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# Combined Effects in Fractional Boundary Value Problem

Sonia Ben Othman *, Safa Dridi, Bilel Khamessi
Département de Mathématiques, Faculté des Sciences de Tunis, Campus universitaire, 2092 Tunis, TUNISIA
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**Abstract:** This paper deals with existence and uniqueness of a positive solution for the fractional boundary value problem

$$\begin{cases} D^{\alpha}u(x) = -a_1(x)u^{\sigma_1} - a_2(x)u^{\sigma_2}, & x \in (0,1), \\ \lim\limits_{x\to 0^+} D^{\alpha-1}u(x) = 0, \ \ u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$, $\sigma_1, \sigma_2 \in (-1, 1)$ and $a_1, a_2$ are nonnegative continuous functions on $(0,1)$ that may be singular at $x = 0$ or $x = 1$. We also give the global behavior of a such solution.

**Keywords:**Fractional differential equation; Dirichlet problem; Positive solution; Schauder fixed point theorem

## 1 Introduction

Fractional differential equations have been recently received much attention. It is caused by the intensive development of the theory of fractional calculus itself and by the applications of such construction in various fields of sciences and engineering, such as control, porus media, chemistry, physics, etc. For example and details, see [8, 9].

In this paper, we consider the following nonlinear fractional problem

$$\begin{cases} D^{\alpha}u(x) = -a_1(x)u^{\sigma_1} - a_2(x)u^{\sigma_2}, & x \in (0,1), \\ \lim\limits_{x\to 0^+} D^{\alpha-1}u(x) = 0, \ \ u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$ and for $i \in \{1, 2\}$, $\sigma_i \in (-1, 1)$, $a_i$ is a nonnegative continuous function on $(0, 1)$ that may be singular at $x = 0$ or $x = 1$ and $D^{\alpha}$ is the Riemann-Liouville fractional derivative. Then, we will adress the question of existence, uniqueness and exact asymptotic behavior of positive solutions to problem (1.1).
For readers convenience, we recall that for a measurable function $v$, the Riemann-Liouville derivative $D^{\beta}v$ of order $\beta > 0$ is defined by

$$D^{\beta}v(x) = \frac{1}{\Gamma(n-\beta)}\Big(\frac{d}{dx}\Big)^n \int_0^x (x-t)^{n-\beta-1}v(t)dt,$$

provided that the right-hand sides is pointwise defined on $(0, 1]$. Here $n = [\beta] + 1$, where $[\beta]$ means the integer part of the number $\beta$ and $\Gamma$ is the Euler Gamma function.

This work is motivated by recent advances in the study of fractional differential equations involving singular or sublinear nonlinearities with different boundary conditions (see [1-5], [7-13, 15] and the references therein). Namely, in [5], the authors investigate the existence and multiplicity of positive solutions of problem

$$\begin{cases} D^{\alpha}u(x) = f(x, u), & x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha \leq 2$ and $f$ is a nonnegative continuous function on $[0, 1] \times [0, \infty)$.

*Corresponding author. *Sonia.benothman@fsb.rnu.tn*


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Recently, in [11], Mâagli et al. considered the following problem

$$\begin{cases} D^{\alpha}u(x) = -a(x)u^{\sigma}, \ \ x \in (0,1), \\ \lim\limits_{x\to 0^+} D^{\alpha-1}u(x) = 0, \ \ u(1) = 0, \end{cases} \tag{1.3}$$

where $-1 < \sigma < 1$ and the function $a$ is required to satisfy some assumptions related to $\mathcal{K}$, the set of all Karamata functions $L$ defined on $(0, \eta]$, by

$$L(t) := c\exp\left(\int_t^{\eta} \frac{z(s)}{s} ds\right),$$

for some $\eta > 1$, where $c > 0$ and $z$ is a continuous function on $[0, \eta]$, with $z(0) = 0$. To describe the result of [11] in more details, we need some notations.

For $\mu \leq \alpha$, $-1 < \sigma < 1$ and $\tilde{L} \in \mathcal{K}$ defined on $(0, \eta]$, $(\eta > 1)$ such that $\int_0^{\eta} \frac{\tilde{L}(t)}{t^{\mu-\alpha+1}} dt < \infty$, we define the function $\Psi_{\tilde{L},\mu,\sigma}$ on $(0,1)$ by

$$\Psi_{\tilde{L},\mu,\sigma}(t) := \begin{cases} 1, & if\ \mu < \sigma + \alpha - 1, \\ \left(\int_t^{\eta} \frac{\tilde{L}(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & if\ \mu = \sigma + \alpha - 1, \\ \left(\tilde{L}(t)\right)^{\frac{1}{1-\sigma}}, & if\ \ \sigma + \alpha - 1 < \mu < \alpha, \\ \left(\int_0^{t} \frac{\tilde{L}(s)}{s} ds\right)^{\frac{1}{1-\sigma}}, & if\ \mu = \alpha. \end{cases}$$

For two nonnegative functions $f$ and $g$ defined on a set $S$, the notation $f(x) \approx g(x)$, $x \in S$ means that there exists $c > 0$ such that $\frac{1}{c} f(x) \leq g(x) \leq c f(x)$, for all $x \in S$.
Throughout this paper, for a nonnegative measurable function $f$ on $(0,1)$, we denote by $G_{\alpha}f$ the potential of $f$ defined on $(0,1)$ by

$$G_{\alpha}f(x) = \int_0^1 G_{\alpha}(x,t) f(t) dt,$$

where

$$G_{\alpha}(x,t) = \frac{1}{\Gamma(\alpha)}[x^{\alpha-2}(1-t)^{\alpha-1} - ((x-t)^+)^{\alpha-1}]$$

is the Green's function for the boundary value problem (1.1), (see [11]).
Also, we denote by $\mathcal{C}_{2-\alpha}([0,1])$ the set of all functions $f$ such that $t \to t^{2-\alpha} f(t)$ is continuous on $[0,1]$. We point out that for any nonnegative function $f$ such that $x \longmapsto (1-x)^{\alpha-1} f(x)$ is continuous and integrable on $(0,1)$, the authors in [11] proved that $G_{\alpha}f$ is the unique function in $\mathcal{C}_{2-\alpha}([0,1])$ satisfying

$$\begin{cases} D^{\alpha}G_{\alpha}f(x) = -f(x), \ \ x \in (0,1), \\ \lim\limits_{x\to 0^+} D^{\alpha-1}G_{\alpha}f(x) = 0 \text{ and } G_{\alpha}f(1) = 0. \end{cases} \tag{1.4}$$

In [11], Mâagli et al. studied problem (1.3) where $a$ verifies
$(\mathbf{H}_0)$ $a \in C((0,1))$ satisfying for each $x \in (0,1)$,

$$a(x) \approx x^{-\lambda}(1-x)^{-\mu} L(x) \tilde{L}(1-x),$$

where $\lambda + (2-\alpha)\sigma \leq 1$, $\mu \leq \alpha$ and $L, \tilde{L} \in \mathcal{K}$ such that

$$\int_0^{\eta} \frac{L(t)}{t^{\lambda+(2-\alpha)\sigma}} dt < \infty \ \text{ and } \int_0^{\eta} \frac{\tilde{L}(t)}{t^{\mu-\alpha+1}} dt < \infty.$$

Based on the Schauder fixed-point theorem, the authors showed in [11] the following result.

**Theorem 1** *Assume that $a$ satisfies $(H_0)$. Then problem (1.3) has a unique positive solution $u \in C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,*

$$u(x) \approx x^{\alpha-2}(1-x)^{\min(1,\frac{\alpha-\mu}{1-\sigma})} \Psi_{\tilde{L},\mu,\sigma}(1-x). \tag{1.5}$$

In this paper, we improve the above result especially when the nonlinearity is the sum of a singular term and a sublinear term.
Let us consider the following assumption.
(**H**) For $i \in \{1,2\}$, $a_i \in C((0,1))$ satisfying for each $x \in (0,1)$,

$$a_i(x) \approx x^{-\lambda_i}(1-x)^{-\mu_i} L_i(x)\tilde{L}_i(1-x),$$

where $\lambda_i + (2-\alpha)\sigma_i \leq 1$, $\mu_i \leq \alpha$ and $L_i, \tilde{L}_i \in \mathcal{K}$ such that

$$\int_0^\eta \frac{L_i(t)}{t^{\lambda_i+(2-\alpha)\sigma_i}} dt < \infty \text{ and } \int_0^\eta \frac{\tilde{L}_i(t)}{t^{\mu_i-\alpha+1}} dt < \infty.$$

**Remark 2** *We need to verify condition $\int_0^\eta \frac{L_i(t)}{t^{\lambda_i+(2-\alpha)\sigma_i}} dt < \infty$ and $\int_0^\eta \frac{\tilde{L}_i(t)}{t^{\mu_i-\alpha+1}} dt < \infty$, only if $\lambda_i + (2-\alpha)\sigma_i = 1$ and $\mu_i = \alpha$. This is due to Lemma 2 below.*

As it turns out, the estimates $(1.5)$ depends closely on $\min(1, \frac{\alpha-\mu}{1-\sigma})$. Also, as it will be seen, the numbers

$$\beta_1 = \min(1, \frac{\alpha-\mu_1}{1-\sigma_1}) \text{ and } \beta_2 = \min(1, \frac{\alpha-\mu_2}{1-\sigma_2}),$$

play an important role in the combined effect of singular and superlinear nonlinearities in $(1.1)$ and lead to a competition. However, without loss of generality, we can suppose that $\frac{\alpha-\mu_1}{1-\sigma_1} \leq \frac{\alpha-\mu_2}{1-\sigma_2}$ and we introduce the function $\theta$ defined on $(0,1)$ by

$$\theta(t) := \begin{cases} t^{\beta_1}\Psi_{\tilde{L}_1,\mu_1,\sigma_1}(t), & if\ \beta_1 < \beta_2, \\ t^{\beta_1}(\Psi_{\tilde{L}_1,\mu_1,\sigma_1}(t) + \Psi_{\tilde{L}_2,\mu_2,\sigma_2}(t)), & if\ \beta_1 = \beta_2. \end{cases} \tag{1.6}$$

Now, we are ready to state our main results.

**Theorem 3** *Assume that the functions $a_1$ and $a_2$ satisfy $(H)$ and put*

$$w(x) = a_1(x)x^{(\alpha-2)\sigma_1}\theta^{\sigma_1}(1-x) + a_2(x)x^{(\alpha-2)\sigma_2}\theta^{\sigma_2}(1-x), \ \ x \in (0,1).$$

*Then we have, for $x \in (0,1)$,*

$$x^{2-\alpha}G_\alpha w(x) \approx \theta(1-x). \tag{1.7}$$

**Theorem 4** *Assume $(H)$. Then problem $(1.1)$ has a unique positive solution $u \in C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,*

$$u(x) \approx x^{\alpha-2}\theta(1-x). \tag{1.8}$$

The outline of the paper is as follows. In Section 2, we state some already known results on Karamata functions which will be useful for our study. Theorem 2 and Theorem 3 are proved respectively in Section 3 and Section 4.
Throughout, the letter $c$ will denote a generic constant which may vary from line to line.

# 2 Preliminary Results

In what follows, we are quoting without proof some fundamental properties of functions belonging to the class $\mathcal{K}$ collected from [6] and [14]. We recall that a function $L$ defined on $(0,\eta]$ belongs to the class $\mathcal{K}$, if

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s} ds\right),$$

for some $\eta > 1$, $c > 0$ and $z$ is a continuous function on $[0,\eta]$ with $z(0) = 0$.

**Proposition 5** *$(i)$ A function $L$ is in $\mathcal{K}$ if and only if $L$ is a positive function in $\mathcal{C}^1((0,\eta])$ such that*

$$\lim_{t\to 0^+}\frac{tL^{'}(t)}{L(t)}=0. \tag{2.1}$$

*$(ii)$ Let $L_1,\ L_2\in\mathcal{K}$ and $p\in\mathbb{R}$. Then we have*

$$L_1+L_2\in\mathcal{K}, L_1L_2\in\mathcal{K} \text{ and } L_1^p\in\mathcal{K}.$$

*$(iii)$ Let $L\in\mathcal{K}$ and $\varepsilon>0$. Then we have*

$$\lim_{t\to 0^+} t^{\varepsilon}L(t)=0.$$

As a standard example of functions belonging to the class $\mathcal{K}$ ( see [14] ), we give

**Example 6** *Let $m\in\mathbb{N}^*$ and $\eta>0$. Let $(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}^m$ and $w$ be a sufficiently large positif real number such that the function*

$$L(t)=\prod_{1\le i\le m}\left(\log_i(\frac{w}{t})\right)^{\mu_i}$$

*is defined and positive on $(0,\eta]$, where $\log_i t=\log\circ....\circ\log t$ $(i\ times)$. Then we have $L\in\mathcal{K}$.*

**Lemma 7** *$(i)$ Let $L$ be a function in $\mathcal{K}$. Then we have*

$$\lim_{t\to 0^+}\frac{L(t)}{\displaystyle\int_t^{\eta}\frac{L(s)}{s}ds}=0.$$

*In particular,*

$$t\longmapsto\int_t^{\eta}\frac{L(s)}{s}ds\in\mathcal{K}.$$

*$(ii)$ If $\displaystyle\int_0^{\eta}\frac{L(s)}{s}ds$ converges, then*

$$\lim_{t\to 0^+}\frac{L(t)}{\displaystyle\int_0^{t}\frac{L(s)}{s}ds}=0.$$

*In particular,*

$$t\longmapsto\int_0^{t}\frac{L(s)}{s}ds\in\mathcal{K}.$$

Applying Karamata's theorem, we get the following.

**Lemma 8** *Let $\gamma\in\mathbb{R}$ and $L$ be a function in $\mathcal{K}$ defined on $(0,\eta]$ for some $\eta>1$. We have*
*$(i)$ If $\gamma>-1$, then $\int_0^{\eta}s^{\gamma}L(s)ds$ converges and*

$$\int_0^t s^{\gamma}L(s)ds\underset{t\to 0^+}{\sim}\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

*$(ii)$ If $\gamma<-1$, then $\int_0^{\eta}s^{\gamma}L(s)ds$ diverges and*

$$\int_t^{\eta} s^{\gamma}L(s)ds\underset{t\to 0^+}{\sim}-\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

**Lemma 9** *(See [6]). Put $r=\max(s^{\frac{1}{1-\sigma_1}},t^{\frac{1}{1-\sigma_2}})$, for $s,t>0$. Then we have*

$$r\le r^{\sigma_1}s+r^{\sigma_2}t\le 2r.$$

**Lemma 10** *(See [6]). Let $L_1, L_2 \in \mathcal{K}$ be defined on $(0,\eta]$, $(\eta > 1)$ and we put for $t \in (0,1)$,*

$$M(t) = \left(\int_t^\eta \frac{L_1(s)}{s} ds\right)^{\frac{1}{1-\sigma_1}} + \left(\int_t^\eta \frac{L_2(s)}{s} ds\right)^{\frac{1}{1-\sigma_2}}.$$

*Then we have for $t \in (0,1)$,*

$$\int_t^\eta \frac{(M^{\sigma_1} L_1 + M^{\sigma_2} L_2)(s)}{s} ds \approx M(t).$$

**Lemma 11** *(See [6]). Let $L_1, L_2 \in \mathcal{K}$ be defined on $(0,\eta]$, $(\eta > 1)$ such that $\int_0^\eta \frac{L_1(s)}{s} ds < \infty$ and $\int_0^\eta \frac{L_2(s)}{s} ds < \infty$. Put*

$$N(t) = \left(\int_0^t \frac{L_1(s)}{s} ds\right)^{\frac{1}{1-\sigma_1}} + \left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{1}{1-\sigma_2}}, \text{ for } t \in (0,1).$$

*Then we have for $t \in (0,1)$,*

$$\int_0^t \frac{(N^{\sigma_1} L_1 + N^{\sigma_2} L_2)(s)}{s} ds \approx N(t).$$

# 3 Proof of Theorem 2

We beginning this section by stating the following result due to [11], that will play a crucial role in the proof of Theorem 2.

**Proposition 12** *(See [11]). Let $L, \tilde{L} \in \mathcal{K}$ and let for $x \in (0,1)$,*

$$a(x) = x^{-\lambda}(1-x)^{-\mu} L(x)\tilde{L}(1-x),$$

*with $\lambda \leq 1$ and $\mu \leq \alpha$. Assume that*

$$\int_0^\eta t^{-\lambda} L(t) dt < \infty \text{ and } \int_0^\eta t^{\alpha-1-\mu} \tilde{L}(t) dt < \infty.$$

*Then we have for $x \in (0,1)$,*

$$x^{2-\alpha} G_\alpha a(x) \approx (1-x)^{\min(1,\alpha-\mu)} \Psi_{\tilde{L},\mu,0}(1-x).$$

**Proof of Theorem 2.** Let $a_1$, $a_2$ be two functions satisfying $(H)$ and $\theta$ be the function given in $(1.6)$. We recall that

$$w(x) = a_1(x) x^{(\alpha-2)\sigma_1} \theta^{\sigma_1}(1-x) + a_2(x) x^{(\alpha-2)\sigma_2} \theta^{\sigma_2}(1-x), \quad x \in (0,1).$$

We first give an explicit form of the function $\theta$.
Indeed, since $\beta_1 < \beta_2$ is equivalent to

$$\frac{\alpha-\mu_1}{1-\sigma_1} < \frac{\alpha-\mu_2}{1-\sigma_2} \text{ and } \sigma_1 + \alpha - 1 < \mu_1,$$

we can write

$$\theta(t)=\begin{cases}\left(\int_0^t \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}, & if\ \mu_1=\alpha \text{ and } \mu_2<\alpha,\\ t^{\frac{\alpha-\mu_1}{1-\sigma_1}}\left(\tilde{L}_1(t)\right)^{\frac{1}{1-\sigma_1}}, & if\ \frac{\alpha-\mu_1}{1-\sigma_1}<\frac{\alpha-\mu_2}{1-\sigma_2} \text{ and } \sigma_1+\alpha-1<\mu_1<\alpha,\\ t^{\frac{\alpha-\mu_1}{1-\sigma_1}}\left((\tilde{L}_1(t)^{\frac{1}{1-\sigma_1}}+(\tilde{L}_2(t))^{\frac{1}{1-\sigma_2}}\right), & if\ \frac{\alpha-\mu_1}{1-\sigma_1}=\frac{\alpha-\mu_2}{1-\sigma_2} \text{ and } \sigma_1+\alpha-1<\mu_1<\alpha,\\ t\left(\left(\int_t^\eta \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}+\left(\int_t^\eta \frac{\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma_2}}\right), & if\ \mu_1=\sigma_1+\alpha-1 \text{ and } \mu_2=\sigma_2+\alpha-1,\\ t\left(1+\left(\int_t^\eta \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}\right), & if\ \mu_1=\sigma_1+\alpha-1 \text{ and } \mu_2<\sigma_2+\alpha-1,\\ t, & if\ \mu_1<\sigma_1+\alpha-1,\\ \left(\int_0^t \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}+\left(\int_0^t \frac{\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma_2}}, & if\ \mu_1=\mu_2=\alpha.\end{cases}$$

Note that throughout the proof, we use Proposition 1 and Lemma 1 to verify that some functions are in $\mathcal{K}$. So, we distinguish the following cases.

**Case 1.** $\mu_1=\alpha$ and $\mu_2<\alpha$.

We have

$$\theta(t)=\left(\int_0^t \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}, \quad t\in(0,1).$$

By calculus, we obtain for $x\in(0,1)$,

$$\begin{aligned} w(x) &\approx x^{-\lambda_1+(\alpha-2)\sigma_1}L_1(x)(1-x)^{-\alpha}\tilde{L}_1(1-x)\theta^{\sigma_1}(1-x)\\ &+ x^{-\lambda_2+(\alpha-2)\sigma_2}L_2(x)(1-x)^{-\mu_2}\tilde{L}_2(1-x)\theta^{\sigma_2}(1-x).\end{aligned}$$

Since $\mu_2<\alpha$ and the functions $t\mapsto(\tilde{L}_1\theta^{\sigma_1})(t)$ and $t\mapsto(\tilde{L}_2\theta^{\sigma_2})(t)$ are in $\mathcal{K}$, we deduce by Proposition 1 $(iii)$, that

$$\begin{aligned} w(x) &\approx x^{-\lambda_1+(\alpha-2)\sigma_1}L_1(x)(1-x)^{-\alpha}\tilde{L}_1(1-x)\theta^{\sigma_1}(1-x)\\ &+ x^{-\lambda_2+(\alpha-2)\sigma_2}L_2(x)(1-x)^{-\alpha}\tilde{L}_1(1-x)\theta^{\sigma_1}(1-x)\\ &:= w_1(x)+w_2(x).\end{aligned}$$

So, by using the fact that

$$\int_0^\eta \frac{\tilde{L}_1(s)\theta^{\sigma_1}(s)}{s}ds=c\left(\int_0^\eta \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}<\infty,$$

we obtain by applying Proposition 2, for $\mu=\alpha$, that

$$\begin{aligned} x^{2-\alpha}G_\alpha w_1(x)+x^{2-\alpha}G_\alpha w_2(x) &\approx \int_0^{1-x}\frac{\tilde{L}_1(s)\theta^{\sigma_1}(s)}{s}ds\\ &\approx \left(\int_0^{1-x}\frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}}=\theta(1-x).\end{aligned}$$

Hence, we conclude that

$$x^{2-\alpha}G_\alpha w(x)\approx\theta(1-x).$$

**Case 2.** $\frac{\alpha-\mu_1}{1-\sigma_1}<\frac{\alpha-\mu_2}{1-\sigma_2}$ and $\sigma_1+\alpha-1<\mu_1<\alpha$.

We have

$$\theta(t)=t^{\frac{\alpha-\mu_1}{1-\sigma_1}}\tilde{L}_1^{\frac{1}{1-\sigma_1}}(t), \quad t\in(0,1).$$

Then we get for $x \in (0,1)$,

$$\begin{aligned} w(x) &\approx x^{-\lambda_1+(\alpha-2)\sigma_1} L_1(x)(1-x)^{-\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}} \tilde{L}_1^{\frac{1}{1-\sigma_1}}(1-x) \\ &+ x^{-\lambda_2+(\alpha-2)\sigma_2} L_2(x)(1-x)^{\frac{\alpha-\mu_1}{1-\sigma_1}\sigma_2-\mu_2} \tilde{L}_2(1-x)\tilde{L}_1^{\frac{\sigma_2}{1-\sigma_1}}(1-x). \end{aligned}$$

Since $\frac{\alpha-\mu_1}{1-\sigma_1} < \frac{\alpha-\mu_2}{1-\sigma_2}$ and the functions $t \mapsto \tilde{L}_1^{\frac{1}{1-\sigma_1}}(t)$ and $t \mapsto (\tilde{L}_2\tilde{L}_1^{\frac{\sigma_2}{1-\sigma_1}})(t)$ are in $\mathcal{K}$, we deduce again by Proposition 1 $(iii)$ that

$$\begin{aligned} w(x) &\approx x^{-\lambda_1+(\alpha-2)\sigma_1} L_1(x)(1-x)^{-\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}} \tilde{L}_1^{\frac{1}{1-\sigma_1}}(1-x) \\ &+ x^{-\lambda_2+(\alpha-2)\sigma_2} L_2(x)(1-x)^{-\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}} \tilde{L}_1^{\frac{1}{1-\sigma_1}}(1-x) \\ &:= w_1(x)+w_2(x). \end{aligned}$$

Using the fact that $\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1} \in (\alpha-1,\alpha)$, then it follows by Lemma 2 that

$$\int_0^\eta s^{\alpha-1-\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}} \tilde{L}_1^{\frac{1}{1-\sigma_1}}(s)ds < \infty.$$

Hence, applying Proposition 2, for $\mu = \frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}$, we obtain that

$$x^{2-\alpha}G_\alpha w_1(x) + x^{2-\alpha}G_\alpha w_2(x) \approx (1-x)^{\alpha-\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}} \tilde{L}_1^{\frac{1}{1-\sigma_1}}(1-x).$$

We conclude that

$$\begin{aligned} x^{2-\alpha}G_\alpha w(x) &\approx (1-x)^{\frac{\alpha-\mu_1}{1-\sigma_1}} \tilde{L}_1^{\frac{1}{1-\sigma_1}}(1-x) \\ &= \theta(1-x). \end{aligned}$$

**Case 3.** $\frac{\alpha-\mu_1}{1-\sigma_1} = \frac{\alpha-\mu_2}{1-\sigma_2}$ and $\sigma_1+\alpha-1 < \mu_1 < \alpha$.
Put $\tilde{L}(t) := \tilde{L}_1^{\frac{1}{1-\sigma_1}}(t) + \tilde{L}_2^{\frac{1}{1-\sigma_2}}(t)$, for $t \in (0,1)$. So, we have

$$\theta(t) = t^{\frac{\alpha-\mu_1}{1-\sigma_1}} \tilde{L}(t), \quad t \in (0,1).$$

By calculus, we obtain for $x \in (0,1)$,

$$\begin{aligned} w(x) &\approx x^{-\lambda_1+(\alpha-2)\sigma_1} L_1(x)(1-x)^{-\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1}} \tilde{L}_1(1-x)\tilde{L}^{\sigma_1}(1-x) \\ &+ x^{-\lambda_2+(\alpha-2)\sigma_2} L_2(x)(1-x)^{-\frac{\mu_1-\alpha}{1-\sigma_1}\sigma_2-\mu_2} \tilde{L}_2(1-x)\tilde{L}^{\sigma_2}(1-x) \\ &:= w_1(x)+w_2(x). \end{aligned}$$

Since $\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1} = \frac{\mu_1-\alpha}{1-\sigma_1}\sigma_2+\mu_2$, $\frac{\mu_1-\alpha\sigma_1}{1-\sigma_1} \in (\alpha-1,\alpha)$ and the functions $t \mapsto (\tilde{L}_1\tilde{L}^{\sigma_1})(t)$ and $t \mapsto (\tilde{L}_2\tilde{L}^{\sigma_2})(t)$ are in $\mathcal{K}$, it follows by Proposition 2, that

$$x^{2-\alpha}G_\alpha w_1(x) \approx (1-x)^{\frac{\alpha-\mu_1}{1-\sigma_1}} \tilde{L}_1(1-x)\tilde{L}^{\sigma_1}(1-x)$$

and

$$x^{2-\alpha}G_\alpha w_2(x) \approx (1-x)^{\frac{\alpha-\mu_1}{1-\sigma_1}} \tilde{L}_2(1-x)\tilde{L}^{\sigma_2}(1-x).$$

Now, using Lemma 3, we conclude that

$$\begin{aligned} x^{2-\alpha}G_\alpha w(x) &\approx (1-x)^{\frac{\alpha-\mu_1}{1-\sigma_1}} \tilde{L}(1-x) \\ &= \theta(1-x). \end{aligned}$$

**Case 4.** $\mu_1 = \sigma_1+\alpha-1$ and $\mu_2 = \sigma_2+\alpha-1$.
Put

$$M(t) := \left(\int_t^\eta \frac{\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma_1}} + \left(\int_t^\eta \frac{\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma_2}}, \quad t \in (0,1).$$

Then we have

$$\theta(t) = tM(t), \quad t \in (0,1).$$

We obtain, for $x \in (0,1)$,

$$\begin{aligned} w(x) &\approx x^{-\lambda_1-(2-\alpha)\sigma_1} L_1(x)(1-x)^{-(\alpha-1)} \tilde{L}_1(1-x) M^{\sigma_1}(1-x) \\ &+ x^{-\lambda_2-(2-\alpha)\sigma_2} L_2(x)(1-x)^{-(\alpha-1)} \tilde{L}_2(1-x) M^{\sigma_2}(1-x) \\ &:= w_1(x) + w_2(x). \end{aligned}$$

Using Lemma 1 $(i)$ and Proposition 1 $(ii)$, we have $t \mapsto (\tilde{L}_1 M^{\sigma_1})(t)$ and $t \mapsto (\tilde{L}_2 M^{\sigma_2})(t)$ belong to $\mathcal{K}$. Then applying Proposition 2, for $\mu = \alpha - 1$, we deduce that

$$x^{2-\alpha} G_\alpha w_1(x) + x^{2-\alpha} G_\alpha w_2(x) \approx (1-x) \int_{1-x}^{\eta} \frac{(M^{\sigma_1} \tilde{L}_1 + M^{\sigma_2} \tilde{L}_2)(s)}{s} ds.$$

Hence, it follows from Lemma 4 that

$$\begin{aligned} x^{2-\alpha} G_\alpha w_1(x) + x^{2-\alpha} G_\alpha w_2(x) &\approx (1-x) M(1-x) \\ &= \theta(1-x). \end{aligned}$$

**Case 5.** $\mu_1 = \sigma_1 + \alpha - 1$ and $\mu_2 < \sigma_2 + \alpha - 1$.
Put

$$b(t) := \left( \int_t^\eta \frac{\tilde{L}_1(s)}{s} ds \right)^{\frac{1}{1-\sigma_1}}, \quad t \in (0,1).$$

Then we have

$$\theta(t) = tb(t), \quad t \in (0,1).$$

By calculus, we obtain, for $x \in (0,1)$,

$$\begin{aligned} w(x) &\approx x^{-\lambda_1-(2-\alpha)\sigma_1} L_1(x)(1-x)^{1-\alpha} \tilde{L}_1(1-x) b^{\sigma_1}(1-x) \\ &+ x^{-\lambda_2-(2-\alpha)\sigma_2} L_2(x)(1-x)^{-\mu_2+\sigma_2} \tilde{L}_2(1-x) b^{\sigma_2}(1-x). \end{aligned}$$

Since $\mu_2 - \sigma_2 < \alpha - 1$ and the functions $t \mapsto (\tilde{L}_1 b^{\sigma_1})(t)$ and $t \mapsto (\tilde{L}_2 b^{\sigma_2})(t)$ are in $\mathcal{K}$, we deduce by Proposition 1 $(iii)$ that

$$\begin{aligned} w(x) &\approx x^{-\lambda_1-(2-\alpha)\sigma_1} L_1(x)(1-x)^{1-\alpha} \tilde{L}_1(1-x) b^{\sigma_1}(1-x) \\ &+ x^{-\lambda_2-(2-\alpha)\sigma_2} L_2(x)(1-x)^{1-\alpha} \tilde{L}_1(1-x) b^{\sigma_1}(1-x) \\ &:= w_1(x) + w_2(x). \end{aligned}$$

Hence, putting $\mu = \alpha - 1$ in Proposition 2, we conclude that

$$\begin{aligned} x^{2-\alpha} G_\alpha w(x) &\approx x^{2-\alpha} G_\alpha w_1(x) + x^{2-\alpha} G_\alpha w_2(x) \\ &\approx (1-x) \left( \int_{1-x}^{\eta} \frac{\tilde{L}_1(s) b^{\sigma_1}(s)}{s} ds \right) \approx \theta(1-x). \end{aligned}$$

**Case 6.** $\mu_1 < \sigma_1 + \alpha - 1$.
We have

$$\theta(t) = t, \quad t \in (0,1).$$

We obtain for $x \in (0,1)$,

$$\begin{aligned} w(x) &\approx x^{-\lambda_1-(2-\alpha)\sigma_1} L_1(x)(1-x)^{-\mu_1+\sigma_1} \tilde{L}_1(1-x) \\ &+ x^{-\lambda_2-(2-\alpha)\sigma_2} L_2(x)(1-x)^{-\mu_2+\sigma_2} \tilde{L}_2(1-x) \\ &:= w_1(x) + w_2(x). \end{aligned}$$

Since $\mu_1 - \sigma_1 < \alpha - 1$, then by applying Proposition 2, for $\mu = \mu_1 - \sigma_1$, we get

$$x^{2-\alpha} G_\alpha w_1(x) \approx (1-x).$$

Moreover, using the fact that $\frac{\alpha-\mu_1}{1-\sigma_1} < \frac{\alpha-\mu_2}{1-\sigma_2}$, we have $\mu_2 - \sigma_2 < \alpha - 1$. Applying again Proposition 2, for $\mu = \mu_2 - \sigma_2$, we get that

$$x^{2-\alpha} G_\alpha w_2(x) \approx (1-x).$$

So, we conclude that

$$\begin{array}{rcl} x^{2-\alpha} G_\alpha w(x) & \approx & (1-x) \\ & = & \theta(1-x). \end{array}$$

**Case 7.** $\mu_1 = \mu_2 = \alpha$.
We have

$$\theta(t) = \left( \int_0^t \frac{\tilde{L}_1(s)}{s} ds \right)^{\frac{1}{1-\sigma_1}} + \left( \int_0^t \frac{\tilde{L}_2(s)}{s} ds \right)^{\frac{1}{1-\sigma_2}}, \quad t \in (0,1).$$

We get, for $x \in (0,1)$,

$$\begin{array}{rcl} w(x) & \approx & x^{-\lambda_1-(2-\alpha)\sigma_1} L_1(x)(1-x)^{-\alpha} \tilde{L}_1(1-x) \theta^{\sigma_1}(1-x) \\ & + & x^{-\lambda_2-(2-\alpha)\sigma_2} L_2(x)(1-x)^{-\alpha} \tilde{L}_2(1-x) \theta^{\sigma_2}(1-x) \\ & := & w_1(x) + w_2(x). \end{array}$$

Since $t \mapsto (\tilde{L}_1 \theta^{\sigma_1})(t)$ and $t \mapsto (\tilde{L}_2 \theta^{\sigma_2})(t)$ belong to $\mathcal{K}$. It follows from Lemma 5 that

$$\int_0^\eta \frac{(\theta^{\sigma_1} \tilde{L}_1 + \theta^{\sigma_2} \tilde{L}_2)(s)}{s} ds \approx \theta(\eta) < \infty.$$

Applying Proposition 2, for $\mu = \alpha$, we deduce that

$$x^{2-\alpha} G_\alpha w_1(x) + x^{2-\alpha} G_\alpha w_2(x) \approx \int_0^{1-x} \frac{(\theta^{\sigma_1} \tilde{L}_1 + \theta^{\sigma_2} \tilde{L}_2)(s)}{s} ds.$$

The result follows from Lemma 5. This ends the proof.

# 4 Proof of Theorem 3

## 4.1 Existence and asymptotic behavior

Let $a_1, a_2$ be two functions satisfying $(H)$. Using Theorem 2, there exists $m > 1$ such that for each $x \in (0,1)$

$$\frac{1}{m} \theta(1-x) \leq x^{2-\alpha} G_\alpha w(x) \leq m \theta(1-x). \tag{4.1}$$

Put $\sigma := \max(|\sigma_1|, |\sigma_2|)$ and $c_0 := m^{\frac{1}{1-\sigma}}$. We consider the closed convex set given by

$$Y := \{v \in \mathcal{C}([0,1]); \frac{1}{c_0} \theta(1-x) \leq v(x) \leq c_0 \theta(1-x)\}.$$

Obviously, the function $x \mapsto x^{2-\alpha} G_\alpha w(x)$ belongs to $\mathcal{C}([0,1])$ and satisfies (4.1). So $Y$ is not empty. In order to use the Schauder's fixed point theorem, we denote for $i \in \{1,2\}$, $\tilde{a}_i(x) = x^{(\alpha-2)\sigma_i} a_i(x)$ and we define the operator $T$ on $Y$ by

$$Tv(x) = x^{2-\alpha} G_\alpha (\tilde{a}_1 v^{\sigma_1} + \tilde{a}_2 v^{\sigma_2})(x).$$

We need to check that the operator $T$ has a fixed point $v$ in $Y$. For this choice of $c_0$, we will prove that $T$ maps $Y$ into itself. Indeed, let $v \in Y$, by using (4.1), we have

$$\begin{aligned} Tv(x) &\leq c_0^{\sigma} x^{2-\alpha} G_\alpha w(x) \\ &\leq c_0^{\sigma} m\theta(1-x) \\ &= c_0\theta(1-x) \end{aligned}$$

and

$$\begin{aligned} Tv(x) &\geq c_0^{-\sigma} x^{2-\alpha} G_\alpha w(x) \\ &\geq \frac{1}{c_0^{\sigma} m}\theta(1-x) \\ &= \frac{1}{c_0}\theta(1-x). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} Tv(x) &= x^{2-\alpha}\int_0^1 G_\alpha(x,t)\,(\tilde{a}_1(t)v^{\sigma_1}(t)+\tilde{a}_2(t)v^{\sigma_2}(t))dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 [(1-t)^{\alpha-1}-x^{2-\alpha}((x-t)^+)^{\alpha-1}]\,(\tilde{a}_1(t)v^{\sigma_1}(t)+\tilde{a}_2(t)v^{\sigma_2}(t))dt \\ &\leq c\int_0^1 (1-t)^{\alpha-1}(\tilde{a}_1(t)\theta^{\sigma_1}(1-t)+\tilde{a}_2(t)\theta^{\sigma_2}(1-t))dt. \end{aligned}$$

Using the fact that $(x,t) \to (1-t)^{\alpha-1}-x^{2-\alpha}((x-t)^+)^{\alpha-1}$ is continuous on $[0,1]\times(0,1)$ and $t \to (1-t)^{\alpha-1}(\tilde{a}_1(t)\theta^{\sigma_1}(1-t)+\tilde{a}_2(t)\theta^{\sigma_2}(1-t))$ is integrable on $(0,1)$, we deduce that the function $Tv$ is in $\mathcal{C}([0,1])$. So $Y$ is invariant under $T$. Also, we conclude that $Tv$ is uniformly bounded.

Now, let us prove that $TY$ is equicontinuous in $[0,1]$. Let $x,y \in [0,1]$ and $v \in Y$, then we have

$$|Tv(x)-Tv(y)| \leq c\int_0^1 |x^{2-\alpha}G_\alpha(x,t)-y^{2-\alpha}G_\alpha(y,t)|\,(\tilde{a}_1(t)\theta^{\sigma_1}(1-t)+\tilde{a}_2(t)\theta^{\sigma_2}(1-t))dt.$$

For every $t \in (0,1)$, we have

$$\begin{aligned} |x^{2-\alpha}G_\alpha(x,t)-y^{2-\alpha}G_\alpha(y,t)| &\leq x^{2-\alpha}G_\alpha(x,t)+y^{2-\alpha}G_\alpha(y,t) \\ &\leq c(1-t)^{\alpha-1}. \end{aligned}$$

As it mentioned, we have $(x,t) \to \left((1-t)^{\alpha-1}-x^{2-\alpha}((x-t)^+)^{\alpha-1}\right)$ is continuous on $[0,1]\times(0,1)$ and $t \to (1-t)^{\alpha-1}(\tilde{a}_1(t)\theta^{\sigma_1}(1-t)+\tilde{a}_2(t)\theta^{\sigma_2}(1-t))$ is integrable on $(0,1)$, then we obtain by Lesbegue's theorem that

$$|Tv(x)-Tv(y)| \longrightarrow 0 \ \text{ as } |x-y| \longrightarrow 0.$$

Hence, by Ascoli's theorem, we conclude that $TY$ is relatively compact in $\mathcal{C}([0,1])$.

Next, we shall prove the continuity of $T$. Let $(v_k)_k$ be a sequence in $Y$ which converges uniformly to $v$ in $Y$. Using the same arguments above, we obtain that for $x \in [0,1]$

$$Tv_k(x) \longrightarrow Tv(x), \text{ as } k \longrightarrow \infty.$$

Since $TY$ is a relatively compact in $\mathcal{C}([0,1])$, we have the uniform convergence, namely

$$\|Tv_k - Tv\|_\infty \longrightarrow 0, \text{ as } k \longrightarrow \infty.$$

Thus, we have proved that $T$ is a compact mapping from $Y$ into itself. It follows by the Schauder fixed point theorem that there exists $v \in Y$ such that $Tv = v$. Put

$$u(x) = x^{\alpha-2}v(x),$$

then $u \in \mathcal{C}_{2-\alpha}([0,1])$ and $u$ satisfies the equation

$$u(x) = G_\alpha(a_1u^{\sigma_1}+a_2u^{\sigma_2})(x).$$

Since the function $t \to (1-t)^{\alpha-1}(a_1(t)u^{\sigma_1}(t)+a_2(t)u^{\sigma_2}(t))$ is continuous and integrable on $(0,1)$, then by $(1.4)$ the function $u$ is a positive continuous solution of problem $(1.1)$.

## 4.2 Uniqueness

Assume that $a_1, a_2$ satisfy $(H)$. Let $u$ and $v$ be two positives solutions of problem $(1.1)$ satisfying $(1.8)$. Then, there exists a constant $m > 1$ such that

$$\frac{1}{m}v \leq u \leq mv.$$

This implies that the set

$$J := \left\{ t \in (1, \infty) : \frac{1}{t}v \leq u \leq tv \right\}$$

is not empty. Now, put $c := \inf J$, then we aim to show that $c = 1$. Suppose that $c > 1$ and we put $\sigma := \max(|\sigma_1|, |\sigma_2|)$. Then we obtain

$$\begin{cases} -D^{\alpha}(c^{\sigma}v - u) = a_1(c^{\sigma}v^{\sigma_1} - u^{\sigma_1}) + a_2(c^{\sigma}v^{\sigma_2} - u^{\sigma_2}) \geq 0, \\ \lim\limits_{x \to 0^+} D^{\alpha-1}(c^{\sigma}v - u)(x) = 0, \ \ (c^{\sigma}v - u)(1) = 0. \end{cases}$$

We conclude by $(1.4)$ that

$$c^{\sigma}v - u = G_{\alpha}(a_1(c^{\sigma}v^{\sigma_1} - u^{\sigma_1}) + a_2(c^{\sigma}v^{\sigma_2} - u^{\sigma_2})) \geq 0.$$

By symmetry, we obtain that $v \leq c^{\sigma}u$. So $c^{\sigma} \in J$. Since $\sigma < 1$ and $c > 1$, we have $c^{\sigma} < c$. This yields to a contradiction with the fact that $c = \inf J$. Hence $c = 1$ and consequently $u = v$.

As an application of Theorem 3, we give the following example.

**Example 13** *Let $\sigma_1 \in (-1, 0)$ and $\sigma_2 \in (0, 1)$. Let $a_1$ and $a_2$ be two positive continuous functions on $(0, 1)$ such that*

$$a_1(x) \approx (1-x)^{-\mu_1} \left( \log(\frac{3}{1-x}) \right)^{-2}$$

*and*

$$a_2(x) \approx (1-x)^{-\mu_2},$$

*where $\mu_1 \leq \alpha$ and $\mu_2 < \alpha$. We suppose that $\frac{\alpha-\mu_1}{1-\sigma_1} \leq \frac{\alpha-\mu_2}{1-\sigma_2}$, then problem $(1.1)$ has a unique positive continuous solution $u$ satisfying for $x \in (0, 1)$,*

$$u(x) \approx x^{\alpha-2} \begin{cases} \left(\log(\frac{3}{1-x})\right)^{\frac{-1}{1-\sigma_1}}, & if\ \mu_1 = \alpha, \\ (1-x)^{\frac{\alpha-\mu_1}{1-\sigma_1}} \left(\log(\frac{3}{1-x})\right)^{\frac{-2}{1-\sigma_1}}, & if\ \sigma_1 + \alpha - 1 < \mu_1 < \alpha, \\ (1-x) \left(\log(\frac{3}{1-x})\right)^{\frac{1}{1-\sigma_2}}, & if\ \mu_1 = \sigma_1 + \alpha - 1 \text{ and } \mu_2 = \sigma_2 + \alpha - 1, \\ (1-x) \left(\log(\frac{3}{1-x})\right)^{\frac{1}{1-\sigma_1}}, & if\ \mu_1 = \sigma_1 + \alpha - 1 \text{ and } \mu_2 < \sigma_2 + \alpha - 1, \\ (1-x), & if\ \mu_1 < \sigma_1 + \alpha - 1. \end{cases}$$

# References


[1] R. P. Agarwal et al. Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. *J. Math. Anal. Appl.* 371(1)(2010): 57-68.

[2] R. P. Agarwal et al. Boundary value problems for differential equations involving Riemann-Liouville fractional derivative on the half line. *Dyn. Contin. Discrete Impulsive Syst.* A. 18(2)(2011): 235-244.

[3] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations. *Appl. Math. Lett.* 23(2010): 390-394.

[4] B. Ahmad and J. J. Nieto. Rieman-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. *Bound. Value Probl.* 36(2011).

[5] Z. Bai et al. Positive solutions for boundary value problem of nonlinear fractional differential equations. *J. Math. Anal. Appl.* 311(2005): 495-505.

[6] R. Chemmam et al. Combined effects in nonlinear singular elliptic problems in a bounded domain, *Advances in Nonlinear Analysis* 1(2012): 301-318.

[7] J. Deng and L. Ma. Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations. *Appl. Math. Lett.* 23(2010): 676-680.

[8] K. Diethelm and A. D. Freed. On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, in Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties. *Springer-Verlag, Heidelberg.* (1999): 217-307.

[9] A. A. Kilbas et al. Theory and Applications of Fractional Differential Equations. *Elsevier, Amsterdam.* (2006).

[10] Y. Liu et al. A sufficient condition for the existence of a positive solution for a nonlinear fractional differential equation with the Riemann-Liouville derivative. *Appl. Math. Lett.* 25(2012): 1986-1992.

[11] H. Mâagli et al. Existence and exact asymptotic behavior of positive solutions for a fractional boundary value problem. *Abstract and Applied Analysis* 6(2013).

[12] I. Podlubny. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calc. App. Anal.* 5(2002): 367-386.

[13] T. Qiu and Z. Bai. Existence of positive solutions for singular fractional differential equations. *Electron. J. Differential Equations* 146(2008): 1-9.

[14] R. Seneta. Regular Varying Functions, Lectures Notes in Math. *Springer-Verlag.* 508 (1976).

[15] Y. Zhao et al. Positive Solutions to boundary value problems of nonlinear fractional differential equations, *Abstract. Appl. Anal.* (2011): 16-25.