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Application of Method of Homotopy Perturbation Technique with Parameters Expanding to Strongly Nonlinear Equations

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Abstract: A coupling method of a homotopy perturbation technique and a parameter expanding technique are used to find solution of strongly nonlinear system. The method is effective for a nonlinear equation with multiplier nonlinear terms, which might have different effects on the solution. A strongly nonlinear equation is used as an example to elucidate the solution procedure. The obtained frequency is of high accuracy and is valid for the whole solution domain.

Keywords: Homotopy perturbation method; Parameter expanding; Nonlinear equation

1 Introduction

In the last decades, the analytical asymptotic technique has confirmed its effectiveness in dealing with nonlinear problems, which is described by nonlinear equations or maps. Various methods of nonlinear analytical techniques for solving nonlinear problems have been proposed, such as, variational approaches[1-5], parameter-expanding method[6-8], parametrized perturbation method[9], homotopy perturbation method[10-14], iteration perturbation method[15],and exp-function method[16]. Recently, Wu presented the variational iteration method to solve the fractional nonlinear equation [17-19].

Among the previous methods presented above, the homotopy perturbation method, which is proposed and development by He[20], can readily eliminate the limitations of classical perturbation techniques,and the solution procedure is very simple, only a few iterations lead to high accurate solutions which are valid for the whole solution domain. The most important step of the homotopy perturbation method is to construct a suitable homotopy equation, which can approximately describe solution properties when homotopy parameter is zero. In this paper, a coupling method of a homotopy technique and a parameter expanding technique are used to solve the nonlinear problems with multiplier nonlinear terms.

2 Homotopy Perturbation Method with Two Expanding Parameters

Consider a general nonlinear equation

$$Lu + Mu + Nu = 0, \tag{1}$$

where L is a linear operator, M and N are nonlinear operators. A critical step for using the homotopy perturbation method is to construct a homotopy equation. In equation (1) there are two nonlinear operators, which might have different effects on the solution. To find potential appropriate solutions for these nonlinear operators, a homotopy equation can be designed as

$$\widetilde{L}u + p_1 Mu + p_2 Nu + p_3(Lu - \widetilde{L}u) = 0, \tag{2}$$

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where $\widetilde{L}$ is the linear operator with some possible unknown parameters, and the solution of $\widetilde{L}u = 0$ can approximately describe the original nonlinear equations. The embedding parameters p_1, p_2, p_3 monotonically increase from zero to unit as the trivial problem $Lu = 0$ is continuous deformed to the original one $Lu + Mu + Nu = 0$.

The solution of equation (2) can be approached as followers

$$u = u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \cdots . \tag{3}$$

Setting $p_i = 1, i = 1, 2, \cdots, k$, one of its potential solution can be found as

$$u = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots . \tag{4}$$

This will lead to an infinite series solution which converges to the exact solution. The method is extensively adopted for classical perturbation method, which is very valid for strongly nonlinear equations.

For example, consider a nonlinear equation

$$u'' + au + bu^3 + cu^5 = 0,\ u(0) = A,\ u'(0) = 0. \tag{5}$$

We can construct the following homotopy equation, that is:

$$u'' + au + p_1 bu^3 + p_2 cu^5 = 0,\ u(0) = A,\ u'(0) = 0. \tag{6}$$

According to (4), solution of equation (5) is expanded in the form

$$u = u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \cdots . \tag{7}$$

Using the parameter-expansion method, the coefficient a of the linear term in equation (5) is also expanded into a series in p, written as:

$$a = \omega_0^2 + p_1 \omega_1 + p_2 \omega_2 + p_1^2 \omega_3 + p_1 p_2 \omega_4 + p_2^2 \omega_5 + \cdots . \tag{8}$$

where ω_0 and $\omega_i, i = 1, 2, \cdots, +\infty$ are unknown constants to be further determined.

Substituting equations (7) and (8) into equation (6), we gets

$$\begin{aligned}(u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \cdots)'' + (\omega_0^2 + p_1 \omega_1 + p_2 \omega_2 + p_1^2 \omega_3 + p_1 p_2 \omega_4 + p_2^2 \omega_5 + \cdots) \cdot \\ (u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \cdots) + p_1 b (u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \cdots)^3 \\ + p_2 c (u_0 + p_1 u_1 + p_2 u_2 + p_1^2 u_3 + p_1 p_2 u_4 + p_2^2 u_5 + \cdots)^5 = 0.\end{aligned} \tag{9}$$

Collecting the same power of $p_1^m p_2^n, m, n = 0, 1, 2, 3, \cdots$, and setting coefficients zero, we can obtain the following equations:

$$u_0'' + \omega_0^2 u_0 = 0,\ u(0) = A,\ u'(0) = 0, \tag{10}$$

$$u_1'' + \omega_0^2 u_1 + \omega_1 u_0 + bu_0^3 = 0, \tag{11}$$

$$u_2'' + \omega_0^2 u_2 + \omega_2 u_0 + cu_0^5 = 0, \tag{12}$$

$$u_3'' + \omega_0^2 u_3 + \omega_1 u_1 + \omega_3 u_0 + 3bu_0^2 u_1 = 0, \tag{13}$$

$$u_4'' + \omega_0^2 u_4 + \omega_1 u_2 + \omega_2 u_1 + \omega_4 u_0 + 3bu_0^2 u_2 + 5cu_0^4 u_1 = 0, \tag{14}$$

$$u_5'' + \omega_0^2 u_5 + \omega_2 u_2 + \omega_5 u_0 + 5cu_0^4 u_2 = 0. \tag{15}$$

$$\cdots$$

The initial conditions for $u_1, u_2, u_3, \cdots$ should satisfy

$$u_1(0) + u_2(0) + u_3(0) + \cdots = 0, \tag{16}$$

$$u_1'(0) + u_2'(0) + u_3'(0) + \cdots = 0. \tag{17}$$

Solving equation (10), we have

$$u_0(t) = A \cos \omega_0 t. \tag{18}$$

Substitution of u_0 into equation (11) results in

$$u_1'' + \omega_0^2 u_1 + A\left(\omega_1 + \frac{3bA^2}{4}\right)\cos\omega_0 t + \frac{bA^3\cos 3\omega_0 t}{4} = 0. \tag{19}$$

No secular term in u_1 requires

$$\omega_1 = -\frac{3bA^2}{4}. \tag{20}$$

A particular solution for equation (19) is readily obtained as

$$u_1 = \frac{bA^3}{32\omega_0^2}\cos 3\omega_0 t. \tag{21}$$

Substitution of u_0 into equation (12), we can get

$$u_2'' + \omega_0^2 u_2 + A\left(\omega_2 + \frac{5cA^4}{8}\right)\cos\omega_0 t + \frac{5cA^5\cos 3\omega_0 t}{16} + \frac{cA^5\cos 5\omega_0 t}{16} = 0. \tag{22}$$

To avoid a secular term in u_2, we set

$$\omega_2 = -\frac{5cA^4}{8}. \tag{23}$$

We obtain a particular solution for equation (22):

$$u_2(t) = \frac{5cA^5}{128\omega_0^2}\cos 3\omega_0 t + \frac{cA^5}{384\omega_0^2}\cos 5\omega_0 t. \tag{24}$$

Substituting the obtained results into equation (13),we obtain equation for u_3

$$u_3'' + \omega_0^2 u_3 + A\left(\omega_3 + \frac{3b^2A^4}{128\omega_0^2}\right)\cos\omega_0 t + \frac{3b^2A^5\cos 3\omega_0 t}{128} + \frac{3b^2A^5\cos 5\omega_0 t}{128\omega_0^2} = 0. \tag{25}$$

Absence of a secular term in u_3 requires

$$\omega_3 = -\frac{3b^2A^4}{128\omega_0^2}. \tag{26}$$

By using a similar calculation as above, we identify ω_4 and ω_5 in view of no secular term in u_4 and u_5 respectively, yielding

$$\omega_4 = -\frac{5bcA^6}{64\omega_0^2}. \tag{27}$$

and

$$\omega_5 = -\frac{95c^2A^8}{1536\omega_0^2}. \tag{28}$$

The solution process can continue without any difficult. Substituting the identified $\omega_i, i = 1,2,3,4,5$ into equation (8) and setting $p_1 = 1$ and $p_2 = 1$, we have

$$a = \omega_0^2 - \frac{3bA^2}{4} - \frac{5cA^4}{8} - \frac{3b^2A^4}{128\omega_0^2} - \frac{5bcA^6}{64\omega_0^2} - \frac{95c^2A^8}{1536\omega_0^2}, \tag{29}$$

which leads to

$$\omega_0 = \sqrt{\frac{\left(\frac{3bA^2}{4} + \frac{5cA^4}{8} + a\right) + \sqrt{\left(\frac{3bA^2}{4} + \frac{5cA^4}{8} + a\right)^2 + 4\left(\frac{3b^2A^4}{128} + \frac{5bcA^6}{64} + \frac{95c^2A^8}{1536}\right)}}{2}}. \tag{30}$$

So we can obtain the approximating solution of equation (5), written as

$$u(t) = A\cos\omega_0 t + \left(\frac{bA^3}{32\omega_0^2} + \frac{5cA^5}{128\omega_0^2}\right)\cos 3\omega_0 t + \frac{cA^5}{384\omega_0^2}\cos 5\omega_0 t + \cdots. \tag{31}$$

3 Results and discussion

If $a = c = 0$, the equation (5) is reduced as

$$u'' + bu^3 = 0. \tag{32}$$

Its exact period can be readily obtained, which reads

$$T_{ex} = 4\sqrt{2}\int_0^{\pi/2} \frac{\sin x dx}{\sqrt{bA^2 \sin^2 x(1+\cos^2 x)}} = \frac{6.743}{\sqrt{b}A}. \tag{33}$$

If only the first order approximate solution of equation (32) is searched for, then from equation (30),we have

$$\omega = \sqrt{\frac{3bA^2}{4}}. \tag{34}$$

Its period can be rewritten as follows

$$T = \frac{7.25}{\sqrt{b}A}. \tag{35}$$

It is obvious that the maximal relative error is less than 7.5%, and the obtained approximate period is vaild for all $b > 0$. To search for n-th order approximate solution, the equation (30) give a reliable way to find its period as follows

$$T = \frac{2\pi}{\omega}. \tag{36}$$

Homotopy perturbation method with parameters expanding is a kind of asymptotic methods, though the higher order approximate solution leads to higher accuracy of the period, and the error for amplitude might become larger. In case the amplitude does not vary with time as illustrated in the above example, we always use frist order approximate solution:

$$u(t) = A\cos(\omega_0 t), \tag{37}$$

where

$$\omega_0 = \sqrt{1 + \frac{3bA^2}{4}}. \tag{38}$$

Comparison of approximate solution,equation (37), with exact solution is shown in figure 1-5.

4 Conclusions

The method is an extension to traditional perturbation method, which is proved to be a powerful mathematical tool to find solution of nonlinear equations. One iteration is enough, and the obtained frequency is of high accuracy. The obtained results are valid for the whole solution domain for $0 < b < \infty, 0 < c < \infty$ and $0 < A < \infty$. The solution process can be used as a paradigm for many other applications.

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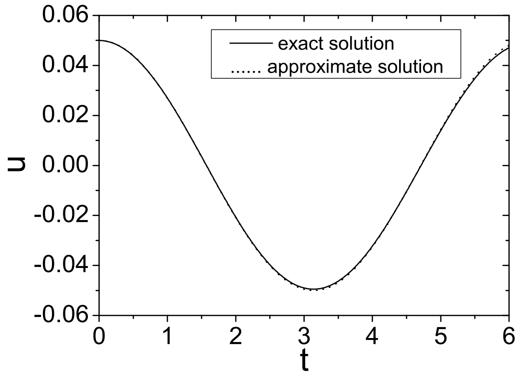

Figure 1: Comparison of approximate solution,equation (37), with exact solution. $a = b = c = 1, A = 0.05$

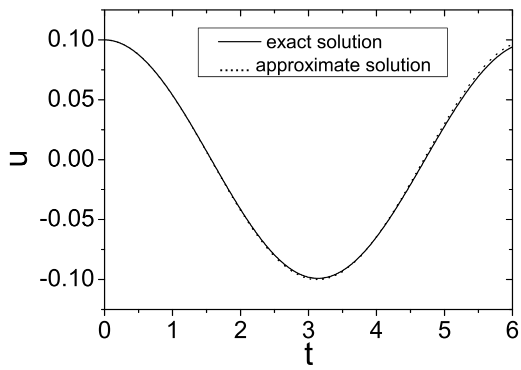

Figure 2: Comparison of approximate solution,equation (37), with exact solution. $a = b = c = 1, A = 0.1$

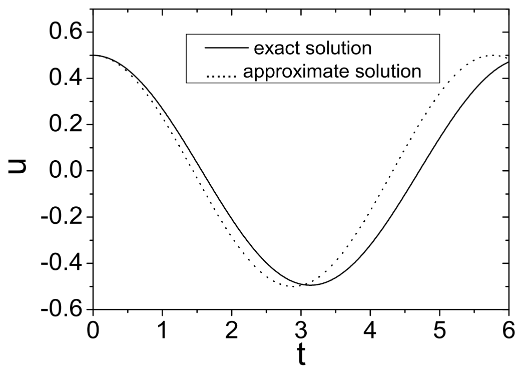

Figure 3: Comparison of approximate solution,equation (37), with exact solution. $a = b = c = 1, A = 0.5$

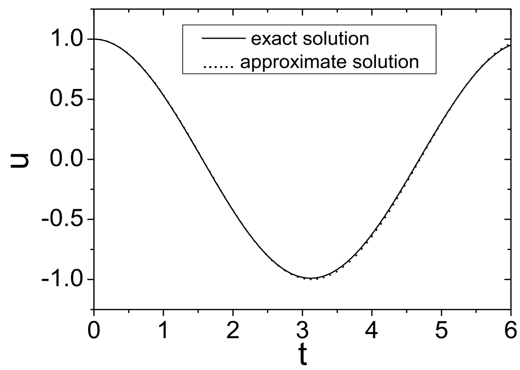

Figure 4: Comparison of approximate solution,equation (37), with exact solution. $a = 1, b = c = 0.01, A = 1$

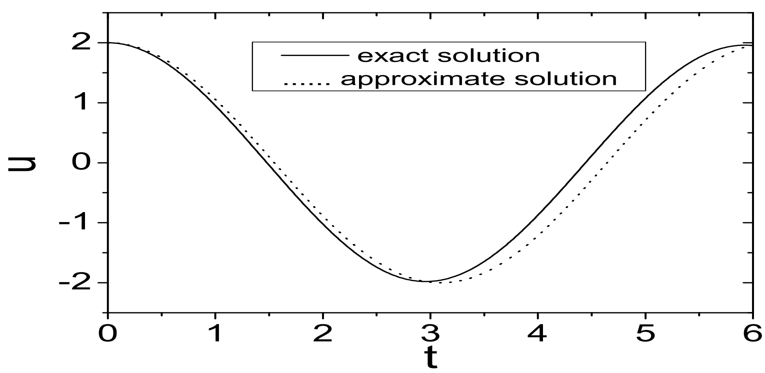

Figure 5: Comparison of approximate solution,equation (37), with exact solution. $a = 1, b = c = 0.01, A = 2$

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