ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.21(2016) No.2, pp.67-79

Parametric Septic Spline Solution for Some Ordinary Differential Equations Occurring in Plate Deflection Theory

Arshad Khan *, Talat Sultana
Department of Mathematics, Jamia Millia Islamia, New Delhi-25, INDIA
(Received 01 November 2011, accepted 24 March 2012)

Abstract: In this paper, we have developed parametric septic spline methods, which reduces to ordinary septic spline as the parameter $\tau \to 0$ for the numerical solution of fourth order linear and nonlinear two point boundary value problems. Using this spline function a few consistency relations are derived for computing approximations to the solution of the problem. Methods of order two, four and six have been obtained which lead to seven diagonal linear system. Boundary equations for existing orders have been developed and truncation error is obtained. Three numerical illustrations are tabulated to demonstrate the practical usefulness of our methods and comparison is made with known methods.

Keywords: Parametric septic splines; Boundary value problems; Boundary equations; Plate deflection theory

1 Introduction

In this paper, we consider two point boundary value problems involving a fourth order differential equation of the form

$$u^{(4)} = f(x, u),\ a \le x \le b, \tag{1.1}$$

subject to the boundary conditions

$$u(a) = \gamma_0,\ u(b) = \gamma_1,\ u'(a) = \delta_0,\ u'(b) = \delta_1, \tag{1.2}$$

where $\gamma_0, \gamma_1, \delta_0, \delta_1$ are finite real arbitrary constants. A particular case of this differential equation often occurs in plate deflection theory such as problem of the bending of a uniformly loaded rectangular plate supported over the entire surface by an elastic foundation rigidly supported along the edge [21].

Finite difference method for the problem (1.1) with boundary conditions (1.2) are given in [1,3,11,12,14,16]. Babuska et al. [3] shown that the resulting error is $O(h^{3/2})$. Usmani [14] has given methods of order two, four and six, while his methods of order two and four lead to the five-diagonal linear systems and the sixth order method leads to a nine-diagonal linear system. The author has also proved that the boundary value problem of the type (1.1-1.2) has a unique solution provided

$$\inf\ f(x) = -\eta\ >\ \frac{-\sigma}{(b-a)^4},\ \ with\ \ \sigma = 500.5639.....$$

Chawla and Katti [12] discussed the construction of finite difference scheme for the two point non-linear boundary value problem involving differential equations of order $2p$, and in particular derived and illustrated methods of order two, four and six, using off step points. A high accuracy quintic spline solution of such problem is given in [13]. Sixth order collocation method based on quintic spline has been developed by Irodotou-Ellina and Houstis [10]. Ramadan et al. [9] developed second and fourth order methods by using nonpolynomial quintic spline. Rashidinia and Aziz [4] has developed parametric quintic spline methods of second, fourth and sixth order. Also Rashidinia and Golbabaee [5] and Siddiqi and Akram [20] generated a difference scheme via quintic spline function for this problem. Usmani [15] also analyzed second and fourth order convergent methods for such problems by using quintic and sextic polynomial spline functions.

* **Corresponding author**. *E-mail address*: akhan1234in@rediffmail.com, talat17m@gmail.com

IJNS.2016.04.15/901

Some authors have solved fourth order two point boundary value problems with different boundary conditions by taking u'' in place of u' [6,7,9,16-19]. Rashidinia and Jalilian [6] and Rashidinia et al. [7] developed non-polynomial and polynomial quintic spline for such problems respectively. Usmani [17] and Usmani and Warsi [19] developed and analyzed second and fourth order convergent methods for the solution of linear fourth order two point boundary value problems by using quartic, quintic and sextic polynomial spline functions respectively. Usmani and Marsden [18] also derived second order finite difference method. Viswanadham et al. [8] have solved fourth order boundary value problem by Galerkin method with quintic B-spline involving two different cases of boundary conditions and the basis functions are redefined into a new set of basis functions which vanish at the boundary where the Dirichlet type of boundary conditions are prescribed.

In this paper, we have developed a new spline function method for fourth order boundary value problems by using parametric septic spline and have solved linear and nonlinear boundary value problems. Analysis of the method shows second, fourth, sixth and eighth order accuracy for arbitrary choices of parameters p_1, p_2, p_3 and p_4.

In section 2, we have described a parametric septic spline method and its consistency relations. Development of new methods for linear and nonlinear boundary value problems are also given in this section as subsection 2.1 and 2.2. section 3 is devoted to the development of the boundary equations. The class of methods are discussed in section 4. The parametric septic spline solution approximating the analytical solution of the fourth order boundary value problem is determined by using the consistency relation involving the fourth order derivative and the values of the spline along with the end conditions in section 5. Truncation error is also discussed for class of methods in this section. In section 6, three examples are considered for the usefulness of the developed method and numerical results are compared with existing methods.

2 Derivation of the method

In order to develop the numerical method for approximating solution of fourth order boundary value problem (1.1) with boundary conditions (1.2), the interval $[a, b]$ is divided into N equal subintervals using the grid points $x_j = a + jh,\ \ j = 0, 1, ..., N$, where

$$x_0 = a, \quad x_N = b, \quad and \quad h = \frac{b-a}{N}. \tag{2.1}$$

A function $S_\Delta(x, \tau)$ of class $C^6[a, b]$ which interpolates $u(x)$ at the mesh point x_j depends on a parameter τ reduces to ordinary septic spline $S_\Delta(x)$ in $[a, b]$ as $\tau \to 0$ is termed as parametric septic spline function. Since the parameter τ can occur in $S_\Delta(x)$ in many ways such a spline is not unique.

If $S_\Delta(x, \tau) = S_\Delta(x)$ is a piecewise function satisfying the following differential equation in the interval $[x_{j-1}, x_j]$

$$S^{(6)}_\Delta(x) - \tau^2 S''_\Delta(x) \quad = \quad (Q_j - \tau^2 M_j)\frac{x - x_{j-1}}{h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{x_j - x}{h} = A_j z + A_{j-1}\bar{z}, \tag{2.2}$$

where

$$z = \frac{x - x_{j-1}}{h},\ \bar{z} = 1 - z,\ A_k = Q_k - \tau^2 M_k,$$

$$S^{(2)}_\Delta(x_k, \tau) = M_k,\ \ S^{(6)}_\Delta(x_k, \tau) = Q_k,\ \ k = j-1, j;\ \ \tau > 0,$$

then it is termed as parametric septic spline II.
Solving equation (2.2), we get

$$\begin{aligned} S_\triangle(x) \quad = \quad & B_1 + B_2 x + B_3 \cosh\sqrt{\tau}x + B_4 \sinh\sqrt{\tau}x + B_5 \cos\sqrt{\tau}x + B_6 \sin\sqrt{\tau}x \\ & -\frac{1}{\tau^2}\left\{(Q_j - \tau^2 M_j)\frac{(x - x_{j-1})^3}{6h} + (Q_{j-1} - \tau^2 M_{j-1})\frac{(x_j - x)^3}{6h}\right\}. \end{aligned} \tag{2.3}$$

To develop the consistency relations between the value of spline and its derivatives at knots, let

$$\left.\begin{aligned} S_\Delta(x_j) = u_j,\ \ S_\Delta(x_{j+1}) = u_{j+1},\\ S''_\Delta(x_j) = M_j,\ \ S''_\Delta(x_{j+1}) = M_{j+1},\\ S^{(4)}_\Delta(x_j) = F_j,\ \ S^{(4)}_\Delta(x_{j+1}) = F_{j+1}.\end{aligned}\right\} \tag{2.4}$$

To define spline in terms of u_j's,M_j's and F_j s, the coefficients introduced in Eq.(2.3) are calculated as

$$\begin{aligned}
B_1 &= u_{j-1} + \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) - \frac{F_{j-1}}{\tau^2}\\
&\quad -\frac{x_{j-1}}{h}\Big[(u_j - u_{j-1}) - \frac{h^2}{6\tau^2}(Q_{j-1} - \tau^2 M_{j-1}) + \frac{h^2}{6\tau^2}(Q_j - \tau^2 M_j) + \frac{1}{\tau^2}(F_{j-1} - F_j)\Big],\\
B_2 &= \frac{1}{h}(u_j - u_{j-1}) + \frac{h}{6\tau^2}\Big[-(Q_{j-1} - \tau^2 M_{j-1}) + (Q_j - \tau^2 M_j)\Big] + \frac{1}{\tau^2 h}(F_{j-1} - F_j),\\
B_3 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\Big[\frac{1}{2}\sinh\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) - \frac{1}{2}\sinh\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\\
&\quad -\frac{1}{\tau}\sinh\sqrt{\tau}x_{j-1}Q_j + \frac{1}{\tau}\sinh\sqrt{\tau}x_j Q_{j-1}\Big],\\
B_4 &= \frac{1}{\tau^2 \sinh\sqrt{\tau}h}\Big[-\frac{1}{2}\cosh\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) + \frac{1}{2}\cosh\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\\
&\quad +\frac{1}{\tau}\cosh\sqrt{\tau}x_{j-1}Q_j - \frac{1}{\tau}\cosh\sqrt{\tau}x_j Q_{j-1}\Big],\\
B_5 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\Big[\sin\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) - \sin\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\Big],\\
B_6 &= \frac{1}{2\tau^2 \sinh\sqrt{\tau}h}\Big[-\cos\sqrt{\tau}x_j\Big(F_{j-1} - \frac{Q_{j-1}}{\tau}\Big) + \cos\sqrt{\tau}x_{j-1}\Big(F_j - \frac{Q_j}{\tau}\Big)\Big].
\end{aligned} \tag{2.5}$$

Substituting these values in (2.3), we get

$$\begin{aligned} S_\Delta(x) &= zu_j + \bar{z}u_{j-1} + \frac{h^2}{6}\Big[p(z)M_j + p(\bar{z})M_{j-1}\Big] + \frac{h^4}{2}\Big[r(z)F_j + r(\bar{z})F_{j-1}\Big]\\ &\quad + \frac{h^6}{6}\Big[q(z)Q_j + q(\bar{z})Q_{j-1}\Big]\end{aligned} \tag{2.6}$$

where

$$p(z) = z^3 - z,\ \ q(z) = \frac{z}{\omega^4} - \frac{z^3}{\omega^4} + \frac{3\sinh\omega z}{\omega^6 \sinh\omega} - \frac{3\sin\omega z}{\omega^6\sin\omega},\ \ r(z) = \frac{-2z}{\omega^4} + \frac{\sinh\omega z}{\omega^4\sinh\omega} + \frac{\sin\omega z}{\omega^4 \sin\omega}, \omega = \sqrt{\tau}h. \tag{2.7}$$

Applying the first, third and fifth derivative continuities at the knots, i.e. $S^{(\mu)}_\Delta(x_j^-) = S^{(\mu)}_\Delta(x_j^+),\ \mu = 1, 3$ and 5, the following consistency relations are derived:

$$\begin{aligned} M_{j+1}+4M_j+M_{j-1} &= \frac{6}{h^2}(u_{j+1}-2u_j+u_{j-1})+3h^2(\alpha_2F_{j+1}+2\beta_2F_j+\alpha_2F_{j-1}) \\ &\quad +h^4(\alpha_1Q_{j+1}+2\beta_1Q_j+\alpha_1Q_{j-1}),\ j=1(1)(N-1), \end{aligned} \tag{2.8}$$

$$\begin{aligned} M_{j+1}-2M_j+M_{j-1} &= \frac{h^2}{6}[(1-\omega^4\alpha_1)F_{j+1}+2(2-\omega^4\beta_1)F_j+(1-\omega^4\alpha_1)F_{j-1}] \\ &\quad -\frac{h^4}{2}(\alpha_2Q_{j+1}+2\beta_2Q_j+\alpha_2Q_{j-1}), j=1(1)(N-1), \end{aligned} \tag{2.9}$$

$$\begin{aligned} & h^2[(1-\omega^4\alpha_1)Q_{j+1}+2(2-\omega^4\beta_1)Q_j+(1-\omega^4\alpha_1)Q_{j-1}] \\ &= 3[(\omega^4\alpha_2+2)F_{j+1}+2(\omega^4\beta_2-2)F_j+(\omega^4\alpha_2+2)F_{j-1}], j=1(1)(N-1), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{\omega^4}+\frac{3}{\omega^5\sinh\omega}-\frac{3}{\omega^5\sin\omega}, \\ \beta_1 &= \frac{2}{\omega^4}-\frac{3}{\omega^5}\coth\omega+\frac{3}{\omega^5}\cot\omega, \\ \alpha_2 &= \frac{-2}{\omega^4}+\frac{1}{\omega^3\sinh\omega}+\frac{1}{\omega^3\sin\omega}, \\ \beta_2 &= \frac{2}{\omega^4}-\frac{1}{\omega^3}\coth\omega-\frac{1}{\omega^3}\cot\omega. \end{aligned} \tag{2.11}$$

As $\tau\to 0$ that is $\omega\to 0$ then $(\alpha_1,\beta_1,\alpha_2,\beta_2)\to(\frac{-31}{2520},\frac{-4}{315},\frac{7}{180},\frac{2}{45})$.

Using equations (2.8)-(2.10), we obtain the following scheme

$$\begin{aligned} & (e_1u_{j-3}+e_2u_{j-2}+e_3u_{j-1}+e_4u_j+e_3u_{j+1}+e_2u_{j+2}+e_1u_{j+3}) \\ &= \frac{h^4}{6}(p_1F_{j-3}+p_2F_{j-2}+p_3F_{j-1}+p_4F_j+p_3F_{j+1}+p_2F_{j+2}+p_1F_{j+3}),\ j=3(1)(N-3), \end{aligned} \tag{2.12}$$

where the coefficients (e_1,e_2,e_3,e_4) and (p_1,p_2,p_3,p_4) of the developed scheme are given by

$$\begin{aligned} e_1 &= 1-3\omega^4\alpha_1+3\omega^8\alpha_1^2-\omega^{12}\alpha_1^3, \\ e_2 &= 4\omega^4\alpha_1-2\omega^4\beta_1-8\omega^8\alpha_1^2+4\omega^8\alpha_1\beta_1-2\omega^{12}\alpha_1^2\beta_1, \\ e_3 &= 7(1-\omega^4\alpha_1)^3-8(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1), \\ e_4 &= 12(1-\omega^4\alpha_1)^2(2-\omega^4\beta_1)-8(1-\omega^4\alpha_1)^3, \\ p_1 &= c_1(1-\omega^4\alpha_1)^2, \\ p_2 &= 2c_1(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+c_2(1-\omega^4\alpha_1)^2-3d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2), \\ p_3 &= (c_1+c_3)(1-\omega^4\alpha_1)^2+6d_1(1-\omega^4\alpha_1)(2-\omega^4\beta_2)+2c_2(1-\omega^4\alpha_1)(2-\omega^4\beta_1) \\ &\quad -3d_2(1-\omega^4\alpha_1)(2+\omega^4\alpha_2), \\ p_4 &= 2c_2(1-\omega^4\alpha_1)^2-6d_1(1-\omega^4\alpha_1)(2+\omega^4\alpha_2)-6d_1(2-\omega^4\beta_1)(2-\omega^4\beta_2) \\ &\quad +2c_3(1-\omega^4\alpha_1)(2-\omega^4\beta_1)+6d_2(1-\omega^4\alpha_1))(2-\omega^4\beta_2). \end{aligned} \tag{2.13}$$

Also

$$
\begin{aligned}
c_1 &= \frac{1}{6}\omega^8\alpha_1^2 - \frac{3}{2}\omega^4\alpha_2^2 - \frac{1}{3}\omega^4\alpha_1 - 6\alpha_1 - 6\alpha_2 + \frac{1}{6},\\
c_2 &= \frac{2}{3}\omega^8\alpha_1^2 + \frac{1}{3}\omega^8\alpha_1\beta_1 - 18\omega^4\alpha_1\alpha_2 - 3\omega^4\alpha_2\beta_2 - 6\omega^4\alpha_2^2 - 2\omega^4\alpha_1 - \frac{1}{3}\omega^4\beta_1 - 12\alpha_1 - 6\beta_2 + \frac{4}{3},\\
c_3 &= \frac{1}{3}\omega^8\alpha_1^2 + \frac{4}{3}\omega^8\alpha_1\beta_1 - 36\omega^4\alpha_1\beta_2 - 12\omega^4\alpha_2\beta_2 - 3\omega^4\alpha_2^2 - \frac{10}{3}\omega^4\alpha_1 - \frac{4}{3}\omega^4\beta_1 + 36\alpha_1\\
&\quad +12\alpha_2 + 12\beta_2 + 3,\\
d_1 &= \omega^4\alpha_2\beta_1 - \omega^4\alpha_1\beta_2 + 6\omega^4\alpha_1^2 - 10\alpha_1 - 2\alpha_2 + 2\beta_1 + \beta_2,\\
d_2 &= 4\omega^4\alpha_2\beta_1 - 4\omega^4\alpha_1\beta_2 + 12\omega^4\alpha_1\beta_1 - 16\alpha_1 - 18\alpha_2 - 4\beta_1 + 4\beta_2.
\end{aligned}
\tag{2.14}
$$

As $\tau \to 0$ that is $\omega \to 0$, we have

$$
\begin{aligned}
&\text{(i)}(e_1, e_2, e_3, e_4) \longrightarrow (1, 0, -9, 16),\\
&\text{(ii)}(c_1, c_2, c_3, d_1, d_2) \longrightarrow \left(\frac{1}{140}, \frac{17}{14}, \frac{249}{70}, \frac{9}{140}, \frac{4}{35}\right),\\
&\text{(iii)}(p_1, p_2, p_3, p_4) \longrightarrow \left(\frac{1}{140}, \frac{6}{7}, \frac{1191}{140}, \frac{604}{35}\right).
\end{aligned}
$$

[**Remark:**] For these values our scheme reduces to the polynomial septic spline for fourth order boundary value problem which is given as equation (7) in G. Akram and S. S. Siddiqi [2].

2.1 Spline solution for linear boundary value problems

Consider a linear fourth order boundary value problem of the form

$$u^{(4)} + g(x)u(x) = q(x),\ a \le x \le b, \tag{2.15}$$

subject to the boundary conditions

$$u(a) = \gamma_0,\ \ u(b) = \gamma_1,\ \ u'(a) = \delta_0,\ \ u'(b) = \delta_1,$$

where $g(x),\ q(x)\ \epsilon\ C[a, b]$ and $u(x)\ \epsilon\ C^6[a, b]$.
On the mesh Δ, the differential equation (2.15) can be discretized by using the spline relation (2.12) to obtain

$$
\begin{aligned}
&\left(e_1 + \frac{h^4}{6}p_1 g_{j-3}\right)u_{j-3} + \left(e_2 + \frac{h^4}{6}p_2 g_{j-2}\right)u_{j-2} + \left(e_3 + \frac{h^4}{6}p_3 g_{j-1}\right)u_{j-1} + \left(e_4 + \frac{h^4}{6}p_4 g_j\right)u_j\\
&+\left(e_3 + \frac{h^4}{6}p_3 g_{j+1}\right)u_{j+1} + \left(e_2 + \frac{h^4}{6}p_2 g_{j+2}\right)u_{j+2} + \left(e_1 + \frac{h^4}{6}p_1 g_{j+3}\right)u_{j+3}\\
&= \frac{h^4}{6}(p_1 q_{j-3} + p_2 q_{j-2} + p_3 q_{j-1} + p_4 q_j + p_3 q_{j+1} + p_2 q_{j+2} + p_1 q_{j+3}),\ \ j = 3(1)(N-3),
\end{aligned}
\tag{2.16}
$$

where $g(x_j) = g_j, q(x_j) = q_j$.

2.2 Spline solution for nonlinear boundary value problems

Consider a nonlinear fourth order boundary value problem of the form

$$u^{(4)} = f(x, u),\ a \le x \le b, \tag{2.17}$$

subject to the boundary conditions

$$u(a) = \gamma_0,\ \ u(b) = \gamma_1,\ \ u'(a) = \delta_0,\ \ u'(b) = \delta_1,$$

where f is a nonlinear function of u. Discretizing the (2.17) by using the spline relation (2.12), we obtain

$$\begin{aligned}&(e_1u_{j-3} + e_2u_{j-2} + e_3u_{j-1} + e_4u_j + e_3u_{j+1} + e_2u_{j+2} + e_1u_{j+3})\\ &= \frac{h^4}{6}(p_1f_{j-3} + p_2f_{j-2} + p_3f_{j-1} + p_4f_j + p_3f_{j+1} + p_2f_{j+2} + p_1f_{j+3}), \quad j = 3(1)(N-3).\end{aligned} \tag{2.18}$$

3 Development of boundary equations

In this section, we have developed the boundary equations. The relation (2.12) gives $(N-5)$ equations in $(N-1)$ unknowns $u_j, [j = 1(1)N-1]$. We require four more equations, two at each end of the range of integration in order to have closed form solution for u_j. For the discretization of the boundary conditions, we define

(i) $$\sum_{k=0}^{4} a_ku_k + b_1hu'_0 + h^4\sum_{k=0}^{5} d_ku_k^{(4)} + t_1 = 0,\ j = 1,$$

(ii) $$\sum_{k=1}^{5} a^*_ku_k + b_2hu'_0 + h^4\sum_{k=1}^{6} d^*_ku_k^{(6)} + t_2 = 0,\ j = 2,$$

(iii) $$\sum_{k=N-5}^{N-1} a^*_ku_k - b^*_{N-2}hu'_N + h^4\sum_{k=N-6}^{N-1} d^*_ku_k^{(4)} + t_{N-2} = 0,\ j = N-2,$$

(iv) $$\sum_{k=N-4}^{N} a_ku_k - b_{N-1}hu'_N + h^4\sum_{k=N-5}^{N} d_ku_k^{(4)} + t_{N-1} = 0,\ j = N-1, \tag{3.1}$$

where $a_k,\ b_1,\ d_k,\ a^*_k,\ b_2,\ d^*_k,\ b_{N-2}$ and b_{N-1} are arbitrary parameters to be determined at $j = 1, 2, N-2, N-1$ for second, fourth and sixth order methods.

4 Class of methods

To obtain the local truncation error $t_j,\ j = 3(1)(N-3)$, associated with the scheme (2.12), we first rewrite it in the form:

$$\begin{aligned}&(e_1u_{j-3} + e_2u_{j-2} + e_3u_{j-1} + e_4u_j + e_3u_{j+1} + e_2u_{j+2} + e_1u_{j+3})\\ &= \frac{h^4}{6}(p_1u^{(4)}_{j-3} + p_2u^{(4)}_{j-2} + p_3u^{(4)}_{j-1} + p_4u^{(4)}_j + p_3u^{(4)}_{j+1} + p_2u^{(4)}_{j+2} + p_1u^{(4)}_{j+3}),\ j = 3(1)(N-3).\end{aligned} \tag{4.1}$$

By expanding (4.1) in Taylor series about x_j, we obtain the following local truncation error t_j as

$$\begin{aligned} t_j \;=\; & (12e_1 + 12e_2 + 12e_3 + 6e_4)u_j + \frac{h^2}{2!}(108e_1 + 48e_2 + 12e_3)u''_j\\ & + \frac{h^4}{4!}(972e_1 + 192e_2 + 12e_3 - 48p_1 - 48p_2 - 48p_3 - 24p_4)u^{(4)}_j\\ & + \frac{h^6}{6!}(8748e_1 + 768e_2 + 12e_3 - 6480p_1 - 2880p_2 - 720p_3)u^{(6)}_j\\ & + \frac{h^8}{8!}(78732e_1 + 3072e_2 + 12e_3 - 272160p_1 - 53760p_2 - 3360p_3)u^{(8)}_j\\ & + \frac{h^{10}}{10!}(708588e_1 + 12288e_2 + 12e_3 - 7348320p_1 - 645120p_2 - 10080p_3)u^{(10)}_j\\ & + \frac{h^{12}}{12!}(6377292e_1 + 49152e_2 + 12e_3 - 155889360p_1 - 6082560p_2 - 23760p_3)u^{(12)}_j\\ & + O(h^{14}). \end{aligned} \tag{4.2}$$

After taking the values $(e_1,\ e_2,\ e_3,\ e_4) = (1, 0, -9, 16)$, the local truncation error t_j is given by

$$t_j = C_4 h^4 u_j^{(4)} + C_6 h^6 u_j^{(6)} + C_8 h^8 u_j^{(8)} + C_{10} h^{10} u_j^{(10)} + C_{12} h^{12} u_j^{(12)} + O(h^{14}), \tag{4.3}$$

where

$$\begin{aligned}
C_4 &= \frac{1}{4!}(864 - 48p_1 - 48p_2 - 48p_3 - 24p_4),\\
C_6 &= \frac{1}{6!}(8640 - 6480p_1 - 2880p_2 - 720p_3),\\
C_8 &= \frac{1}{8!}(78624 - 272160p_1 - 53760p_2 - 3360p_3),\\
C_{10} &= \frac{1}{10!}(708480 - 7348320p_1 - 645120p_2 - 10080p_3),\\
C_{12} &= \frac{1}{12!}(6377184 - 155889360p_1 - 6082560p_2 - 23760p_3).
\end{aligned} \tag{4.4}$$

By using the above equation and by eliminating the coefficients of the various powers of h for different choices of parameters p_1, p_2, p_3 and p_4, we obtain the following class of methods:

4.1 Second order methods

The values of unknown coefficients of boundary equations for second order at each end are given by

(i) $At\ \ j = 1,\ (a_0, a_1, a_2, a_3, a_4) = (-\frac{37}{18}, 1, \frac{7}{2}, \frac{-31}{9}, 1),$
$(b_1, d_0, d_1, d_2, d_3, d_4, d_5) = (-\frac{5}{3}, \frac{13}{12}, -\frac{5}{2}, 0, 0, 0, 0).$

(ii) $At\ \ j = 2,\ (a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = (\frac{633}{113}, -\frac{1592}{113}, \frac{1518}{113}, -\frac{672}{113}, 1),$
$(b_2, d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = (\frac{120}{113}, \frac{137}{113}, 0, 0, 0, 0, 0).$

(iii) $At\ \ j = N-2,\ (a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*) = (1, -\frac{672}{113}, \frac{1518}{113}, -\frac{1592}{113}, \frac{633}{113}),$
$(b_{N-2}, d_{N-6}^*, d_{N-5}^*, d_{N-4}^*, d_{N-3}^*, d_{N-2}^*, d_{N-1}^*) = (\frac{120}{113}, 0, 0, 0, 0, 0, \frac{137}{113}).$

(iv) $At\ \ j = N-1,\ (a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = (1, -\frac{31}{9}, \frac{7}{2}, 1, -\frac{37}{18}),$
$(b_{N-1}, d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = (-\frac{5}{3}, 0, 0, 0, 0, -\frac{5}{2}, \frac{13}{12}).$

(4.5)

The local truncation error is

$$t_j = \begin{cases} (\frac{91}{72})h^6 u_j^{(6)} + O(h^7), & j = 1, N-1, \\ (\frac{5}{6})h^6 u_j^{(6)} + O(h^7), & j = 2, N-2. \end{cases} \tag{4.6}$$

Case 1: For $(p_1, p_2, p_3, p_4) = (\frac{3}{2}, 0, 0, 33)$, the truncation error is given by

$$t_j = \left(-\frac{3}{2}\right)h^6 u_j^{(6)} + O(h^8),\ \ j = 3(1)N-3. \tag{4.7}$$

Case 2: For $(p_1, p_2, p_3, p_4) = (1, 0, 0, 34)$, the truncation error is given by

$$t_j = (3)h^6 u_j^{(6)} + O(h^8),\ \ j = 3(1)N-3. \tag{4.8}$$

4.2 Fourth order methods

The values of unknown coefficients of boundary equations for fourth order at each end are given by

(i) $At\ \ j = 1,\ (a_0, a_1, a_2, a_3, a_4) = (-\frac{767}{3}, 416, -204, \frac{128}{3}, 1),$
$(b_1, d_0, d_1, d_2, d_3, d_4, d_5) = (-140, -\frac{4}{3}, -\frac{76}{3}, -\frac{28}{3}, 0, 0, 0).$

(ii) $At\ \ j = 2,\ (a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = (-\frac{449}{6}, \frac{649}{4}, -\frac{233}{2}, \frac{337}{12}, 1),$
$(b_2, d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = (-\frac{35}{2}, -\frac{1319}{144}, -\frac{779}{36}, -\frac{959}{144}, 0, 0, 0).$

(iii) $At\ \ j = N-2,\ (a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*) = (1, \frac{337}{12}, -\frac{233}{2}, \frac{649}{4}, -\frac{449}{6}),$

$(b_{N-2}, d^*_{N-6}, d^*_{N-5}, d^*_{N-4}, d^*_{N-3}, d^*_{N-2}, d^*_{N-1}) = (-\frac{35}{2}, 0, 0, 0, -\frac{959}{144}, -\frac{779}{36}, -\frac{1319}{144})$.
(iv) $At\ \ j = N-1,\ (a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = (1, \frac{128}{3}, -204, 416, -\frac{767}{3})$,
$(b_{N-1}, d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = (-140, 0, 0, 0, -\frac{28}{3}, -\frac{76}{3}, -\frac{4}{3})$.

(4.9)

The local truncation error is

$$t_j = \begin{cases} (\frac{1}{180})h^8 u_j^{(8)} + O(h^9), j = 1, N-1, \\ \\ (\frac{365}{3002})h^8 u_j^{(8)} + O(h^9), j = 2, N-2. \end{cases} \tag{4.10}$$

Case 1: For $(p_1, p_2, p_3, p_4) = (-\frac{77}{144}, \frac{269}{64}, 0, \frac{8255}{288})$, the truncation error is given by

$$t_j = \left(-\frac{43}{960}\right)h^8 u_j^{(8)} + O(h^{10}),\ j = 3(1)N-3. \tag{4.11}$$

Case 2: For $(p_1, p_2, p_3, p_4) = (-\frac{5}{9}, \frac{17}{4}, 0, \frac{515}{18})$, the truncation error is given by

$$t_j = \left(\frac{1}{30}\right)h^8 u_j^{(8)} + O(h^{10}),\ j = 3(1)N-3. \tag{4.12}$$

4.3 Sixth order methods

The value of unknown coefficients of boundary equations for sixth order at each end are given by
(i) $At\ \ j = 1,\ (a_0, a_1, a_2, a_3, a_4) = (4, -\frac{98}{11}, \frac{93}{11}, -\frac{50}{11}, 1)$,
$(b_1, d_0, d_1, d_2, d_3, d_4, d_5) = (\frac{18}{11}, \frac{228}{13735}, \frac{180}{1453}, -\frac{349}{622}, -\frac{371}{2183}, -\frac{19}{23671}, \frac{1}{2200})$.
(ii) $At\ \ j = 2,\ (a^*_1, a^*_2, a^*_3, a^*_4, a^*_5) = (0, -\frac{47}{20}, \frac{57}{13}, -\frac{93}{26}, 1)$,
$(b_2, d^*_1, d^*_2, d^*_3, d^*_4, d^*_5, d^*_6) = (-\frac{3}{13}, -\frac{291}{2374}, -\frac{498}{1129}, -\frac{1761}{2309}, -\frac{236}{1595}, -\frac{91}{11346}, \frac{26}{16693})$.
(iii) $At\ \ j = N-2,\ (a^*_{N-5}, a^*_{N-4}, a^*_{N-3}, a^*_{N-2}, a^*_{N-1}) = (1, -\frac{93}{26}, \frac{57}{13}, -\frac{47}{20}, 0)$,
$(b_{N-2}, d^*_{N-6}, d^*_{N-5}, d^*_{N-4}, d^*_{N-3}, d^*_{N-2}, d^*_{N-1}) = (-\frac{3}{13}, \frac{26}{16693}, -\frac{91}{11346}, -\frac{236}{1595}, -\frac{1761}{2309}, -\frac{498}{1129}, -\frac{291}{2374})$.
(iv) $At\ \ j = N-1,\ (a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = (1, -\frac{50}{11}, \frac{93}{11}, -\frac{98}{11}, 4)$,
$(b_{N-1}, d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = (\frac{18}{11}, \frac{1}{2200}, -\frac{19}{23671}, -\frac{371}{2183}, -\frac{349}{622}, \frac{180}{1453}, \frac{228}{13735})$.

(4.13)

The local truncation error is

$$t_j = \begin{cases} (\frac{40}{44507})h^{10} u_j^{(10)} + O(h^{11}),\ j = 1, N-1, \\ \\ (-\frac{2093}{1621})h^{10} u_j^{(10)} + O(h^{11}),\ j = 2, N-2. \end{cases} \tag{4.14}$$

Case 1: For $(p_1, p_2, p_3, p_4) = (\frac{1}{5}, -\frac{1}{4}, \frac{56}{5}, \frac{137}{10})$, the truncation error is given by

$$t_j = \left(\frac{299}{1779}\right)h^{10} u_j^{(10)} + O(h^{12}),\ j = 3(1)N-3. \tag{4.15}$$

Case 2: For $(p_1, p_2, p_3, p_4) = (\frac{61}{300}, -\frac{27}{300}, \frac{45}{4}, \frac{409}{30})$, the truncation error is given by

$$t_j = \left(\frac{254}{1487}\right)h^{10} u_j^{(10)} + O(h^{12}),\ j = 3(1)N-3. \tag{4.16}$$

4.4 Eighth order method

Case: For $(p_1, p_2, p_3, p_4) = (\frac{446}{2131}, -\frac{617}{2018}, \frac{2506}{221}, \frac{3338}{247})$, the truncation error is given by

$$t_j = \left(-\frac{294}{5711}\right)h^{12} u_j^{(12)} + O(h^{14}),\ j = 3(1)N-3. \tag{4.17}$$

However the boundary equations for eighth-order method are not constructed and these can be obtained by adapting the above procedure.

5 Spline solution

The parametric septic spline solution of the problem (1.1) subject to the boundary conditions (1.2) is based on the system of linear equations given by equations (2.12) and (3.1).

Let $U = (u_j)$, $\overline{U} = (\overline{u_j})$, $V = (v_j)$ and $T = (t_j)$ be the column vectors. Here, $U, \overline{U}, T$ denotes the exact solution, approximate solution and truncation error respectively. Also $E = (e_j) = U - \overline{U}$, $j = 1, 2, ...(N-1)$; is the discretization error. Then we can write the standard matrix equations in the form

(i) $LU = V + T$,
(ii) $L\overline{U} = V$, (5.1)
(iii) $LE = T$.

Also we have $L = A + h^4 BG$ and $G = diag(g_j)$. (5.2)
For the case of nonlinear two-point boundary value problem $u^{(4)} = f(x, u)$, system(5.1)(ii) becomes

$$A\overline{U}^{(j+1)} - h^4 Bf(x, \overline{U}^j) = W, \qquad j = 0, 1, 2, ... \tag{5.3}$$

which can be solved numerically by using an iterative technique (See section 6, example 3). W is an $(N-1)$ column vector given by $W = (w_j)$, $j = 1(1)(N-1)$. The septa-diagonal matrix A is given by

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & & & & & & \\ a_1^* & a_2^* & a_3^* & a_4^* & a_5^* & & & & & \\ e_2 & e_3 & e_4 & e_3 & e_2 & e_1 & & & & \\ e_1 & e_2 & e_3 & e_4 & e_3 & e_2 & e_1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & e_1 & e_2 & e_3 & e_4 & e_3 & e_2 & e_1 \\ & & & & e_1 & e_2 & e_3 & e_4 & e_3 & e_2 \\ & & & & & a_{N-5}^* & a_{N-4}^* & a_{N-3}^* & a_{N-2}^* & a_{N-1}^* \\ & & & & & & a_{N-4} & a_{N-3} & a_{N-2} & a_{N-1} \end{bmatrix} \tag{5.4}$$

and the matrix B has the form

$$B = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & & & & & & \\ -d_1^* & -d_2^* & -d_3^* & -d_4^* & -d_5^* & -d_6^* & & & & & \\ p_2/6 & p_3/6 & p_4/6 & p_3/6 & p_2/6 & p_1/6 & & & & & \\ p_1/6 & p_2/6 & p_3/6 & p_4/6 & p_3/6 & p_2/6 & p_1/6 & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & p_1/6 & p_2/6 & p_3/6 & p_4/6 & p_3/6 & p_2/6 & p_1/6 \\ & & & & p_1/6 & p_2/6 & p_3/6 & p_4/6 & p_3/6 & p_2/6 \\ & & & & -d_{N-6}^* & -d_{N-5}^* & -d_{N-4}^* & -d_{N-3}^* & -d_{N-2}^* & -d_{N-1}^* \\ & & & & & -d_{N-5} & -d_{N-4} & -d_{N-3} & -d_{N-2} & -d_{N-1} \end{bmatrix}, \tag{5.5}$$

where, we have taken the values of $(e_1, e_2, e_3, e_4) = (1, 0, -9, 16)$ for different orders.

Moreover,

$$\begin{aligned}
v_1 &= -a_0\gamma_0 - b_1h\delta_0 - h^4d_0(q_0 - g_0\gamma_0) - h^4(d_1q_1 + d_2q_2 + d_3q_3 + d_4q_4 + d_5q_5),\ j = 1,\\
v_2 &= -b_2h\delta_0 - h^4(d_1^*q_1 + d_2^*q_2 + d_3^*q_3 + d_4^*q_4 + d_5^*q_5 + d_6^*q_6),\ j = 2,\\
v_3 &= -e_1\gamma_0 + \frac{h^4}{6}p_1(q_0 - g_0\gamma_0) + \frac{h^4}{6}(p_2q_1 + p_3q_2 + p_4q_3 + p_3q_4 + p_2q_5 + p_1q_6),\ j = 3,\\
v_j &= \frac{h^4}{6}(p_1q_{j-3} + p_2q_{j-2} + p_3q_{j-1} + p_4q_j + p_3q_{j+1} + p_2q_{j+2} + p_1q_{j+3}),\ j = 4(1)(N-4),\\
v_{N-3} &= -e_1\gamma_1 + \frac{h^4}{6}p_1(q_N - g_N\gamma_1) + \frac{h^4}{6}(p_1q_{N-6} + p_2q_{N-5} + p_3q_{N-4} + p_4q_{N-3} + p_3q_{N-2}\\
&\quad + p_2q_{N-1}),\ j = N-3,\\
v_{N-2} &= b_{N-2}h\delta_1 - h^4(d_{N-6}^*q_{N-6} + d_{N-5}^*q_{N-5} + d_{N-4}^*q_{N-4}\\
&\quad + d_{N-3}^*q_{N-3} + d_{N-2}^*q_{N-2} + d_{N-1}^*q_{N-1}),\ j = N-2,\\
v_{N-1} &= -a_N\gamma_1 + b_{N-1}h\delta_1 - h^4d_N(q_N - g_N\gamma_1) - h^4(d_{N-5}q_{N-5} + d_{N-4}q_{N-4}\\
&\quad + d_{N-3}q_{N-3} + d_{N-2}q_{N-2} + d_{N-1}q_{N-1}),\ j = N-1.
\end{aligned} \tag{5.6}$$

The form of vector W, we have

$$\begin{aligned}
w_1 &= -a_0\gamma_0 - b_1h\delta_0 - h^4d_0f(x_0,\gamma_0),\ j = 1,\\
w_2 &= -b_2h\delta_0,\ j = 2,\\
w_3 &= -e_1\gamma_0 + \frac{h^4}{6}p_1f(x_0,\gamma_0),\ j = 3,\\
w_j &= 0, \quad j = 4(1)(N-4),\\
w_{N-3} &= -e_1\gamma_1 + \frac{h^4}{6}p_1f(x_N,\gamma_1),\ j = N-3,\\
w_{N-2} &= b_{N-2}h\delta_1,\ j = N-2,\\
w_{N-1} &= -a_N\gamma_1 + b_{N-1}h\delta_1 - h^4d_Nf(x_N,\gamma_1),\ j = N-1.
\end{aligned} \tag{5.7}$$

We have the following cases :
(i) For second order methods the truncation error is $\| T \| = O(h^6)$. It follows that $\| E \| = O(h^2)$.
(ii) For fourth order methods the truncation error is $\| T \| = O(h^8)$. It follows that $\| E \| = O(h^4)$.
(iii) For sixth order method the truncation error is $\| T \| = O(h^{10})$. It follows that $\| E \| = O(h^6)$.
(iv) For eighth order method the truncation error is $\| T \| = O(h^{12})$. It follows that $\| E \| = O(h^8)$.

6 Numerical examples and results

To illustrate the applicability and efficiency of our method and also to compare our results with existing methods, we consider three numerical examples for linear and nonlinear fourth order two point boundary value problems. These examples are solved by the presented methods with step length $h = 2^{-m},\ m = 3,4,5$ and the maximum absolute errors in numerical solutions are listed in tables 1 to 7. We also compared our results with the methods in references [4-6,11,12,14,16,20] for second, fourth and sixth order methods. All calculations are performed in MATLAB 7.

Example 1: We consider the following linear fourth order boundary value problem [4-6,14,16,20,21]

$$u^{(4)} + xu = -(8 + 7x + x^3)e^x,\ \ 0 \le x \le 1,\ \ u(0) = 0, u(1) = 0, u'(0) = 0, u'(1) = -e. \tag{6.1}$$

The analytical solution for this example is

$$u(x) = x(1-x)e^x. \tag{6.2}$$

The maximum absolute errors for this example and comparison with other methods for second, fourth and sixth order methods are tabulated in tables 1-3.

Table 1. Maximum absolute errors for example 1

h	$1/8$	$1/16$	$1/32$
Our second order methods			
$(p_1, p_2, p_3, p_4) = (3/2, 0, 0, 33)$	2.07×10^{-4}	1.64×10^{-5}	5.24×10^{-6}
$(p_1, p_2, p_3, p_4) = (1, 0, 0, 34)$	3.48×10^{-4}	4.65×10^{-5}	1.08×10^{-5}
[4]	3.49×10^{-4}	8.59×10^{-5}	2.14×10^{-5}
[5]	3.49×10^{-4}	8.59×10^{-5}	2.14×10^{-5}
[14]	3.49×10^{-4}	8.59×10^{-5}	2.14×10^{-5}
[20]	5.38×10^{-4}	8.81×10^{-5}	1.64×10^{-5}
[6]	1.74×10^{-3}	4.32×10^{-4}	1.08×10^{-4}
[16]	2.09×10^{-2}	5.27×10^{-3}	1.33×10^{-3}

Table 2. Maximum absolute errors for example 1

h	$1/8$	$1/16$	$1/32$
Our fourth order methods			
$(p_1, p_2, p_3, p_4) = (-77/144, 269/64, 0, 8255/288)$	9.41×10^{-9}	3.82×10^{-9}	2.82×10^{-10}
$(p_1, p_2, p_3, p_4) = (-5/9, 17/4, 0, 515/18)$	9.93×10^{-8}	4.06×10^{-9}	2.25×10^{-10}
[4]	7.83×10^{-8}	5.16×10^{-9}	3.25×10^{-10}
[14]	7.83×10^{-8}	5.16×10^{-9}	3.25×10^{-10}
[6]	5.49×10^{-7}	2.83×10^{-8}	1.67×10^{-9}
[16]	2.70×10^{-5}	1.09×10^{-6}	4.54×10^{-8}

Table 3. Maximum absolute errors for example 1

h	$1/8$	$1/16$	$1/32$
Our sixth order methods			
$(p_1, p_2, p_3, p_4) = (1/5, -1/4, 56/5, 137/10)$	1.69×10^{-9}	1.31×10^{-9}	1.63×10^{-10}
$(p_1, p_2, p_3, p_4) = (61/300, -27/100, 45/4, 409/30)$	1.75×10^{-9}	1.31×10^{-9}	1.63×10^{-10}
[6]	2.47×10^{-9}	3.93×10^{-11}	3.25×10^{-13}
[11]	1.91×10^{-7}	3.12×10^{-9}	4.98×10^{-11}
[16]	3.11×10^{-7}	2.74×10^{-9}	2.13×10^{-11}

Example 2: We consider the following linear fourth order boundary value problem [4-6,12,14,16]

$$u^{(4)} - xu = -(11 + 9x + x^2 - x^3)e^x, \ -1 \le x \le 1,$$
$$u(-1) = 0, u(1) = 0, u'(-1) = 2e^{-1}, u'(1) = -2e. \tag{6.3}$$

The analytical solution for this example is

$$u(x) = (1 - x^2)e^x. \tag{6.4}$$

The maximum absolute errors for this example and comparison with other methods for second, fourth and sixth order methods are tabulated in tables 4-6.

Table 4. Maximum absolute errors for example 2

h	$1/8$	$1/16$	$1/32$
Our second order methods			
$(p_1, p_2, p_3, p_4) = (3/2, 0, 0, 33)$	6.73×10^{-4}	2.23×10^{-4}	5.64×10^{-5}
$(p_1, p_2, p_3, p_4) = (1, 0, 0, 34)$	1.98×10^{-3}	4.51×10^{-4}	1.12×10^{-4}
[4]	1.45×10^{-2}	3.58×10^{-3}	9.00×10^{-4}
[6]	4.9×10^{-2}	1.5×10^{-2}	4.3×10^{-3}
[12]	7.5×10^{-2}	1.9×10^{-2}	4.7×10^{-3}

Table 5. Maximum absolute errors for example 2

h	$1/8$	$1/16$	$1/32$
Our fourth order methods			
$(p_1, p_2, p_3, p_4) = (-77/144, 269/64, 0, 8255/288)$	4.45×10^{-7}	4.41×10^{-8}	2.99×10^{-9}
$(p_1, p_2, p_3, p_4) = (-5/9, 17/4, 0, 515/18)$	8.58×10^{-7}	4.06×10^{-8}	2.33×10^{-9}
[14]	8.52×10^{-7}	5.43×10^{-8}	3.41×10^{-9}
[4]	1.27×10^{-5}	8.25×10^{-7}	2.98×10^{-8}
[6]	5.85×10^{-5}	3.8×10^{-6}	2.5×10^{-7}
[12]	9.8×10^{-5}	5.0×10^{-6}	2.9×10^{-7}
[16]	1.43×10^{-3}	6.03×10^{-5}	2.69×10^{-6}

Table 6. Maximum absolute errors for example 2

h	$1/8$	$1/16$	$1/32$
Our sixth order methods			
$(p_1, p_2, p_3, p_4) = (1/5, -1/4, 56/5, 137/10)$	7.98×10^{-8}	4.13×10^{-9}	4.01×10^{-10}
$(p_1, p_2, p_3, p_4) = (61/300, -27/100, 45/4, 409/30)$	8.09×10^{-8}	4.15×10^{-9}	4.03×10^{-10}
[4]	3.31×10^{-7}	5.15×10^{-9}	8.10×10^{-11}
[12]	6.9×10^{-7}	2.2×10^{-8}	4.0×10^{-10}
[6]	8.2×10^{-7}	1.8×10^{-8}	3.5×10^{-10}
[16]	1.35×10^{-3}	1.40×10^{-5}	1.16×10^{-7}

Example 3: We consider the following nonlinear fourth order boundary value problem [8]

$$u^{(4)} = 6e^{-4u} - 12(1 + x)^{-4},\ 0 \le x \le 1,\ \ u(0) = 0, u(1) = ln2, u'(0) = 1, u'(1) = 0.5. \tag{6.5}$$

The analytical solution for this example is

$$u(x) = ln(1 + x). \tag{6.6}$$

The maximum absolute errors for this example is compared with [8] and tabulated in table 7.

Table 7. Maximum absolute errors for example 3

x	0.2	0.4	0.6	0.8
Our second order methods				
$(p_1, p_2, p_3, p_4) = (3/2, 0, 0, 33)$	3.59×10^{-5}	2.18×10^{-5}	8.09×10^{-6}	2.36×10^{-6}
$(p_1, p_2, p_3, p_4) = (1, 0, 0, 34)$	5.57×10^{-5}	6.42×10^{-5}	4.55×10^{-5}	1.68×10^{-5}
Our fourth order methods				
$(p_1, p_2, p_3, p_4) = (-77/144, 269/64, 0, 8255/288)$	1.96×10^{-7}	3.64×10^{-7}	2.95×10^{-7}	1.09×10^{-7}
$(p_1, p_2, p_3, p_4) = (-5/9, 17/4, 0, 515/18)$	2.67×10^{-7}	5.14×10^{-7}	4.24×10^{-7}	1.58×10^{-7}
Our sixth order methods				
$(p_1, p_2, p_3, p_4) = (1/5, -1/4, 56/5, 137/10)$	1.20×10^{-7}	8.83×10^{-8}	4.46×10^{-8}	1.70×10^{-8}
$(p_1, p_2, p_3, p_4) = (61/300, -27/100, 45/4, 409/30)$	1.22×10^{-7}	8.90×10^{-8}	4.48×10^{-8}	1.71×10^{-8}
[8]	7.59×10^{-7}	2.41×10^{-6}	4.91×10^{-6}	3.09×10^{-6}

Conclusion

Parametric septic spline method is developed for the approximate solution of fourth order two point boundary value problems. A class of methods are presented for solving such problems. This shows that our class of methods are better in the sense of accuracy and applicability. This have been verified by the maximum absolute errors given in tables 1-7.

References

[1] C. P. Katti. Five-diagonal sixth order methods for two-point boundary value problems involving fourth order differential equations. *Math. Comput.* 152(35)(1980): 1177–1179.

[2] G. Akram and S. S. Siddiqi. End conditions for interpolatory septic splines. *Int. J. Comput. Math.* 12(82)(2005): 1525–1540.

[3] I. V. O. Babuska et al. Numerical processes in differential equation. *Wiley (Interscience), New York*. (1966).

[4] J. Rashidinia and T. Aziz. Quintic spline solution of fourth order two-point boundary value problem. *Int. J. Appl. Sci. Comput.* 3(1997): 191–197.

[5] J. Rashidinia and A. Golbabaee. Convergence of numerical solution of a fourth order boundary value problem. *Appl. Math. Comput.* 171(2005): 1296–1305.

[6] J. Rashidinia and R. Jalilian. Nonpolynomial spline for solution of boundary value problems in plate deflection theory. *Int. J. Comput. Math.* 84(2007): 1483–1494.

[7] J. Rashidinia et al. Quintic spline solution of boundary value problem in plate deflection theory. *Comput. Sci. Eng. Elec. Eng.* 16(2009): 53–59.

[8] K. N. S. Kasi Viswanadham et al. Numerical solution of fourth order boundary value problems by Galerkin method with quintic B-splines. *Int. J. Nonlin. Sci.* 10(2010): 222–230.

[9] M. A. Ramadan et al. Quintic nonpolynomial spline solutions for fourth order two point boundary value problem. *Commun. Nonlin. Sci. Numer. Simul.* 14(2009): 1105–1114.

[10] M. Irodotou-Ellina and E. N. Houstis. An $O(h^6)$ quintic spline collocation method for fourth order two point boundary value problems. *BIT*. 28(1988): 288–301.

[11] M. K. Jain et al. Numerical solution of a fourth order ordinary differential equation. *J. Eng. Math.* 11(1977): 373–380.

[12] M. M. Chawla and C. P. Katti. Finite difference methods for two-point boundary value problems involving high order differential equations. *BIT*. 19(1979): 27–33.

[13] M. M. Chawla and R. Subramanian. High accuracy quintic spline solution of fourth order two-point boundary value problems. *Int. J. Comput. Math.* 31(1989): 87–94.

[14] R. A. Usmani. Discrete variable methods for a boundary value problem with engineering applications. *Math. Comput.* 32(1978): 1087–1096.

[15] R. A. Usmani. Smooth spline approximations for the solution of a boundary value problem with engineering applications. *J. Comput. Appl. Math.* 6(1980): 93–97.

[16] R. A. Usmani. Finite difference methods for a certain two-point boundary value problem. *Indian J. Pure Appl. Math.* 14(1983): 398–411.

[17] R. A. Usmani. The use of quartic splines in the numerical solution of a fourth-order boundary value problem. *J. Comput. Appl. Math.* 44(1992): 187–199.

[18] R. A. Usmani and M. J. Marsden. Numerical solution of some ordinary differential equations occurring in plate deflection theory. *J. Eng. Math.* 9(1975): 1–10.

[19] R. A. Usmani and S. A. Warsi. Smooth spline solutions for boundary value problems in plate deflection theory. *Comp. Maths. Appls.* 6(1980): 205–211.

[20] S. S. Siddiqi and G. Akram. Quintic spline solutions of fourth order boundary value problems. *Int. J. Numer. Anal. Model.* 5(1)(2005): 101–111.

[21] S. Timoshenko and S. W. Krieger. Theory of Plates and Shells. *New York: McGraw-Hill* (1959).