ISSN 1749-3889 (print), 1749-3897 (online)
**International Journal of Nonlinear Science**
Vol.21(2016) No.1,pp.60-64

# A Note On Topological Conjugacy For Perpetual Points

Awadhesh Prasad *
Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India
*(Received 28 May 2015, accepted 4 September 2015)*

**Abstract:** Recently a new class of critical points, termed as *perpetual points*, where acceleration becomes zero but the velocity remains non-zero, is observed in nonlinear dynamical systems. In this work we show whether a transformation also maps the perpetual points to another system or not. We establish mathematically that a linearly transformed system is topologicaly conjugate, and hence does map the perpetual points. However, for a nonlinear transformation, various other possibilities are also discussed. It is noticed that under a linear diffeomorphic transformation, perpetual points are mapped, and accordingly, eigenvalues are preserved.

**Keywords:** perpetual points; topological conjugacy

## 1 Introduction

Consider a general dynamical system specified by the equations

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}) \tag{1}$$

where $\mathbf{X} = (x_1, x_2, ..., x_n)^T \in \Sigma$ is $n$-dimensional vector of dynamical variables and $\mathbf{f} = (f_1(\mathbf{X}), f_2(\mathbf{X}), ...f_n(\mathbf{X}))^T$ specifies the evolution equations (velocity vector) of the system. Here $T$ stands for transpose of the vector. Acceleration of the system can be obtained by taking time derivative of Eq. (1), viz.,

$$\begin{aligned} \ddot{\mathbf{X}} &= D_{\mathbf{X}^T}\mathbf{f}(\mathbf{X}) \cdot \mathbf{f}(\mathbf{X}) \\ &= \mathbf{F}(\mathbf{X}) \end{aligned} \tag{2}$$

where $\mathbf{F} = D_{\mathbf{X}^T}\mathbf{f}(\mathbf{X}) \cdot \mathbf{f}(\mathbf{X})^\dagger$ is termed as acceleration vector.

The fixed points ($\mathbf{X}_{FP}$) of system, Eq. (1), are the ones where velocity and acceleration are simultaneously zero. Using linear stability analysis the dynamics near the fixed points is determined by the eigenvalues $\lambda = \{\lambda_1, \lambda_2, ...., \lambda_n\}$ of the Jacobian $D_{\mathbf{X}^T}\mathbf{f}(\mathbf{X})$. Similarly, the perpetual points ($\mathbf{X}_{PP}$) are the ones where acceleration is zero while velocity remains finite (nonzero). The eigenvalues, $\mu = \{\mu_1, \mu_2, ...., \mu_n\}$ of $D_{\mathbf{X}^T}\mathbf{F}(\mathbf{X})$ determine whether velocity is either extremum or inflection type. The understanding of motion around perpetual points is studied recently [1] where connection between $\lambda$ and $\mu$ are discussed along with its various applications [1, 2].

Note that an understanding of the topological conjugation between different systems is important in almost all branch of sciences. This, in fact, will suggest as to whether the behavior of one system is similar to that of the other or not. If there exists conjugacy between the systems then the systems behavior of one can immediately be derived from the knowledge of the system behavior of conjugate system. Usually, under certain conditions, such conjugate systems can be transformed into each other using an appropriate transformation. Suppose there is transformation, $\mathbf{h}(\mathbf{X})$, from variables $\mathbf{X}$ to $\mathbf{Y}$, i.e.,

$$\mathbf{Y} = \mathbf{h}(\mathbf{X}) \tag{3}$$

*Corresponding author. *E-mail address*: awadhesh@physics.du.ac.in


IJNS.2016.02.16/900

†Here $D_{\mathbf{X}}$ stands for derivative with respect to $\mathbf{X}$.

where $\mathbf{Y} = (y_1, y_2, ..., y_n)^T \in \Omega$ and $\mathbf{h} = (h_1(\mathbf{X}), h_2(\mathbf{X}), ....h_n(\mathbf{X}))^T$, then the transformed version of Eqs. (1) and (2) take the forms

$$\begin{aligned} \dot{\mathbf{Y}} &= \mathbf{g}(\mathbf{Y}) = \mathbf{g}(\mathbf{h}(\mathbf{X})) && (4)\\ \text{and} \quad \ddot{\mathbf{Y}} &= \mathbf{G}(\mathbf{Y}) = \mathbf{G}(\mathbf{h}(\mathbf{X})), && (5) \end{aligned}$$

respectively. Here $\mathbf{g}(\mathbf{Y})$ and $\mathbf{G}(\mathbf{Y})$ are the velocity and the acceleration vectors of the transformed system. The dynamics near the fixed points is very important to understand the system behavior. Rigorous studies have been done to understand the individual as well as conjugate systems around the fixed points. We recapitulate such works in Definition 2.1 and Theorem 2.1 mainly for the sake of completeness. The details can be found in any standard books on dynamical systems [3–7]. In the present work we show the topological conjugacy of perpetual points between the systems, Eqs. (2) and (5). The results are discussed and representative examples are presented in Sec. II. We summarize the results and highlight their importance in Sec. III

## 2 Results and Discussions

**Definition 1** *The dynamical systems $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$ and $\dot{\mathbf{Y}} = \mathbf{g}(\mathbf{Y})$, are topologically conjugate if there exists a homeomorphism* $\mathbf{h} : \Sigma \to \Omega$ for their corresponding flows $\phi_t(\mathbf{X}) \in R \times \Sigma$ and $\psi_t(\mathbf{Y}) \in R \times \Omega$, such that for each $\mathbf{X} \in \Sigma$ and $t \in R$*

$$\psi_t(\mathbf{h}(\mathbf{X})) = \mathbf{h}(\phi_t(\mathbf{X})). \quad (6)$$

*Alternatively, one can recast it as*

$$\begin{aligned} \psi \circ \mathbf{h} &= \mathbf{h} \circ \phi && (7)\\ \text{or} \quad \psi &= \mathbf{h} \circ \phi \circ \mathbf{h}^{-1} && (8) \end{aligned}$$

*where the composition symbol, $\circ$, means $\psi \circ \mathbf{h} = \psi_t(\mathbf{h}(\mathbf{X}))$, etc.*

**Theorem 1** *If the velocity vectors $\mathbf{f}(\mathbf{X})$ and $\mathbf{g}(\mathbf{Y})$ are topologically conjugate under $\mathbf{h}(\mathbf{X})$ then fixed points of $\mathbf{f}(\mathbf{X})$ are mapped to the fixed points of $\mathbf{g}(\mathbf{Y})$.*

**Proof.** From the Definition 1, the time derivative of Eq. (6) gives

$$\begin{aligned} \dot{\psi}_t(\mathbf{h}(\mathbf{X})) &= D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \dot{\phi}_t(\mathbf{X}) && (9)\\ i.e.\mathbf{g}(\mathbf{h}(\mathbf{X})) &= D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \mathbf{f}(\mathbf{X}). && (10) \end{aligned}$$

Here, $D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X})$ is a Jacobian of size $n \times n$. Since the velocity vector $\mathbf{f}(\mathbf{X}) = 0$ at the fixed points and hence Eq. (10) follows $\mathbf{g}(\mathbf{Y}) = 0$. Note that $\mathbf{g}(\mathbf{Y}) = 0$ is also possible for $D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = 0$ whether $\mathbf{f}(\mathbf{X}) \neq 0$ or $\mathbf{f}(\mathbf{X}) = 0$. This suggests that for the case of nonlinear $\mathbf{h}(\mathbf{X})$, particularly when the possibility of $D_{\mathbf{X}}\mathbf{h}(\mathbf{X}) = 0$ exists, some new fixed points may be created† in addition to that of the $\mathbf{f}(\mathbf{X}) = 0$. ■

**Theorem 2** *If the velocity vectors $\mathbf{f}(\mathbf{X})$ and $\mathbf{g}(\mathbf{Y})$ are topologically conjugate under homeomorphic* linear *$\mathbf{h}(\mathbf{X})$ then the perpetual points of $\mathbf{f}(\mathbf{X})$ are mapped to the perpetual points of $\mathbf{g}(\mathbf{Y})$.*

**Proof.** The time derivative of Eq. (10) gives

$$\begin{aligned} \ddot{\psi}_t(\mathbf{h}(\mathbf{X})) &= D^2_{t\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \dot{\phi}_t(\mathbf{X}) + D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \ddot{\phi}_t(\mathbf{X}) && (11)\\ i.e.\mathbf{G}(\mathbf{h}(\mathbf{X})) &= D^2_{t\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \mathbf{f}(\mathbf{X}) + D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \mathbf{F}(\mathbf{X}). && (12) \end{aligned}$$

Here $D^2_{t\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = (\mathbf{I} \otimes \dot{\phi}_t^T(\mathbf{X})) \cdot D^2_{\mathbf{X}\mathbf{X}^T}\mathbf{h}(\mathbf{X})$ where $\otimes$ is direct product while $\mathbf{I}$ and $D^2_{\mathbf{X}\mathbf{X}^T}\mathbf{h}(\mathbf{X})$ are identity and

* $\mathbf{h}(\mathbf{X})$ should be one-to-one, onto, continuous, and has its continuous inverse also.

† Due to nonlinear function, $\mathbf{h}(\mathbf{X})$, the order of polynomial of $\mathbf{Y}$ in $\mathbf{g}(\mathbf{Y})$ may be higher than that of the $\mathbf{X}$ in $\mathbf{f}(\mathbf{X})$.

Hessian matrices of dimensions $n \times n$ and $nn \times n$ respectively [1, 8]. If $\mathbf{h}(\mathbf{X})$ is linear then $D^2_{\mathbf{X}\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = 0$. Therefore, Eq. (12) can be recast as

$$\mathbf{G}(\mathbf{h}(\mathbf{X})) = D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \mathbf{F}(\mathbf{X}). \tag{13}$$

Since $D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \neq 0$ for linear $\mathbf{h}(\mathbf{X})$, therefore the perpetual points corresponding to $\mathbf{F}(\mathbf{X}) = 0$ are mapped to $\mathbf{G}(\mathbf{Y}) = 0$. This completes the proof. ∎

**Remark 3** *If $\mathbf{h}(\mathbf{X})$ is nonlinear then the term $D^2_{t\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot \mathbf{f}(\mathbf{X})$ may not be zero at perpetual points (where $\mathbf{f}(\mathbf{X}) \neq 0$). If this term is not zero then systems, $\mathbf{F}(\mathbf{X})$ and $\mathbf{G}(\mathbf{Y})$, are not topologically equivalent, and hence a set of new perpetual points may be created for transformed system, $\mathbf{G}(\mathbf{Y})$. Note that, for nonlinear $\mathbf{h}(\mathbf{X})$, a set of new perpetual points are also possible in transformed system if $D^2_{t\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = 0$ and $D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = 0$ for $\mathbf{F}(\mathbf{X}) \neq 0$. However if $D^2_{t\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = 0$, $D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = 0$ and $\mathbf{F}(\mathbf{X}) = 0$ then the both $\mathbf{G}(\mathbf{Y}) = 0$ and $\mathbf{g}(\mathbf{Y}) = 0$, and hence the perpetual points corresponding to $\mathbf{f}(\mathbf{X}) = 0$ get mapped to the new fixed points of $\mathbf{g}(\mathbf{Y})$.*

**Theorem 4** *If the velocity vectors $\mathbf{f}(\mathbf{X})$ and $\mathbf{g}(\mathbf{Y})$ are topologically conjugate under diffeomorphic* linear $\mathbf{h}$ then the eigenvalues $\lambda$ and $\mu$ for the respective fixed and perpetual points, corresponding to the velocity and acceleration vectors, are preserved.*

**Proof.** The derivative of Eq. (10) with respect to $\mathbf{X}$ gives

$$D_{\mathbf{h}^T}\mathbf{g}(\mathbf{h}(\mathbf{X})) \cdot D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = (\mathbf{I} \otimes \mathbf{f}^T(\mathbf{X})) \cdot D^2_{\mathbf{X}\mathbf{X}^T}\mathbf{h}(\mathbf{X}) + D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot D_{\mathbf{X}^T}\mathbf{f}(\mathbf{X}). \tag{14}$$

Here, $D_{\mathbf{h}^T}\mathbf{g}(\mathbf{h}(\mathbf{X})) = D_{\mathbf{Y}^T}\mathbf{g}(\mathbf{Y})$. At fixed point $\mathbf{f}(\mathbf{X}_{FP}) = 0$, hence

$$D_{\mathbf{Y}^T}\mathbf{g}(\mathbf{Y}) = D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot D_{\mathbf{X}^T}\mathbf{f}(\mathbf{X}) \cdot (D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}))^{-1}. \tag{15}$$

This shows that the set of eigenvalues $\lambda$ is same at the fixed points for the both flows.

Similarly, the derivative of Eq. (13) with respect to $\mathbf{X}$ gives

$$D_{\mathbf{Y}^T}\mathbf{G}(\mathbf{Y}) \cdot D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) = (\mathbf{I} \otimes \mathbf{F}^T(\mathbf{X})) \cdot D^2_{\mathbf{X}\mathbf{X}^T}\mathbf{h}(\mathbf{X}) + D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot D_{\mathbf{X}^T}\mathbf{F}(\mathbf{X}). \tag{16}$$

At perpetual points $\mathbf{F}(\mathbf{X}_{PP}) = 0$, hence

$$D_{\mathbf{Y}^T}\mathbf{G}(\mathbf{Y}) = D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}) \cdot D_{\mathbf{X}^T}\mathbf{F}(\mathbf{X}) \cdot (D_{\mathbf{X}^T}\mathbf{h}(\mathbf{X}))^{-1} \tag{17}$$

This shows that the set of eigenvalues $\mu$ is same at the perpetual points for the both flows. ∎

**Example 5** *In order to demonstrate the mapping and eigenvalues of/at perpetual points we consider a simple, analytically tractable, dynamical system*

$$\dot{x} = x^2 - A^2, \tag{18}$$

*where $A$ is a parameter. It has two fixed points, $x_{FP} = \pm A$, which are respectively stable and unstable with $\lambda = \pm 2A$ i.e., initial conditions $x(0) < A$ lead to $x_{FP} = -A$ otherwise settle at infinity. Its time derivative gives acceleration,*

$$\ddot{x} = 2x(x^2 - A^2), \tag{19}$$

*which gives one perpetual point, $x_{PP} = 0$. The velocity at this perpetual points is maximum, $A^2$, while the eigenvalue is $\mu = -2A^2$ [1].*

*Now let us transform this system using linear function,*

$$y = h(x) = \alpha x + \beta \tag{20}$$

*$\mathbf{h}(\mathbf{X})$ should be homeomorphic* as well as $\mathbf{h}$ and $\mathbf{h}^{-1}$ have continuous derivatives.

*where $\alpha$ and $\beta$ are parameters. The first and second order time derivatives give, velocity and acceleration for the transformed system, as*

$$\dot{y} = \frac{(y-\beta)^2 - \alpha^2 A^2}{\alpha} \tag{21}$$

$$\text{and} \quad \ddot{y} = \frac{2}{\alpha^2}(y-\beta)[(y-\beta)^2 - \alpha^2 A^2], \tag{22}$$

*respectively, The fixed and the perpetual points of this transformed system are $y_{FP} = \beta \pm \alpha A$ and $y_{PP} = \beta$ respectively. Note that these are the transformed points corresponding to Eqs. (18) and (19) under the transformation, Eq. (20). The eigenvalues $\lambda = \pm 2A$ and $\mu = -2A^2$ are the same as those of the Eqs. (18) and (19) respectively. Therefore under the linear transformation the fixed and perpetual points are mapped and the eigenvalues are preserved [cf. Theorems 1, 2 and 3].*

**Example 6** *Consider the same system as described by Eq. (18). However, with nonlinear transformation, say,*

$$y = h(x) = x^2 \tag{23}$$

*the velocity and acceleration for the transformed system become*

$$\dot{y} = 2\sqrt{(y)}(y - A^2) \tag{24}$$

$$\text{and} \quad \ddot{y} = 2(3y - A^2)(y - A^2). \tag{25}$$

*Note that the range of $y$ is $y \geq 0$. Its fixed points are $A^2$ and $0$ where the former is the mapping of fixed points of Eq. (18) while latter one is newly created. The eigenvalues, $\lambda$, for these fixed points are $\pm 2A$ and $\pm\infty$ respectively which depend on the sign of square root. The perpetual point is $A^2/3$ which preimage is not the $x_{PP} = 0$ of Eq. (18), and its eigenvalues, $\mu$ is $-4A^2$. These suggest that due to nonlinear transformation new fixed or perpetual points points may be created [cf. Remarks 1 and 2].*

# 3 Summary

In this work we established the theorems for the topological conjugacy of the perpetual points under linear transformation. The eigenvalues also get preserved under diffeomorphic linear transformation. However if the transformation is nonlinear then this topological feature of the dynamical systems remains restricted. Examples corresponding to the linear and nonlinear transformations are presented to confirm the conjugacy.

The concept of topological conjugacy is very important for comparing dynamical systems which are structurally different (of various mathematical models) but have similar (equivalent) dynamics. This becomes even more important for nonlinear systems which are not solvable analytically. For such systems global dynamical behavior are normally guessed/analyzed from its linearized version, e.g., understanding the global dynamical behavior of the full system is determined by assembling the nearby (local) motions around all the fixed points present in the system. Note that, as the motion at fixed points is stationary while extremum (or inflection type) velocity at perpetual points is nonzero, the understanding of the dynamics of oscillating systems near perpetual points are necessary to understand the full system. Therefore, the results of topological conjugacy presented in this paper are of immediate use for understanding the global dynamics of the nonlinear systems.

It may be mentioned that many problems/theorems, similar to those studied in the context of fixed points, now need extension for the studies of perpetual points e.g. Hartman-Grobman theorem, $C^k$-conjugacy etc.

# Acknowledgments

Author thanks, R. S. Kaushal, M. D. Shrimali, B. Biswal, N. Kuznetsov, and R. Ramaswamy for critical comments on this manuscript. The financial supports from the DST, Govt. of India and Delhi University Research & Development Grant are gratefully acknowledged.

# References

[1] A. Prasad. Existence of perpetual points in nonlinear dynamical systems and its applications. *Int. J. Bif. Chaos*. 25(2015): 5–10.

[2] D. Dudkowski et al. Perpetual points and hidden attractors in dynamical systems. *Phys. Lett. A*. 379(2015): 2591–2596.

[3] D. K. Arrowsmith and C. M. Place. An Introduction to Dynamical Systems. *Cambridge University Press, Cambridge.* (1990).

[4] J. Guckenheimer and P. Holes. Booklet title. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields *Springer-Verlag, New York.* (1985).

[5] Y. Kuznetsov. Elements of Applied Bifurcation Theory. *Springer, New York.* (2004).

[6] J. D. Meiss. Differential Dynamical Systems. *SIAM, Philadelphia.* (2007).

[7] S. Wiggins. Introduction to Applied Nonlinear Dynamical Systems and Chaos. *Springer-Verlag, New York.* (1990).

[8] J. R. Magnus and H. Neudecker. Matrix Differential Calculus. *John Wiley & Sons, England.* (1999).