u_1 u_{2x} + u_0^2 u_{3x}] \\ & +6[u_0 u_{2xxx} + u_1 u_{1xxx} + u_2 u_{0xxx}] + 3[u_{0x}u_{2xx} + u_{1x}u_{1xx} + u_{2x}u_{0xx}] \\ & \cdots \end{aligned} \tag{33}$$

Comparing the coefficient of like powers of p, we have

$$p^0 : u_0(x,t) = 10k^2(3\operatorname{sech}^2(kx) - 1)$$

$$p^1 : u_1(x,t) = 5760k^7 \operatorname{sech}^2(kx)\tanh(kx)\frac{t^\beta}{\Gamma(1+\beta)}$$

$$p^2 : u_2(x,t) = 552960k^{12}\operatorname{sech}^2(kx)(1 - 3\tanh^2(kx))\frac{t^{2\beta}}{\Gamma(1+2\beta)}$$

$$\cdots$$

So the series solution is

$$\begin{aligned} u(x,t) = {} & 10k^2(3\operatorname{sech}^2(kx) - 1) + 5760k^7\operatorname{sech}^2(kx)\tanh(kx)\frac{t^\beta}{\Gamma(1+\beta)} + \\ & 552960k^{12}\operatorname{sech}^2(kx)(1 - 3\tanh^2(kx))\frac{t^{2\beta}}{\Gamma(1+2\beta)} \end{aligned} \tag{34}$$

4 Conclusions

In this work, homotopy perturbation sumudu transform method (HPSTM) has been successfully applied to approximate solution of some nonlinear partial differential equations. On comparing the results of this method with HPM, it is observed HPSTM is extremely simple and easy to handle the nonlinear terms. The main advantage of this method is to overcome the lack of satisfied initial conditions and to construct homotopy, which is a difficult task in case of HPM. Maple 13 package is used to calculate series obtained from iteration. Further the method needs much less computational work which shows fast convergent for solving nonlinear system of partial differential equations.

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