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Fractal Interpolation Surfaces and Perturbations on Vertical Scaling Factors

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Abstract: A hyperbolic iterated function system (IFS) whose attractor is the graph of a continuous function $f : [a,\ b] \times [c,\ d] \to \mathbb{R}$ that interpolates the given data in $\mathbb{R}^3$ is considered. The function f is called the bivariate fractal interpolation function (FIF) arising from the IFS and the graph of f which is a surface in $\mathbb{R}^3$ is called the bivariate fractal interpolation surface (FIS). By introducing a small perturbation in the vertical scaling factors that are involved in the construction of the IFS, a perturbed iterated function system in $\mathbb{R}^3$ is obtained. Due to perturbation in the vertical scaling factors, there is a change in the FIF. In this paper, the change in the FIF due to the change in the scaling factors of the IFS is primarily studied. An upper bound for the error estimation between the original FIF and the perturbed FIF is also found.

Keywords: Fractal interpolation surface; Bivariate iterated function system; vertical scaling factor; perturbation; error estimation.

1 Introduction

The concept of fractal interpolation function (FIF) which is a special type of continuous function that interpolates the given data was introduced by Barnsley [1]. The construction of a FIF is based on the Read-Bajraktarević operator acting on suitable function spaces and the graph of the FIF is the attractor of an iterated function system (IFS) consisting of contraction mappings. The FIF is widely applied in various areas of science and engineering. For example, the fractal interpolation surfaces which are the graphs of the fractal interpolation functions, are used in rendering rough surfaces in computer graphics in more realistic manner.

Dalla [2] gave a method of construction of bivariate fractal interpolation function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ that interpolates given points $\{\ (x_0,\ y_0,\ z_{00}),\ \cdots,\ (x_N,\ y_M,\ z_{NM})\ \}$ in $\mathbb{R}^3$. The graph of f is known as the bivariate fractal interpolation surface (FIS) which is the attractor of the bivariate IFS.

For more flexible fractal interpolation functions Chand and Kapoor [15, 16] constructed coalescence hidden variable fractal interpolation functions (CHFIFs) and its smoothness is analyzed in [8]. Using operator approximation technique, Kapoor and Prasad [9, 10] studied smoothness and stability of coalescence hidden variable fractal interpolation surfaces (CHFISs).

Using variable scaling functions, a new class of IFS has been introduced and a more general FIF is constructed in [11]. Analytical properties of bivariate fractal interpolation functions are studied in [12] as a generalization of [13]. A new method of construction of the FISs with function vertical scaling factors on a rectangular domain can be found in [14].

H-Y Wang and coworkers studied the perturbation and error analysis for bivariate FIFs by introducing perturbation functions in the variables x and y in [6] and for one variable FIFs in [5]. On the other hand there are studies on how the vertical scaling factor affects the bounds of the affine fractal interpolation function [3] and the bounds of the attractor of the iterated function system with double vertical scaling factor [4]. Xu and Feng [7] studied the problem of perturbing the vertical scaling factors of one variable FIF and gave a condition under which the perturbed IFS meet the fractal interpolation continuous condition.

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In the present paper, we study the problem of perturbing the vertical scaling factor of bivariate FIFs. The conditions for which the new perturbed IFS to satisfy the continuous condition are obtained. Finally, an upper bound for the error estimation between the two FIFs are also given.

Let a set of data point $T = \{(x_i,\ y_j,\ z_{ij})\ :\ i = 0,\ 1,\ \ldots,\ N;\ \ j = 0,\ 1,\ \ldots,\ M\}$ be given in $\mathbb{R}^3$ with $N > 1$ and $M > 1$. Let $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ and $c = y_0 < y_1 < y_2 < \cdots < y_M = d$. Set $I = [a,\ b]$, $J = [c,\ d]$, $I_i = [x_{i-1},\ x_i]$, $J_j = [y_{j-1},\ y_j]$ and $D_{ij} = I_i \times J_j$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Assume that the interpolation points are such that each of the sets

$$\begin{aligned} \{(x_0,\ y_j,\ z_{0j}) \ &:\ j = 0,\ 1,\ \ldots,\ M\} \\ \{(x_N,\ y_j,\ z_{Nj}) \ &:\ j = 0,\ 1,\ \ldots,\ M\} \\ \{(x_i,\ y_0,\ z_{i0}) \ &:\ i = 0,\ 1,\ \ldots,\ N\} \\ \{(x_i,\ y_M,\ z_{iM}) \ &:\ i = 0,\ 1,\ \ldots,\ N\} \end{aligned}$$

is colinear.

Let $D = I \times J$ and define $w_{ij} : D \times \mathbb{R} \to \mathbb{R}^3$ by

$$w_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_i x + b_i \\ c_j y + d_j \\ e_{ij}x + f_{ij}y + g_{ij}xy + s_{ij}z + k_{ij} \end{pmatrix} = \begin{pmatrix} u_i(x) \\ v_j(y) \\ F_{ij}(x, y, z) \end{pmatrix} \tag{1}$$

where the constants a_i, b_i, c_j, d_j, e_{ij}, f_{ij}, g_{ij}, k_{ij} are defined by the equations

$$w_{ij}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{j-1} \\ z_{(i-1)(j-1)} \end{pmatrix},\ w_{ij}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = \begin{pmatrix} x_i \\ y_{j-1} \\ z_{i(j-1)} \end{pmatrix},$$

$$w_{ij}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_j \\ z_{(i-1)j} \end{pmatrix},\ w_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = \begin{pmatrix} x_i \\ y_j \\ z_{ij} \end{pmatrix}.$$

Set $\phi_{ij}(x,\ y) = e_{ij}x + f_{ij}y + g_{ij}xy + k_{ij}$ for $(x,\ y) \in D$. Then, $F_{ij}(x,\ y,\ z) = s_{ij}z + \phi_{ij}(x,\ y)$ for $(x,\ y,\ z) \in D \times \mathbb{R}$. The functions ϕ_{ij}'s are bivariate Lipschitz functions. The constants s_{ij}'s are called the vertical scaling factors. Let $0 < |s_{ij}| < 1$. Then

$$\{D \times \mathbb{R},\ w_{ij} \equiv (u_i,\ v_j,\ F_{ij})\ :\ i = 1,\ \ldots,\ N;\ \ j = 1,\ \ldots,\ M\} \tag{2}$$

constitute an IFS.

The IFS has a unique attractor G which is the graph of a continuous function $f : D \to \mathbb{R}$ that interpolates the given data [2]. The function f is called the fractal interpolation function (FIF) for the IFS (2), and f satisfies the fixed point equation

$$f(x,\ y)\ =\ s_{ij}\ f(u_i^{-1}(x),\ v_j^{-1}(y))\ +\ \phi_{ij}(u_i^{-1}(x),\ v_j^{-1}(y)) \qquad \text{for all } (x,\ y) \in D_{ij}, \tag{3}$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$.

2 Perturbation in Scaling Factors

We introduce a perturbation in the scaling factors of the IFS (2) as follows.
Define $T_{ij}\ :\ D \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$ by

$$T_{ij}(x,\ y,\ z)\ =\ (s_{ij} + \delta_{ij})z\ +\ \phi_{ij}(x,\ y)$$

where δ_{ij}'s satisfy the condition $0 < |s_{ij} + \delta_{ij}| < 1$.
Then, we construct a new IFS

$$\{D \times \mathbb{R},\ (u_i,\ v_j,\ T_{ij})\ :\ i = 1,\ \ldots,\ N;\ \ j = 1,\ \ldots,\ M\}\,. \tag{4}$$

We call the IFS (4) as the perturbed IFS of the given IFS (2).
In the sequel, we denote the matrix consisting of the scaling factors as $S = (s_{ij})$ and the matrix consisting of the perturbations added to the scaling factors as $\Delta = (\delta_{ij})$.

Example 1 *Let f be a fractal interpolation function whose graph is the attractor of the IFS $\{D\times\mathbb{R},\ w_{ij}\equiv(u_i,\ v_j,\ F_{ij})\ :\ i=1,\ \ldots,\ 3;\ \ j=1,\ \ldots,\ 3\}$ that interpolates points given in the following table*

y/x	0	64	128	192
0	100	100	100	100
64	100	130	130	140
128	100	140	130	140
192	100	100	130	140

For this IFS, the vertical scaling factors are taken as

$$S=\begin{pmatrix} 0.45 & 0.4 & -0.45 \\ 0.35 & -0.15 & 0.15 \\ -0.3 & 0.25 & -0.45 \end{pmatrix}.$$

and the computed functions F_{ij}'s are given by

$$\begin{aligned}
F_{11}(x,\ y,\ z) &= 0.45z+0.0003255xy+55\\
F_{12}(x,\ y,\ z) &= 0.4z+0.1563x-0.000434xy+60\\
F_{13}(x,\ y,\ z) &= -0.45z+0.1563x+0.0007595xy+145\\
F_{21}(x,\ y,\ z) &= 0.35z+0.1563y-0.0001085xy+65\\
F_{22}(x,\ y,\ z) &= -0.15z+0.0521x-0.0001085xy+145\\
F_{23}(x,\ y,\ z) &= 0.15z+0.0521y-0.0001628xy+115\\
F_{31}(x,\ y,\ z) &= -0.3z+0.2083y-0.0007595xy+130\\
F_{32}(x,\ y,\ z) &= 0.25z-0.2083x-0.0521y+0.0008138xy+115\\
F_{33}(x,\ y,\ z) &= -0.45z+0.0521y+0.0004883xy+175
\end{aligned}$$

Then, it is easy to check that the continuity condition are satisfied for the given data. For example, $F_{33}(0,0,100)=F_{22}(192,192,140)=F_{23}(192,0,100)=F_{32}(0,192,100)=130=z_{22}$, and so on.

In the above IFS, the vertical scaling factors S are perturbed and the new vertical scaling factors S^* are obtained as follows.

$$S^*=S+\Delta=\begin{pmatrix} 0.45 & 0.4 & -0.45 \\ 0.35 & -0.15 & 0.15 \\ -0.3 & 0.25 & -0.45 \end{pmatrix}+\begin{pmatrix} -0.15 & 0.1 & 0.15 \\ 0.15 & -0.15 & 0.05 \\ -0.1 & 0.05 & 0.15 \end{pmatrix}=\begin{pmatrix} 0.3 & 0.5 & -0.3 \\ 0.5 & -0.3 & 0.2 \\ -0.4 & 0.3 & -0.3 \end{pmatrix}$$

Then the perturbed IFS $\{D\times\mathbb{R},\ (u_i,\ v_j,\ T_{ij})\ :\ i=1,\ \ldots,\ 3;\ \ j=1,\ \ldots,\ 3\}$ is constructed where,

$$\begin{aligned}
T_{11}(x,\ y,\ z) &= 0.3z+0.0003255xy+55\\
T_{12}(x,\ y,\ z) &= 0.5z+0.1563x-0.000434xy+60\\
T_{13}(x,\ y,\ z) &= -0.3z+0.1563x+0.0007595xy+145\\
T_{21}(x,\ y,\ z) &= 0.5z+0.1563y-0.0001085xy+65\\
T_{22}(x,\ y,\ z) &= -0.3z+0.0521x-0.0001085xy+145\\
T_{23}(x,\ y,\ z) &= 0.2z+0.0521y-0.0001628xy+115\\
T_{31}(x,\ y,\ z) &= -0.4z+0.2083y-0.0007595xy+130\\
T_{32}(x,\ y,\ z) &= 0.3z-0.2083x-0.0521y+0.0008138xy+115\\
T_{33}(x,\ y,\ z) &= -0.3z+0.0521y+0.0004883xy+175
\end{aligned}$$

Observe that $T_{33}(0,0,100)=145$, $T_{22}(192,192,140)=109$ and hence $T_{33}(0,0,100)\neq T_{22}(192,192,140)$. Therefore, the continuity condition is not satisfied by the perturbed IFS.

In the following, we find conditions so that the perturbed IFS will satisfy the continuity condition.

Theorem 2 *Consider the perturbed IFS (4) of the IFS (2). Let $M_{ij}(x,\ y,\ z) = T_{ij}(x,\ y,\ z) + \lambda_{ij}$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$, where λ_{ij}'s are real constants. Assume that $\lambda_{11} = 0$.*

1. *If λ_{ij} satisfies*

$$M_{ij}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$

then

$$\lambda_{ij} \ = \ \left(\sum_{k=1}^{i-1}\delta_{kj}\right)\ z_{N0} \ - \ \left(\sum_{k=2}^{i}\delta_{kj}\right)\ z_{00} \ + \ \lambda_{1j}\ . \tag{5}$$

2. *If λ_{ij} satisfies*

$$M_{ij}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix} = M_{i(j+1)}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$

then

$$\lambda_{ij} \ = \ \left(\sum_{k=1}^{j-1}\delta_{ik}\right)\ z_{0M} \ - \ \left(\sum_{k=2}^{j}\delta_{ik}\right)\ z_{00} \ + \ \lambda_{i1}\ . \tag{6}$$

3. *If λ_{ij} satisfies*

$$M_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = M_{i(j+1)}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix}$$

then

$$\lambda_{ij} \ = \ \left(\sum_{k=1}^{j-1}\delta_{ik}\right)\ z_{NM} \ - \ \left(\sum_{k=2}^{j}\delta_{ik}\right)\ z_{N0} \ + \ \lambda_{i1}\ . \tag{7}$$

4. *If λ_{ij} satisfies*

$$M_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix}$$

then

$$\lambda_{ij} \ = \ \left(\sum_{k=1}^{i-1}\delta_{kj}\right)\ z_{NM} \ - \ \left(\sum_{k=2}^{i}\delta_{kj}\right)\ z_{0M} \ + \ \lambda_{1j}\ . \tag{8}$$

5. *If λ_{ij} satisfies*

$$M_{i(j+1)}\begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = M_{(i+1)j}\begin{pmatrix} x_0 \\ y_M \\ z_{0M} \end{pmatrix}$$

then

$$\lambda_{(i+1)j} \ = \ \delta_{i(j+1)} z_{N0} \ - \ \delta_{(i+1)j}\ z_{0M} \ + \ \lambda_{i(j+1)}\ . \tag{9}$$

6. *If λ_{ij} satisfies*

$$M_{ij}\begin{pmatrix} x_N \\ y_M \\ z_{NM} \end{pmatrix} = M_{(i+1)(j+1)}\begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$

then

$$\lambda_{(i+1)(j+1)} \ = \ \delta_{ij}\ z_{NM} \ - \ \delta_{(i+1)(j+1)}\ z_{00} \ + \ \lambda_{ij}\ . \tag{10}$$

Then the changed IFS

$$\{D \times \mathbb{R},\ (u_i,\ v_j,\ M_{ij})\ :\ i = 1,\ \ldots,\ N;\ \ j = 1,\ \ldots,\ M\} \tag{11}$$

satisfies the fractal interpolation continuous condition. We denote the function whose graph is the attractor of the IFS (11) by $f_\delta(x,\ y)$.

Proof. 1. The fractal interpolation continuous condition

$$M_{ij} \begin{pmatrix} x_N \\ y_0 \\ z_{N0} \end{pmatrix} = M_{(i+1)j} \begin{pmatrix} x_0 \\ y_0 \\ z_{00} \end{pmatrix}$$

gives

$$\begin{aligned} \delta_{ij}\, z_{N0} \ +\ \lambda_{ij} &= \delta_{(i+1)j}\, z_{00} \ +\ \lambda_{(i+1)j}\ , \\ \lambda_{(i+1)j} &= \delta_{ij}\, z_{N0} \ -\ \delta_{(i+1)j}\, z_{00} \ +\ \lambda_{ij}\ . \end{aligned}$$

Now,

$$\begin{aligned} \lambda_{ij} &= \delta_{(i-1)j}\, z_{N0} \ -\ \delta_{ij}\, z_{00} \ +\ \lambda_{(i-1)j}\ , \\ \lambda_{(i-1)j} &= \delta_{(i-2)j}\, z_{N0} \ -\ \delta_{(i-1)j}\, z_{00} \ +\ \lambda_{(i-2)j}\ , \\ \cdots &= \cdots \\ \lambda_{2j} &= \delta_{1j}\, z_{N0} \ -\ \delta_{2j}\, z_{00} \ +\ \lambda_{1j}\ . \end{aligned}$$

Substituting the values of $\lambda_{1j}, \ldots, \lambda_{(i-1)j}$ in λ_{ij} we get

$$\lambda_{ij} = \left(\sum_{k=1}^{i-1} \delta_{kj}\right)\ z_{N0} \ -\ \left(\sum_{k=2}^{i} \delta_{kj}\right)\ z_{00} \ +\ \lambda_{1j}\ .$$

The proofs of the remaining other cases are similar to the case 1. ■
Note:

1. Putting $j = 1$ in (8) and substituting λ_{i1} in (7) we get

$$\lambda_{ij} = \left(\sum_{k=1}^{j-1} \delta_{ik} \ +\ \sum_{k=1}^{i-1} \delta_{k1}\right)\ z_{NM} \ -\ \left(\sum_{k=2}^{j} \delta_{ik}\right)\ z_{N0} \ -\ \left(\sum_{k=2}^{i} \delta_{k1}\right)\ z_{0M} \ +\ \lambda_{11}\ . \tag{12}$$

   Putting $i = 1$ in (7) and substituting λ_{1j} in (8) we get

$$\lambda_{ij} = \left(\sum_{k=1}^{i-1} \delta_{kj} + \sum_{k=1}^{j-1} \delta_{1k}\right)\ z_{NM} \ -\ \left(\sum_{k=2}^{i} \delta_{kj}\right)\ z_{0M} \ -\ \left(\sum_{k=2}^{j} \delta_{1k}\right)\ z_{N0} \ +\ \lambda_{11}\ . \tag{13}$$

2. For the boundary points $\{(x_i,\ y_0,\ z_{i0})\ :\ i = 0,\ 1,\ \ldots,\ N\}$, Equation (5) is used to check the continuity of M_{ij}'s $(1 \leq i \leq N,\ j = 1)$.

3. For the boundary points $\{(x_0,\ y_j,\ z_{0j})\ :\ j = 0,\ 1,\ \ldots,\ M\}$, Equation (6) is used to check the continuity of M_{ij}'s $(i = 1,\ 1 \leq j \leq M)$.

4. For the boundary points $\{(x_N,\ y_j,\ z_{Nj})\ :\ j = 0,\ 1,\ \ldots,\ M\}$, Equation (12) is used to check the continuity of M_{ij}'s.

5. For the boundary points $\{(x_i,\ y_M,\ z_{iM})\ :\ i = 0,\ 1,\ \ldots,\ N\}$, Equation (13) is used to check the continuity of M_{ij}'s.

6. For other fractal interpolation points, Equation (9) or (10) or (12) is used to determine λ_{ij}.

Example 3 *Consider the IFS and the perturbed IFS given in Example 1. Using Theorem 2, we compute λ_{ij}'s, M_{ij}'s and show that the fractal interpolation continuity conditions are satisfied. Now*
Since $\lambda_{11} = 0$, we have $M_{11}(192, 192, 140) = 109 + \lambda_{11} = 109 + 0 = 109$.
Using (12), we get $\lambda_{12} = -31$ and $M_{12}(192, 0, 100) = 140 + \lambda_{12} = 109$.
Using (9), we get $\lambda_{21} = -36$ and $M_{21}(0, 192, 100) = 145 + \lambda_{21} = 109$.
Using (10), we get $\lambda_{22} = -6$ and $M_{22}(0, 0, 100) = 115 + \lambda_{22} = 109$.

Using (12), we get $\lambda_{12} = -31$ and $M_{12}(192, 192, 140) = 144 + \lambda_{12} = 113$.
Using (12), we get $\lambda_{13} = -32$ and $M_{13}(192, 0, 100) = 145 + \lambda_{13} = 113$.
Using (9), we get $\lambda_{22} = -2$ and $M_{22}(0, 192, 100) = 115 + \lambda_{22} = 113$.
Using (10), we get $\lambda_{23} = -22$ and $M_{23}(0, 0, 100) = 135 + \lambda_{23} = 113$

Using (12), we get $\lambda_{21} = -36$ and $M_{21}(192, 192, 140) = 161 + \lambda_{21} = 125$.
Using (12), we get $\lambda_{22} = 0$ and $M_{22}(192, 0, 100) = 125 + \lambda_{22} = 125$.
Using (9), we get $\lambda_{31} = -5$ and $M_{31}(0, 192, 100) = 130 + \lambda_{31} = 125$.
Using (10), we get $\lambda_{32} = -20$ and $M_{32}(0, 0, 100) = 145 + \lambda_{32} = 125$.

Using (12), we get $\lambda_{22} = 0$ and $M_{22}(192, 192, 140) = 109 + \lambda_{22} = 109$.
Using (12), we get $\lambda_{23} = -26$ and $M_{23}(192, 0, 100) = 135 + \lambda_{23} = 109$.
Using (9), we get $\lambda_{32} = -26$ and $M_{32}(0, 192, 100) = 135 + \lambda_{32} = 109$.
Using (10), we get $\lambda_{33} = -36$ and $M_{33}(0, 0, 100) = 145 + \lambda_{33} = 109$.

It can be easily checked that the continuity condition of M_{ij}'s in (11) are satisfied on the boundary points of the rectangular domain by using the value of λ_{ij} in (11) computed from (5) or (6) or (12) or (13).

Therefore, it is concluded that the IFS (11) satisfies the fractal interpolation continuous condition when adding corresponding λ_{ij}.

3 Error Estimation

In this section, the estimate on error between the FIF $f(x, y)$ of the IFS (2) and the FIF $f_\delta(x, y)$ of the perturbed IFS (11) is computed.

Set $\mathcal{N} = \{1, 2, \ldots, N\}$ and $\mathcal{M} = \{1, 2, \ldots, M\}$. For $i_k \in \mathcal{N}$, $k = 1, 2, \ldots, n$, define $u_{i_1 i_2 \cdots i_n}(x) = u_{i_1} \circ u_{i_2} \circ \cdots u_{i_n}(x)$; and for $j_k \in \mathcal{M}$, $k = 1, 2, \ldots, n$ define $v_{j_1 j_2 \cdots j_n}(y) = v_{j_1} \circ v_{j_2} \circ \cdots \circ v_{j_n}(y)$.

Lemma 4 *Let $f(x, y)$ and $f_\delta(x, y)$ be the bivariate FIFs generated by the IFS (2) and (11) respectively. Then for any $(x, y) \in D$, $i_k \in \mathcal{N}$, $j_k \in \mathcal{M}$, $k = 1, 2, \ldots, n$, we have,*

$$u_{i_1 i_2 \cdots i_n}(x) = \left(\prod_{k=1}^{n} a_{i_k}\right) x + \sum_{r=1}^{n} \left(\prod_{k=1}^{r-1} a_{i_k}\right) b_{i_r}, \tag{14}$$

$$v_{j_1 j_2 \cdots j_n}(y) = \left(\prod_{k=1}^{n} c_{j_k}\right) y + \sum_{r=1}^{n} \left(\prod_{k=1}^{r-1} c_{j_k}\right) d_{j_r}, \tag{15}$$

$$\begin{aligned} f_\delta(u_{i_1 i_2 \cdots i_n}(x), v_{j_1 j_2 \cdots j_n}(y)) &= \left(\prod_{k=1}^{n} (s_{i_k j_k} + \delta_{i_k j_k})\right) f_\delta(x, y) + \left(\prod_{k=1}^{n-1} (s_{i_k j_k} + \delta_{i_k j_k})\right) \phi_{i_n j_n}(x, y) \\ &+ \sum_{r=1}^{n-1} \left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k})\right) \phi_{i_r j_r}(\overline{x}, \overline{y}) \\ &+ \sum_{r=1}^{n} \left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k})\right) \lambda_{i_r j_r} \end{aligned}$$

where

$$\overline{x} = u_{i_{r+1}\dots i_n}(x) = \left(\prod_{k=1}^{n-r} a_{i_{r+k}}\right) x + \sum_{l=1}^{n-r}\left(\prod_{k=1}^{l-1} a_{i_{r+k}}\right) b_{i_{r+l}}\ , \tag{16}$$

and

$$\overline{y} = v_{j_{r+1}\dots j_n}(y) = \left(\prod_{k=1}^{n-r} c_{j_{r+k}}\right) y + \sum_{l=1}^{n-r}\left(\prod_{k=1}^{l-1} c_{j_{r+k}}\right) d_{j_{r+l}}\ . \tag{17}$$

Proof.

$$\begin{aligned} u_{i_1 i_2 \dots i_n}(x) &= u_{i_1} \circ_{i_2} \circ \cdots \circ u_{i_n}(x) \\ &= u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_{n-1}}(a_{i_n}x + b_{i_n}) \\ &= u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_{n-2}}(a_{i_{n-1}}(a_{i_n}x + b_{i_n}) + b_{i_{n-1}}) \\ \vdots \ \ \vdots \ \ &\vdots \quad \ \vdots \\ &= \left(\prod_{k=1}^{n} a_{i_k}\right)\ x\ +\ \sum_{r=1}^{n}\left(\prod_{k=1}^{r-1} a_{i_k}\right)\ b_{i_r}. \end{aligned}$$

Similarly,

$$v_{j_1 j_2 \cdots j_n}(y) = \left(\prod_{k=1}^{n} c_{j_k}\right)\ x\ +\ \sum_{r=1}^{n}\left(\prod_{k=1}^{r-1} c_{j_k}\right)\ d_{j_r}\ .$$

Now

$$f(x,y) = s_{ij} f(u_i^{-1}(x), v_j^{-1}(y)) + \phi_{ij}(u_i^{-1}(x), v_j^{-1}(y)), \forall (x,y) \in D_{ij},$$

for $i = 1,\ 2,\ \dots,\ N,\ j = 1,\ 2,\ \dots,\ M.$
Then

$$f(u_{i_1}(x), v_{j_1}(y)) = s_{i_1 j_1} f(x,y) + \phi_{i_1 j_1}(x,y)$$

$$\begin{aligned} f(u_{i_1 i_2}(x), v_{j_1 j_2}(y)) &= s_{i_1 j_1} f(u_{i_2}(x), v_{j_2}(y)) + \phi_{i_1 j_1}(u_{i_2}(x), v_{j_2}(y)) \\ &= s_{i_1 j_1}(s_{i_2 j_2} f(x,y) + \phi_{i_2 j_2}(x,y)) + \phi_{i_1 j_1}(u_{i_2}(x), v_{j_2}(y)) \\ &= s_{i_1 j_1} s_{i_2 j_2} f(x,y) + s_{i_1 j_1}\phi_{i_2 j_2}(x,y) + \phi_{i_1 j_1}(u_{i_2}(x), v_{j_2}(y)) \end{aligned}$$

$$\begin{aligned} f(u_{i_1 i_2 i_3}(x), v_{j_1 j_2 i_3}(y)) &= s_{i_1 j_1} s_{i_2 j_2} f(u_{i_3}(x), v_{j_3}(y)) + s_{i_1 j_1}\phi_{i_2 j_2}(u_{i_3}(x), v_{j_3}(y)) \\ &+ \phi_{i_1 j_1}(u_{i_2 i_3}(x), v_{j_2 j_3}(y)) \\ &= s_{i_1 j_1} s_{i_2 j_2} s_{i_3 j_3} f(x,y) + s_{i_1 j_1} s_{i_2 j_2}\phi_{i_3 j_3}(x,y) \\ &+ s_{i_1 j_1}\phi_{i_2 j_2}(u_{i_3}(x), v_{j_3}(y)) + \phi_{i_1 j_1}(u_{i_2 i_3}(x), v_{j_2 j_3}(y)) \end{aligned}$$

In general,

$$\begin{aligned} f(u_{i_1 i_2 \dots i_n}(x), v_{j_1 j_2 \dots j_n}(y)) &= \left(\prod_{k=1}^{n} s_{i_k j_k}\right) f(x,y) + \left(\prod_{k=1}^{n-1} s_{i_k j_k}\right)\phi_{i_n j_n}(x,y) \\ &+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} s_{i_k j_k}\right)\phi_{i_r j_r}(u_{i_{r+1}\dots i_n}(x), v_{j_{r+1}\dots j_n}(y)). \end{aligned}$$

Therefore

$$\begin{aligned} f(u_{i_1 i_2 \dots i_n}(x), v_{j_1 j_2 \dots j_n}(y)) &= \left(\prod_{k=1}^{n} s_{i_k j_k}\right) f(x,y) + \left(\prod_{k=1}^{n-1} s_{i_k j_k}\right)\phi_{i_n j_n}(x,y) \\ &+ \sum_{r=1}^{n-1}\left(\prod_{k=1}^{r-1} s_{i_k j_k}\right)\phi_{i_r j_r}(\overline{x}, \overline{y}). \end{aligned} \tag{18}$$

Now

$$\begin{aligned} f_\delta(u_{i_1 i_2 \dots i_n}(x), v_{j_1 j_2 \dots j_n}(y)) &= \prod_{k=1}^{n} (s_{i_k j_k} + \delta_{i_k j_k})\, f_\delta(x,y) + \left(\prod_{k=1}^{n-1} (s_{i_k j_k} + \delta_{i_k j_k}) \right) (\phi_{i_n j_n}(x,y) + \lambda_{i_n j_n}) \\ &+ \sum_{r=1}^{n-1} \left(\prod_{k=1}^{r-1} s_{i_k j_k} + \delta_{i_k j_k} \right) (\phi_{i_r j_r}(\overline{x},\ \overline{y}) + \lambda_{i_r j_r}) \\ &= \prod_{k=1}^{n} (s_{i_k j_k} + \delta_{i_k j_k})\, f_\delta(x,y) + \left(\prod_{k=1}^{n-1} (s_{i_k j_k} + \delta_{i_k j_k}) \right) \phi_{i_n j_n}(x,y) \\ &+ \sum_{r=1}^{n-1} \left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k}) \right) \phi_{i_r j_r}(\overline{x},\ \overline{y}) \\ &+ \sum_{r=1}^{n} \left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k}) \right) \lambda_{i_r j_r}\ . \end{aligned} \tag{19}$$

■

Theorem 5 *Let $f(x,y)$ and $f_\delta(x,y)$ be the bivariate FIFs generated with the IFS (2) and (11) respectively. For any given $(x,y) \in D$ let $\{i_k\}$, $i_k \in \mathcal{N}$, be the sequence such that x satisfies*

$$x = \sum_{r=1}^{\infty} (\prod_{k=1}^{r-1} a_{i_k}) b_{i_r} \tag{20}$$

and let $\{j_k\}$, $j_k \in \mathcal{M}$ be another sequence such that y satisfies

$$y = \sum_{r=1}^{\infty} (\prod_{k=1}^{r-1} c_{j_k}) d_{j_r}\ . \tag{21}$$

Then

$$\begin{aligned} f_\delta(x,y) - f(x,y) &= \sum_{r=1}^{\infty} \left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k}) - \prod_{k=1}^{r-1} s_{i_k j_k} \right) \phi_{i_r j_r}(x', y') \\ &+ \sum_{r=1}^{\infty} \left(\prod_{k=1}^{r-1} (s_{i_k j_k} + \delta_{i_k j_k}) \right) \lambda_{i_r j_r} \end{aligned} \tag{22}$$

where $x' = \sum_{l=1}^{\infty} (\prod_{k=1}^{l-1} a_{i_{r+k}}) b_{i_{r+l}}$, $y' = \sum_{l=1}^{\infty} (\prod_{k=1}^{l-1} c_{j_{r+k}}) d_{j_{r+l}}$.

Proof. Since $u_i : I \to I_i,\ i = 1,\ 2 \dots, N$ as

$$u_i(x) = a_i x + b_i$$

where $a_i = \frac{x_i - x_{i-1}}{x_N - x_0}$, is contractive on the closed interval I, the sequence of sets $\{u_{i_1 i_2 \cdots i_n}(I)\}$ is monotonically decreasing with diameter tends to zero as $n \to \infty$. Hence by the Cantor's intersection theorem, $\bigcap_{n=1}^{\infty} u_{i_1 i_2 \cdots i_n}(I)$ consists of a single point in I for any sequence $\{i_k\}, i_k \in \mathcal{N}$. Then for any given $x \in I$, there exists a sequence $\{i_k\}, i_k \in \mathcal{N}$ such that

$$\{x\} = \bigcap_{n=1}^{\infty} u_{i_1 i_2 \cdots i_n}(I) = \lim_{n \to \infty} u_{i_1 i_2 \cdots i_n}(I)\ .$$

Since each a_{i_k} in (14) obeys $0 < a_{i_k} < 1$, applying (14), x can be expressible as

$$x = \lim_{n \to \infty} u_{i_1 i_2 \cdots i_n}(x^*) = \sum_{r=1}^{\infty} (\prod_{k=1}^{r-1} a_{i_k}) b_{i_r}\ ,$$

where x^* is arbitrarily chosen in I. Similarly for any given $y \in J$, there exists another sequence $\{j_k\}$, $j_k \in \mathcal{M}$, such that y satisfies (21) .
Using (16), (17) and (18), $f(x\ ,\ y)$ can be expressed for any $(x\ ,\ y) \in D$ as

$$\begin{aligned} f(x\ ,\ y) &= \lim_{n\to\infty} f(u_{i_1 i_2 \cdots i_n}(x^*), v_{j_1 j_2 \cdots j_n}(y^*)) \\ &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1} s_{i_k j_k}\right)\phi_{i_r j_r}\left(\sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} a_{i_{r+k}}\right) b_{i_{r+l}}, \sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} c_{j_{r+k}}\right) d_{j_{r+l}}\right) \\ &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1} s_{i_k j_k}\right)\phi_{i_r j_r}(x', y') \end{aligned} \quad (23)$$

where $x' = \sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} a_{i_{r+k}}\right) b_{i_{r+l}}$, $y' = \sum_{l=1}^{\infty}\left(\prod_{k=1}^{l-1} c_{j_{r+k}}\right) d_{j_{r+l}}$, and $(x^*,\ y^*)$ are chosen arbitrarily in D.
From (16), (17), it is clear that $(x',\ y')$ belongs to D.
Similarly, by (19), we can get

$$f_\delta(x\ ,\ y) = \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_k j_k} + \delta_{i_k j_k})\right)\phi_{i_r j_r}(x', y') + \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_k j_k} + \delta_{i_k j_k})\right)\lambda_{i_r j_r}\ . \quad (24)$$

Therefore by (23) and (24)

$$\begin{aligned} f_\delta(x,y) - f(x,y) &= \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_k j_k} + \delta_{i_k j_k}) - \prod_{k=1}^{r-1} s_{i_k j_k}\right)\phi_{i_r j_r}(x', y') \\ &+ \sum_{r=1}^{\infty}\left(\prod_{k=1}^{r-1}(s_{i_k j_k} + \delta_{i_k j_k})\right)\lambda_{i_r j_r} \end{aligned}$$

which completes the proof. ■

Corollary 6 *Let $f(x,y)$ and $f_\delta(x,y)$ be the bivariate IFS generated by the IFS (2) and (11) respectively. Let $s = \max\{|s_{ij}| : i \in \mathcal{N}, j \in \mathcal{M}\} < 1$, $0 < \delta = \max\{|\delta_{ij}|, i \in \mathcal{N}, j \in \mathcal{M}\}$ such that $s + \delta < 1$ and $A = \max\{\|\phi_{ij}\| : i \in \mathcal{N}, j \in \mathcal{M}\}$ where $\|\phi_{ij}\| = \max\{|\phi_{ij}(x,y)| : (x,y) \in D\}$. Then*

$$|f_\delta(x,y) - f(x,y)| \le \frac{\delta\left\{A + (1-s)(N+M)\left(|z_{NM}| + |z_{N0}| + |z_{0M}| + |z_{00}|\right)\right\}}{(1-s)(1-s-\delta)}$$

Proof. Equations (5), (6), (9), (10), (12) and (13) together give,

$$|\lambda_{nm}| \le (N+M)\delta\left(|z_{NM}| + |z_{N0}| + |z_{0M}| + |z_{00}|\right)\ .$$

Therefore, by (22),

$$\begin{aligned} |f_\delta(x,y) - f(x,y)| &\le \frac{A}{1-s-\delta} - \frac{A}{1-s} + \frac{(N+M)\delta\left(|z_{NM}| + |z_{N0}| + |z_{0M}| + |z_{00}|\right)}{1-s-\delta} \\ &= \frac{\delta\left\{A + (1-s)(N+M)\left(|z_{NM}| + |z_{N0}| + |z_{0M}| + |z_{00}|\right)\right\}}{(1-s)(1-s-\delta)}\ . \end{aligned}$$

This completes the proof. ■

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