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Discrete Control Systems of Fractional Order

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Abstract: The present paper analyzes fractional order discrete time control system using a new transform method. The comparison for unit step response of a fractional order control system in continuous and discrete time domains is presented.
Keywords: Fractional order; control system; transfer function; unit step response; time domain

1 Fractional Order Control

Fractional calculus is a new branch of mathematics that deals with derivatives and integrals of arbitrary orders. It is a powerful tool in the description of memory and hereditary properties of different phenomena. Many scientists have paid a lot of attention due to its interesting applications in various fields of science and engineering, such as viscoelasticity, diffusion, neurology, control theory and statistics [16].

The most frequently used definitions for fractional derivative are the Grünwald-Letnikov (GL), the Riemann-Liouville (RL) and the Caputo notions.

Definition 1 *[16] Let f be piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval of $[0,\infty)$, m is a positive integer and α is a real number such that $m-1<\alpha\leq m$. For $t\geq 0$, the α^{th}-order*

1. *Riemann-Liouville type fractional derivative of f is defined by*

$$({}^{RL}D^{\alpha}f)(t)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t\frac{f(\xi)}{(t-\xi)^{\alpha-m+1}}d\xi. \tag{1}$$

2. *Grünwald-Letnikov type fractional derivative of f is defined by*

$$({}^{GL}D^{\alpha}f)(t)=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{[\frac{t}{h}]}(-1)^j\binom{\alpha}{j}f(t-jh). \tag{2}$$

3. *Caputo type fractional derivative of f is defined by*

$$({}^{C}D^{\alpha}f)(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^{(m)}(\xi)}{(t-\xi)^{\alpha-m+1}}d\xi. \tag{3}$$

Remark 1 *For a wide class of functions, the above three definitions are equivalent [16]. In particular,*

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1. *Riemann-Liouville and Caputo derivatives are equivalent under homogeneous initial conditions.*
2. *Grünwald-Letnikov and Riemann-Liouville derivatives are equivalent for a class of functions having* $(m-1)$ *continuous derivatives.*

The use of fractional order derivatives and integrals in control theory shows better results than integer order approaches. Oustaloup [15] initiated the study of fractional order control systems. He demonstrated that fractional order controllers outperform their integer order counterparts [16].

Now we consider a simple fractional order control system with a single input $u(t)$ and a single output $y(t)$ [16] as shown in Figure 1.

Figure 1: Simple Control System

It can be described by a linear non-homogeneous fractional differential equation

$$a_p(D^{\alpha_p}y)(t)+a_{p-1}(D^{\alpha_{p-1}}y)(t)+\cdots+a_1(D^{\alpha_1}y)(t)+a_0(D^{\alpha_0}y)(t)=u(t) \tag{4}$$

where D^α denotes the α^{th}-order Riemann-Liouville or Caputo derivative, a_i $(i=0,1,2,\cdots,p)$ are constants and α_i $(i=0,1,2,\cdots,p)$ are rational numbers. With out loss of generality we may assume that $m-1<\alpha_p\leq m$, $\alpha_p>\alpha_{p-1}>\cdots>\alpha_1>\alpha_0>0$, $\alpha_i-\alpha_{i-1}\leq 1$ for all $i=1,2,\cdots,p$ and $0<\alpha_0\leq 1$.

The Laplace transform [16] of $(D^\alpha y)(t)$ is given by

$$L[(D^\alpha y)(t)](s)=\int_0^\infty e^{-st}(D^\alpha y)(t)dt=s^\alpha L[y(t)](s)-\sum_{j=0}^{m-1}s^{\alpha-k-1}y^{(k)}(0). \tag{5}$$

Under homogeneous initial conditions, the transfer function of the corresponding fractional order system (4) is given by

$$G(s)=\frac{1}{a_ps^{\alpha_p}+a_{p-1}s^{\alpha_{p-1}}+\cdots+a_1s^{\alpha_1}+a_0s^{\alpha_0}}. \tag{6}$$

2 Discrete Fractional Nabla Calculus

The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is new. The analogous theory for discrete fractional nabla calculus was initiated and properties of the theory of fractional sums and differences were established [6–8, 11, 13, 14, 18].

Here we introduce basic definitions and results concerning discrete fractional nabla calculus. Let $h>0$ be a real number and $t_n=nh$, $n\in\mathbb{N}_0^+$ be the mesh points, where $\mathbb{N}_0^+=\{0,1,2,\cdots\}$. Assume that f is a function defined on this mesh and put $f_n=f(t_n)$. The first order backward h-difference of $f(t_n)$ is defined by

$$(\nabla_h f)(t_n)=\frac{f_n-f_{n-1}}{h},\quad n=1,2,\cdots \tag{7}$$

Definition 2 *[2] The extended binomial coefficient* $\binom{t}{n}$, *(*$t\in\mathbb{R}$, $n\in\mathbb{Z}$*) is given by*

$$\binom{t}{n}=\begin{cases}\frac{\Gamma(t+1)}{\Gamma(t-n+1)\Gamma(n+1)} & n>0,\\ 1 & n=0,\\ 0 & n<0.\end{cases} \tag{8}$$

Lemma 2 *[16] For any a, $b \in \mathbb{R}$, the quotient expansion of two gamma functions at infinityis is given by*

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} = t^{a-b}\Big[1 + O\Big(\frac{1}{t}\Big)\Big],\ (|t| \to \infty).$$

Definition 3 *[2] Let $\alpha \in \mathbb{R}^+$. The α^{th}-order fractional nabla sum of $f(t_n)$ is defined by*

$$(\nabla_h^{-\alpha} f)(t_n) = h^{\alpha} \sum_{j=0}^{n-1} (-1)^j \binom{-\alpha}{j} f(t_{n-j}). \tag{9}$$

Using fractional nabla sum, fractional nabla difference can be achieved as follows.

Definition 4 *[2] Let $\alpha \in \mathbb{R}^+$ and m be a positive integer such that $m-1 < \alpha \le m$. The α^{th}-order Riemann-Liouville and Caputo type fractional nabla differences of $f(t_n)$ are defined by*

$$({}^{RL}\nabla_h^{\alpha} f)(t_n) = (\nabla_h^m \nabla_h^{-(m-\alpha)} f)(t_n) \tag{10}$$

and

$$({}^{C}\nabla_h^{\alpha} f)(t_n) = (\nabla_h^{-(m-\alpha)} \nabla_h^m f)(t_n) \tag{11}$$

respectively. For $\alpha = 0$, we set $({}^{RL}\nabla_h^{\alpha} f)(t_n) = ({}^{C}\nabla_h^{\alpha} f)(t_n) = f(t_n)$.

Theorem 3 *The equivalent forms of (10) and (11) are*

$$({}^{RL}\nabla_h^{\alpha} f)(t_n) = h^{-\alpha} \sum_{j=0}^{n-1} (-1)^j \binom{\alpha}{j} f_{n-j} \tag{12}$$

and

$$({}^{C}\nabla_h^{\alpha} f)(t_n) = h^{-\alpha} \sum_{j=0}^{n-1} (-1)^j \binom{\alpha}{j} f_{n-j} + \sum_{k=0}^{m-1} h^{k-\alpha} (-1)^n \binom{\alpha-k-1}{n-1} [\nabla_h^k f_n]_{n=0} \tag{13}$$

respectively.

Proof. Consider the Riemann-Liouville type fractional nabla difference for $0 < \alpha < 1$. We have

$$\begin{aligned}
({}^{RL}\nabla_h^{\alpha} f)(t_n) &= (\nabla_h \nabla_h^{-(1-\alpha)} f)(t_n) \\
&= \nabla_h \Big[h^{1-\alpha} \sum_{j=0}^{n-1} (-1)^j \binom{\alpha-1}{j} f(t_{n-j})\Big] \\
&= h^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-1}{j} (-1)^j f_{n-j} - h^{-\alpha} \sum_{j=0}^{n-2} (-1)^j \binom{\alpha-1}{j} f_{n-j-1} \\
&= h^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-1}{j} (-1)^j f_{n-j} + h^{-\alpha} \sum_{j=1}^{n-1} (-1)^j \binom{\alpha-1}{j-1} f_{n-j} \\
&= h^{-\alpha} f_n + h^{-\alpha} \sum_{j=1}^{n-1} (-1)^j \Big[\binom{\alpha-1}{j} + \binom{\alpha-1}{j-1}\Big] f_{n-j} \\
&= h^{-\alpha} \sum_{j=0}^{n-1} (-1)^j \binom{\alpha}{j} f_{n-j}.
\end{aligned} \tag{14}$$

Now we assume that $1 < \alpha < 2$. Using (14), we have

$$\begin{aligned}({}^{RL}\nabla_h^{\alpha}f)(t_n) &= (\nabla_h^2\nabla_h^{-(2-\alpha)}f)(t_n)\\ &= \nabla_h\Big[\nabla_h\Big[h^{2-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-2}{j}f(t_{n-j})\Big]\Big]\\ &= \nabla_h\Big[h^{1-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}f(t_{n-j})\Big]\\ &= h^{-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha}{j}f_{n-j}.\end{aligned}$$

Proceeding in a similar way we can prove (12). Consider the Caputo type fractional nabla difference for $0 < \alpha < 1$. We have

$$\begin{aligned}({}^{C}\nabla_h^{\alpha}f)(t_n) &= (\nabla_h^{-(1-\alpha)}\nabla_h f)(t_n)\\ &= h^{1-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}(\nabla_h f)(t_{n-j})\\ &= h^{1-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}\Big[\frac{f_{n-j}-f_{n-j-1}}{h}\Big]\\ &= h^{-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}f_{n-j} - h^{-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}f_{n-j-1}\\ &= h^{-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}f_{n-j} + h^{-\alpha}\sum_{j=1}^{n}(-1)^j\binom{\alpha-1}{j-1}f_{n-j}\\ &= h^{-\alpha}f_n + h^{-\alpha}\sum_{j=1}^{n-1}(-1)^j\Big[\binom{\alpha-1}{j}+\binom{\alpha-1}{j-1}\Big]f_{n-j} + h^{-\alpha}\binom{\alpha-1}{n-1}(-1)^n f_0\\ &= h^{-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha}{j}f_{n-j} + h^{-\alpha}\binom{\alpha-1}{n-1}(-1)^n f_0. \qquad (15)\end{aligned}$$

Suppose $1 < \alpha < 2$. Using (15), we get

$$\begin{aligned}({}^{C}\nabla_h^{\alpha}f)(t_n) &= (\nabla_h^{-(2-\alpha)}\nabla_h^2 f)(t_n)\\ &= h^{2-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-2}{j}(\nabla_h(\nabla_h f))(t_{n-j})\\ &= h^{1-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha-1}{j}(\nabla_h f)(t_{n-j}) + h^{1-\alpha}\binom{\alpha-2}{n-1}(-1)^n[\nabla_h f_n]_{n=0}\\ &= h^{-\alpha}\sum_{j=0}^{n-1}(-1)^j\binom{\alpha}{j}f_{n-j} + h^{-\alpha}\binom{\alpha-1}{n-1}(-1)^n f_0 + h^{1-\alpha}\binom{\alpha-2}{n-1}(-1)^n[\nabla_h f_n]_{n=0}.\end{aligned}$$

Proceeding in a similar way we can prove (13). ■

Remark 4 *We observe that, under homogeneous initial conditions, both Riemann-Liouville and Caputo differences are equivalent. Consequently,*

$$(D^{\alpha}f)(t) = \lim_{h\to 0}({}^{RL}\nabla_h^{\alpha}f)(t_n) = \lim_{h\to 0}({}^{C}\nabla_h^{\alpha}f)(t_n). \qquad (16)$$

3 Discrete Fractional Order Control

In this section, we consider a fractional order digital control system obtained by discretizing the continuous time domain of a fractional order control system, based on finite difference approximation of fractional derivatives obtained in (16).

Analogous to (4), the corresponding fractional order digital control system can be described by a linear non-homogeneous fractional nabla difference equation

$$[a_p \nabla_h^{\alpha_p} + a_{p-1} \nabla_h^{\alpha_{p-1}} + \cdots + a_1 \nabla_h^{\alpha_1} + a_0 \nabla_h^{\alpha_0}] y_n = u_n \tag{17}$$

with a single input signal u_n and a single output signal y_n. Here ∇_h^{α} denotes the α^{th}-order Riemann-Liouville or Caputo fractional nabla difference, a_i ($i = 0, 1, 2, \cdots, p$) are constants and α_i ($i = 0, 1, 2, \cdots, p$) are rational numbers. With out loss of generality we may assume that $m - 1 < \alpha_p \leq m$, $\alpha_p > \alpha_{p-1} > \cdots > \alpha_1 > \alpha_0 > 0$, $\alpha_i - \alpha_{i-1} \leq 1$ for all $i = 1, 2, \cdots, p$ and $0 < \alpha_0 \leq 1$.

The analysis of a discrete fractional order control system or simply fractional order digital control system using Z-transforms can be found in the literature [12, 17, 19]. In the present article, we analyze the same using a discrete Laplace transform (N-transform) which is the Laplace transform for the fractional nabla difference on the time scale of integers [3, 4].

Definition 5 *[7, 11, 14] For a function $f_n : \mathbb{N}_0^+ \to \mathbb{R}$, the N-transform of f_n is defined by*

$$N[f_n] = \sum_{j=1}^{\infty} (1 - z)^{j-1} f_j = F(z), \tag{18}$$

for each $z \in \mathbb{C}$ for which the series converges.

Definition 6 *[11] A function f_n is of exponential order r, $r > 0$ if there exists a constant $A > 0$ such that $|f_n| \leq Ar^{-n}$ for sufficiently large n.*

The following lemma discusses the convergence of N-transform.

Lemma 5 *[11] Suppose f_n is of exponential order r, $r > 0$. Then $N[f_n]$ exists for all $z \in \mathbb{C}$ such that $|\frac{1-z}{1-r}| < 1$.*

Theorem 6 *[14] Let f_n is of exponential order r, $r > 0$ and $k \in \mathbb{N}_0^+$. Then, for all $z \in \mathbb{C}$ such that $|\frac{1-z}{1-r}| < 1$,*

1. (Shifting Theorem) $N[f_{n-k}] = (1 - z)^k N[f_n]$.
2. (Shifting Theorem) $N[f_{n+k}] = (1 - z)^{-k} \Big[N[f_n] - f_1 - (1 - z)^1 f_2 - \cdots - (1 - z)^{k-1} f_k \Big]$.
3. $N\Big[\nabla_h^{-\alpha} f_n\Big] = h^{\alpha} z^{-\alpha} N[f_n]$.
4. Under zero initial conditions, $N\Big[{}^{RL}\nabla_h^{\alpha} f_n\Big] = N\Big[{}^{C}\nabla_h^{\alpha} f_n\Big] = h^{-\alpha} z^{\alpha} N[f_n]$.
5. $N\Big[\binom{n+a-2}{n-1}\Big] = \frac{1}{z^a}$.

Taking N-transform on both sides of (17), under the assumption of zero initial conditions, we obtain the transfer function of the corresponding fractional order digital control system as

$$G(z) = \frac{1}{a_p z^{\alpha_p} + a_{p-1} z^{\alpha_{p-1}} + \cdots + a_1 z^{\alpha_1} + a_0 z^{\alpha_0}}. \tag{19}$$

4 Examples

Now, we illustrate the application of N-transform method to obtain unit step responses of some fractional order digital control systems.

The unit step sequence is defined as

$$\mu(n) = \begin{cases} 1, & \text{for } n > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The N-Transform is

$$N[\mu(n)] = \sum_{j=1}^{\infty}(1-z)^{j-1}\mu(j) = \sum_{j=1}^{\infty}(1-z)^{j-1} = \frac{1}{z}. \tag{20}$$

Example 7 *Consider a digital control system described by the fractional nabla difference equation*

$$a\nabla_h^{\alpha} y_n = u_n, \quad 0 < \alpha \leq 1, \tag{21}$$

with zero initial conditions.

Applying N-Transform on both sides of (21), we get

$$\begin{aligned} ah^{-\alpha}z^{\alpha}Y(z) &= U(z) \\ \text{or } \frac{Y(z)}{U(z)} &= \frac{1}{a}h^{\alpha}z^{-\alpha}. \end{aligned} \tag{22}$$

Letting $U(z) = \frac{1}{z}$, we obtain

$$Y(z) = \frac{1}{a}h^{\alpha}z^{-\alpha-1}.$$

Applying inverse N-transform on both sides, we get

$$y(n) = \frac{1}{a}h^{\alpha}N^{-1}[z^{-\alpha-1}] = \frac{1}{a}h^{\alpha}\binom{n+\alpha-1}{n-1} \tag{23}$$

as the unit step response of the system.

Example 8 *Consider a discrete time control system described by the fractional nabla difference equation*

$$[a_1\nabla_h^{\alpha_1} + a_0\nabla_h^{\alpha_0}]y_n = u_n, \quad 0 < \alpha_0 < \alpha_1 \leq 1, \tag{24}$$

with zero initial conditions.

Applying N-Transform on both sides of (24), we get

$$\begin{aligned} [a_1h^{-\alpha_1}z^{\alpha_1} + a_0h^{-\alpha_0}z^{\alpha_0}]Y(z) &= U(z) \\ \text{or } \frac{Y(z)}{U(z)} &= = \frac{1}{a_1}h^{\alpha_1}\frac{1}{z^{\alpha_1}[1+\frac{a_0}{a_1}h^{(\alpha_1-\alpha_0)}z^{-(\alpha_1-\alpha_0)}]}. \end{aligned} \tag{25}$$

Letting $U(z) = \frac{1}{z}$, we obtain

$$\begin{aligned} Y(z) &= \frac{1}{a_1}h^{\alpha_1}z^{-(1+\alpha_1)}\Big[1+\frac{a_0}{a_1}h^{(\alpha_1-\alpha_0)}z^{-(\alpha_1-\alpha_0)}\Big]^{-1} \\ &= \frac{1}{a_1}\sum_{j=0}^{\infty}(-1)^j\Big[\frac{a_0}{a_1}\Big]^j\Big[h^{j(\alpha_1-\alpha_0)+\alpha_1}\Big]\Big[z^{-(j(\alpha_1-\alpha_0)+(1+\alpha_1))}\Big]. \end{aligned}$$

Applying inverse N-transform on both sides, we get

$$\begin{aligned} y_n &= \frac{1}{a_1}\sum_{j=0}^{\infty}(-1)^j\Big[\frac{a_0}{a_1}\Big]^j\Big[h^{j(\alpha_1-\alpha_0)+\alpha_1}\Big]N^{-1}\Big[z^{-(j(\alpha_1-\alpha_0)+(1+\alpha_1))}\Big] \\ &= \frac{1}{a_1}\sum_{j=0}^{\infty}(-1)^j\Big[\frac{a_0}{a_1}\Big]^j\Big[h^{j(\alpha_1-\alpha_0)+\alpha_1}\Big]\binom{n+j(\alpha_1-\alpha_0)+\alpha_1-1}{n-1} \end{aligned} \tag{26}$$

as the unit step response of the system.

5 Conclusion

Finally, we compare unit step responses of a fractional order control system both in continuous and discrete time domains.

Example 9 *[16] Consider a simple continuous time control system described by a two term fractional differential equation with zero initial condition as*

$$a(D^{\alpha}y)(t) + by(t) = u(t). \tag{27}$$

Using Laplace transform method [16], the unit step response of (27) in time domain is

$$y(t) = \frac{1}{a}t^{\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k\frac{t^{k\alpha}}{\Gamma(k\alpha+\alpha+1)}. \tag{28}$$

Now we consider the corresponding discrete time control system governed by a fractional nabla difference equation with zero initial condition

$$a\nabla_h^{\alpha}y_n + by_n = u_n. \tag{29}$$

Using (26), the unit step response of (29) in time domain is

$$y_n = \frac{1}{a}\sum_{k=0}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k h^{k\alpha+\alpha}\binom{n+k\alpha+\alpha-1}{n-1}. \tag{30}$$

Remark 10 *In the limit of $h \to 0$, $n \to \infty$ with $t = nh$ fixed, the solution y_n obtained in (30) converges to the solution $y(t)$ obtained in (28).*

The proof follows from the quotient expansion of two gamma functions at infinityis. Consider

$$\begin{aligned} y_n &= \frac{1}{a}\sum_{k=0}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k h^{k\alpha+\alpha}\binom{n+k\alpha+\alpha-1}{n-1} \qquad (31)\\ &= \frac{1}{a}\sum_{k=0}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k h^{k\alpha+\alpha}\frac{\Gamma(n+k\alpha+\alpha)}{\Gamma(k\alpha+\alpha+1)\Gamma(n)}\\ &= \frac{1}{a}\sum_{k=0}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k h^{k\alpha+\alpha}\frac{n^{k\alpha+\alpha}}{\Gamma(k\alpha+\alpha+1)}\\ &= \frac{1}{a}t^{\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\frac{b}{a}\right)^k\frac{t^{k\alpha}}{\Gamma(k\alpha+\alpha+1)}\\ &= y(t). \end{aligned}$$

The comparison of unit step responses of discrete and continuous time systems for different values of h and t are presented in Figures 2 - 4.

6 Conclusion

We conclude that the unit step responses of a fractional order control system can be approximated by a digital control system of fractional order.

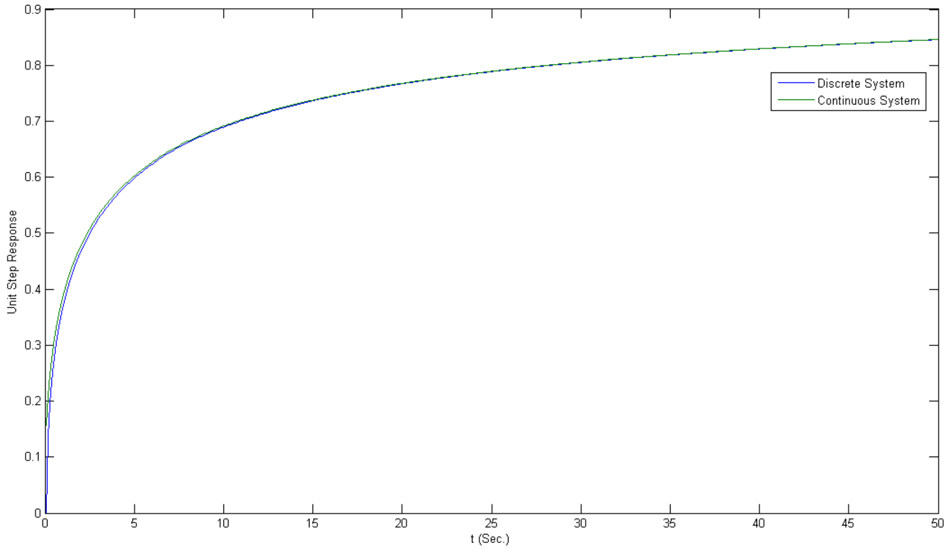

Figure 2: Calculations for h = 0.1 and t = 50

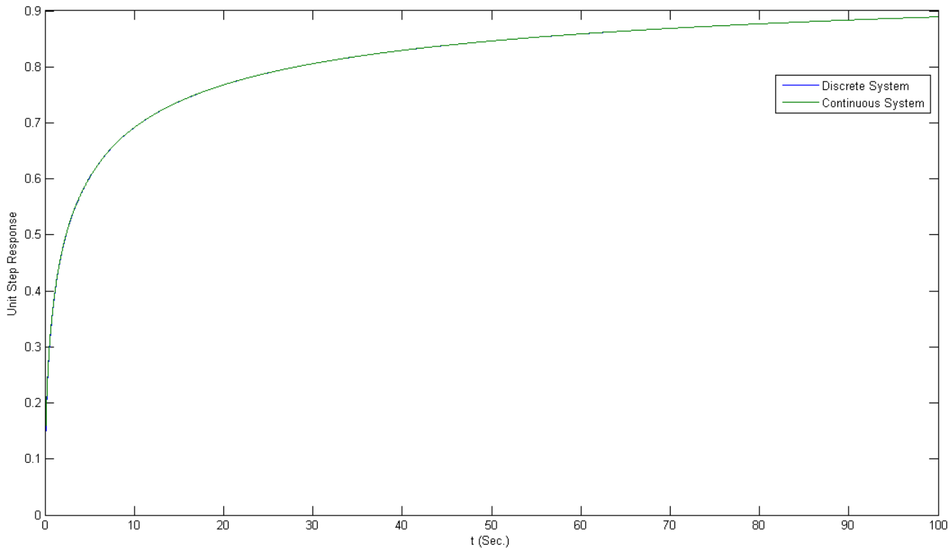

Figure 3: Calculations for h = 0.01 and t = 100

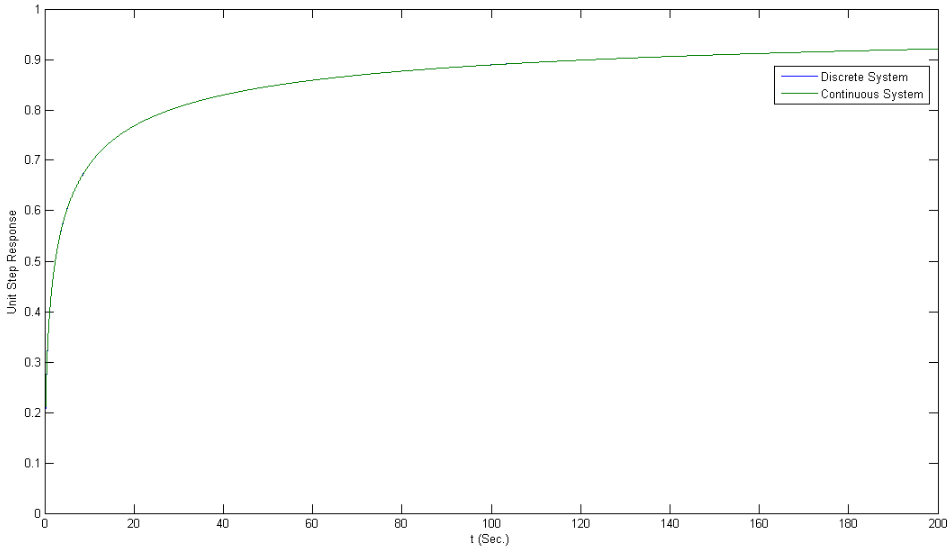

Figure 4: Calculations for h = 0.005 and t = 200

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