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Approximate Controllability of Semilinear Fractional Stochastic Control System

Anurag Shukla *, N. Sukavanam, D. N. Pandey
Department of Mathematics, Indian Institute of Technology Roorkee Roorkee-247667, India
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Abstract: The objective of this paper is to present some sufficient conditions for approximate controllability of semilinear fractional stochastic control system. The results are hold only for fractional order α such that $1/2 < \alpha < 1$, when the nonlinear function is Lipschitz continuous. Sufficient conditions are obtained by separating the given fractional semilinear stochastic system into two systems namely a semilinear fractional system and a fractional linear stochastic system. To prove our results, the Schauder fixed point theorem is applied. At the end, examples are given to show the effectiveness of the result.

Keywords: Approximate Controllability, Semilinear System, Stochastic System, Reachable set, Fundamental solution, Finite delay.

1 Introduction

Let X and U be the Hilbert spaces and $Z = L_2[0, b; X]$,$Z_h = L_2[-h, b; X]$, $0 < h < b$ and $Y = L_2[0, b; U]$ be function spaces. $\mathbb{R}^k$ denotes $k-$ dimensional real Euclidean space. Let (Ω, ζ, P) be the probability space with a probability measure P on Ω and a filtration $\{\zeta_t | t \in [0, b]\}$ generated by a Wiener Process $\{\omega(s) : 0 \le s \le t\}$.
We consider the semilinear stochastic control system of the form:

$$\begin{aligned} {}^cD_t^{\alpha} x(t) &= Ax(t) + Bu(t) + f(t, x(t)) + \frac{d\omega(t)}{dt},\ t > 0. \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

where the state function $x \in Z$; $A : D(A) \subseteq X \to X$ is a closed linear operator which generates a strongly continuous semigroup $T(t)$ on X; $B : Y \to Z$ is a bounded linear operator; function $f : [0, b] \times X \to X$ is a nonlinear operator such that, f is measurable with respect to t for all $x \in Z$ and continuous with respect to x for almost all $t \in [0, b]$. $Cov(x, y)$ denotes the covariance of random variables x and y. The initial condition x_0 is a Gaussian random variable with $cov x_0 = P_0$, ω is X-valued Wiener process, $\omega(0) = 0$, $\mathbb{E}\omega(t) = 0$, where $\mathbb{E}$ denotes the expectation. $Cov\omega(t) = M't$ where M' is the nuclear operator on X. Control $u(t)$ takes values in U for each $t \in [0, b]$.

Fractional order semilinear equations are abstract formulations for many problem arising in engineering and physics. The potential applications of fractional calculus are in diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory and several more.

The approximately controllability of the stochastic systems of integer order ($\alpha = 1, 2$) has been proved in [13, 17] among others. However there are only few papers which deal with the approximate controllability of fractional order stochastic systems. In [16] Sakthivel et al. proved the approximate controllability by assuming that the C_0- semigroup $T(t)$ is compact and the nonlinear function is continuous and uniformly bounded. Anurag et. al in $[2 - 5]$ obtained sufficient conditions for controllability of semilinear stochastic systems using fixed point theorems.

The main objective of this paper is to provide different sufficient conditions for the approximate controllability of fractional order semilinear stochastic system. To prove the results we use the technique similar to that of [13, 15] with suitable modifications so as to be compatible with fractional order stochastic systems. The uniform boundedness of nonlinear function assumed by other authors is replaced by Lipschitz continuity.

***Corresponding author**. *E-mail address*: anuragshukla259@gmail.com

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In sections (5) we also studied sufficient conditions for fractional semilinear stochastic system with finite delay.

By splitting the system (1.1), we get the following pair of coupled systems

$$\begin{aligned} {}^cD_t^\alpha y(t) &= Ay(t) + Bv(t) + f(t, y(t) + z(t)); \quad 0 \le t \le b. \\ y(0) &= y_0. \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} {}^cD_t^\alpha z(t) &= Az(t) + Bw(t) + \frac{d\omega(t)}{dt}; \quad 0 \le t \le b. \\ z(0) &= x_0 - y_0. \end{aligned} \tag{1.3}$$

The system represented by (1.3) is linear stochastic system with delay in state and for each realization $z(t)$ of system (1.3), the system given by (1.2) is a deterministic fractional system. Thus the solution $y(t)$ of the semilinear system (1.2) depends on the solution $z(t)$ of linear fractional stochastic system (1.3). The functions v and w are $Y-$valued control function, such that $u = v + w$.

It can be easily seen that, the solution $x(t)$ of the semilinear stochastic system (1.1) is given by $y(t) + z(t)$ where $y(t)$ and $z(t)$ are the solutions of the systems (1.2) and (1.3), respectively.

2 Preliminaries

Lemma 1. *In* [13]*, for $z \in Z$ and $n \in N_0(L)$; the following inequality holds*

$$||n||_Z \le (1 + c)||z||_Z \tag{2.1}$$

where c is such that $||G|| \le c$.

Proof: *Let $a \in N_0^\perp(L)$, then $Ga = a_0 = (n_0 + a) \in \overline{R(B)}$, for some $n_0 \in N_0(L)$ (n_0 is chosen such that $a_0 = n_0 + a$ satisfies the equation(3.1)). Further, since $z \in Z$ has unique decomposition namely, $z = n_1 + a$ where $n_1 \in N_0(L)$ and $a \in N_0^\perp(L)$, we get*

$$z = (n_1 - n_0) + (n_0 + a)$$

which gives a unique representation for z as follows

$$z = n + q : n \in N_0(L),\ q = a_0 \in \overline{R(B)},$$

where $a_0 = Ga$ and $n = n_1 - n_0$.

Now, $z = n_1 + a \Rightarrow ||z||^2 = ||n_1||^2 + ||a||^2$,
which implies that

$$||a|| \le ||z|| \tag{2.2}$$

Also consider

$$\begin{aligned} z = n + q : n \in N_0(L),\ q \in \overline{R(B)}, \\ \Rightarrow ||n|| = ||z - Ga||, \\ \Rightarrow ||n|| \le ||z|| + c||a||. \end{aligned} \tag{2.3}$$

From the equations (3.3) and (3.4), we conclude that

$$||n||_Z \le (1 + c)||z||_Z.$$

□

Definition 2. *The fractional integral of order α with the lower limit 0 for a function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(.)$ is gamma function.

Definition 3. *Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$$^{L}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}ds,\ t>0,\ n-1<\alpha<n.$$

Definition 4. *The Caputo derivative of order α for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$$^{c}D^{\alpha}f(t) = ^{L}D^{\alpha}\Bigg(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^k(0)\Bigg),\ t>0,\ n-1<\alpha<n.$$

If $f(t) \in C^n[0,\infty)$, then

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^n(s)ds = I^{n-\alpha}f^n(s),\ t>0,\ n-1<\alpha<n.$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$\mathbb{L}\{^{c}D^{\alpha}f(t);s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0);\ \ n-1\leq\alpha<n.$$

We define the mild solution of the systems (1.1) as

$$\begin{aligned} x(t) &= S_{\alpha}(t)x_0 + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)\{Bu(s)+f(s,x(s))\}ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)d\omega(s),\ t>0. \end{aligned} \tag{2.4}$$

the mild solution of the semilinear system (1.2), can be written as

$$y(t) = S_{\alpha}(t)y_0 + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)\{Bv(s)+f(s,y(s)+z(s))\}ds. \tag{2.5}$$

and the mild solution of the linear stochastic system (1.3), can be written as

$$z(t) = S(t)_{\alpha}(x_0-y_0) + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)Bw(s)ds + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)d\omega(s). \tag{2.6}$$

Consider the fractional linear system corresponding to the system (1.2), given by

$$\begin{aligned} ^{c}D_t^{\alpha}p(t) &= Ap(t)+Br(t),\quad t>0. \\ p(t) &= y_0. \end{aligned} \tag{2.7}$$

The mild solution of the above linear system is expressed as

$$p(t) = S_{\alpha}(t)y_0 + \int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)Br(s)ds\ t>0. \tag{2.8}$$

where

$$S_{\alpha}(t)x = \int_0^{\infty}\phi_{\alpha}(\theta)T(t^{\alpha}\theta)xd\theta.$$

$$T_{\alpha}(t)x = \alpha\int_0^{\infty}\theta\phi_{\alpha}(\theta)T(t^{\alpha}\theta)xd\theta.$$

Here $\phi_{\alpha}(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\varphi_{\alpha}(\theta^{-1/\alpha})$ is the probability density function defined on $(0,\infty)$, that is $\phi_{\alpha}(\theta) \geq 0$, and $\int_0^{\infty}\phi_{\alpha}(\theta)d\theta = 1$.We define $\varphi_{\alpha}(\theta)$ as

$$\varphi_{\alpha}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}sin(n\pi\alpha),\ \theta\in(0,\infty).$$

Definition 5. *The set given by* $K_T(f) = \{x(T) \in X : x \in Z_h\}$ *where* x *is a mild solution of* (1.1) *corresponding to control* $u \in Y$ *is called Reachable set of the system* (1.1).

Definition 6. *The system* (1.1) *is said to be approximately controllable if* $K_T(f)$ *is dense in* X*, means* $\overline{K_T(f)} = X$.

3 Controllability Results

In this section, sufficient conditions are established for the approximate controllability of control system (1.1) in the case when the nonlinear function f satisfies Lipschitz continuity and growth conditions.

The following conditions are assumed:

(H_1) For every $p \in Z$ there exists a $q \in \overline{R(B)}$ such that $Lp = Lq$ where the operator $L : Z \to X$ is defined as

$$Lx = \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s)x(s)ds$$

(H_2) The semigroup $\{T(t), t \geq 0\}$ generated by A is compact on X and there is a constant $M_1 \geq 0$ such that $||T(t)|| \leq M_1$.

(H_3) Nonlinear function $f(t,x)$ satisfies Lipschitz continuity on Z.i.e

$$||f(t,x_1) - f(t,x_2)|| \leq l_p||x_1 - x_2||, \quad l_p > 0 \quad for\ all\ x_1, x_2 \in Z.$$

(H_4) $f(t,x)$ satisfies linear growth condition, that is,

$$||f(t,x)|| \leq a_1 + b_1||x||,$$

where a_1 and b_1 are constants.

(H_5) $Db_1 < 1$

where the constant b_1 appear in the assumption (H_4) . The constant D is defined in Lemma (4).

Throughout this paper $D(A)$, $R(A)$ and $N_0(A)$ denotes the domain, range and null space of operator A, respectively.

Let $G : N_0^\perp(L) \to \overline{R(B)}$ be an operator defined as follows

$$Ga = a_0$$

where $a \in N_0^\perp(L)$ and a_0 is the unique minimum norm element in the set $\{a + N_0(L)\} \bigcap \overline{R(B)}\}$ satisfying the following condition

$$||Ga|| = ||a_0|| = min\left[||e|| : e \in \{a + N_0(L)\} \bigcap \overline{R(B)}\}\right] \tag{3.1}$$

The operator G is well defined, linear and continuous $(see\ [8],\ Lemma\ 1)$. From continuity of G, it follows that $||Ga|| \leq c||a||_Z$, for some constant $c \geq 0$.

Since $Z = N_0(L) + \overline{R(B)}$ as is evident from condition $(H1)$, any element $z \in Z$ can be expressed as

$$z = n + q : n \in N_0(L), q \in \overline{R(B)}$$

Lemma 7. *In* [13]*, for* $z \in Z$ *and* $n \in N_0(L)$*; the following inequality holds*

$$||n||_Z \leq (1+c)||z||_Z \tag{3.2}$$

where c is such that $||G|| \leq c$.

Proof: *Let* $a \in N_0^\perp(L)$*, then* $Ga = a_0 = (n_0 + a) \in \overline{R(B)}$*, for some* $n_0 \in N_0(L)$ *(*n_0 *is chosen such that* $a_0 = n_0 + a$ *satisfies the equation(3.1)). Further, since* $z \in Z$ *has unique decomposition namely,* $z = n_1 + a$ *where* $n_1 \in N_0(L)$ *and* $a \in N_0^\perp(L)$*, we get*

$$z = (n_1 - n_0) + (n_0 + a)$$

which gives a unique representation for z *as follows*

$$z = n + q : n \in N_0(L),\ q = a_0 \in \overline{R(B)},$$

where $a_0 = Ga$ *and* $n = n_1 - n_0$.
Now, $z = n_1 + a \Rightarrow ||z||^2 = ||n_1||^2 + ||a||^2$,
which implies that

$$||a|| \leq ||z|| \tag{3.3}$$

Also consider

$$\begin{aligned} z = n + q : n \in N_0(L),\ q \in \overline{R(B)}, \\ \Rightarrow ||n|| = ||z - Ga||, \\ \Rightarrow ||n|| \leq ||z|| + c||a||. \end{aligned} \tag{3.4}$$

From the equations (3.3) *and* (3.4)*, we conclude that*

$$||n||_Z \leq (1 + c)||z||_Z$$

□

Let us introduce some operators in the following way:
$K : Z \to Z$ defined by

$$(Kz)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) z(s) ds$$

Now, let M_0 be the subspace of Z (see[12]) such that

$$M_0 = \Big\{ m \in Z : m(t) = (Kn)(t),\ n \in N_0(L) \quad 0 \leq t \leq b$$

It can be noted that $m(b) = 0$ for all $m \in M_0$.
For each solution $p(t)$ of the system (2.4) with control r and for each realization $z(t)$ of the system (1.3), define the random operator $f_p : \overline{M_0} \to M_0$ as

$$f_p = Kn, \quad t > 0 \tag{3.5}$$

where n is given by the unique decomposition

$$F(p + z + m) = n + q : n \in N_0(L),\ q \in \overline{R(B)}, \tag{3.6}$$

where $F : Z \to Z$ given by

$$(Fx)(t) = f(t, x(t)), \qquad 0 \leq t \leq b$$

Here F is the Nemytskii operator of f. It is easy to see that F satisfies Lipschitz continuity (H_3) and linear growth conditions (H_4).

4 Main Results

Lemma 8. ([15]) *For any fixed* $t \geq 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are bounded linear operators.*

Proof: *For any fixed* $t \geq 0$*, since the* C_0*-semigroup* $T(t)$ *is a linear operator, it is easy to see that* $S_\alpha(t)$ *and* $T_\alpha(t)$ *are also linear operators. For* $\xi \in [0, 1]$*, according to* ([])*, direct calculation gives that*

$$\int_0^\infty \frac{1}{\theta^\xi} \psi_\alpha(\theta) d\theta = \frac{\Gamma(1 + \frac{\xi}{a})}{\Gamma(1 + \xi)}$$

Then we have

$$\int_0^\infty \theta^\xi \phi_\alpha(\theta) d\theta = \int_0^\infty \frac{1}{\theta^{\alpha\xi}} \psi_\alpha(\theta) d\theta = \frac{\Gamma(1 + \xi)}{\Gamma(1 + \alpha\xi)}$$

In the case $\xi = 1$, we have

$$\int_0^\infty \theta^\xi \phi_\alpha(\theta)d\theta = \int_0^\infty \frac{1}{\theta^{\alpha\xi}}\psi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\xi)}$$

For any $x \in V$, we have

$$||S_\alpha(t)x|| = \left|\left|\int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)xd\theta\right|\right| \le M||x||$$

where M be a positive constant such that $||T(t)|| \le M$, for all $t \ge 0$ and

$$||T_\alpha(t)x|| = \left|\left|\alpha\int_0^\infty \theta\phi_\alpha(\theta)T(t^\alpha\theta)xd\theta\right|\right| \le \frac{M\alpha}{\Gamma(1+\alpha)}||x||.$$

This completes the proof. □

Now for sufficient conditions of approximate controllability for system (1.3). Let us introduce some operators and lemmas.
The mild solution of system (1.3) is

$$z(t) = S_\alpha(t)(x_0 - y_0) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bw(s)ds + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)d\omega(s). \tag{4.1}$$

Define the operator $L_0^b : L_2[0,b;U] \to L_2[\Omega,\zeta_t,X]$, the controllability operator $\Pi_s^b : L_2[\Omega,\zeta_t,X] \to L_2[\Omega,\zeta_t,X]$ associated with (4.1), and the controllability operator $\Gamma_s^b : X \to X$ associated with the corresponding deterministic system of (4.1) as

$$L_0^b = \int_0^b (b-s)^{\alpha-1}T_\alpha(b-s)Bw(s)ds \tag{4.2}$$

$$\Pi_s^b\{.\} = \int_s^b (b-s)^{2(\alpha-1)}T_\alpha^*(b-t)BB^*T_\alpha^*(b-t)\mathbb{E}\{.|\zeta_t\}dt. \tag{4.3}$$

$$\Gamma_s^b = \int_s^b (b-s)^{2(\alpha-1)}T_\alpha^*(b-t)BB^*T_\alpha^*(b-t)dt. \tag{4.4}$$

It is easy to see that the operators $L_0^b, \Pi_s^b, \Gamma_s^b$ are linear bounded operators, and the adjoint $(L_0^b)^* : L_2[\Omega,\zeta_t,X] \to L_2[0,b;U]$ of L_0^b is defined by

$$(L_0^b)^* = B^*T_\alpha^*(b-t)(t-s)^{\alpha-1}\mathbb{E}\{z|\zeta_t\}.$$
$$\Pi_0^b = L_0^b(L_0^b)^*.$$

Before studying the approximate controllability of system (1.3), let us first investigate the relation between Π_s^b and Γ_s^b, $s \le r < b$ and resolvent operator $R(\lambda,\Pi_s^b) = (\lambda I + \Pi_s^b)^{-1}$ and $R(\lambda,\Gamma_r^b) = (\lambda I + \Gamma_r^b)^{-1}$, $s \le r < b$ for $\lambda > 0$, respectively.

Lemma 9. *For every $z \in L_2[\Omega,\zeta_t,X]$there exists $\varphi(.) \in L_2^\zeta(0,b;\mathbb{L}(\mathbb{R}^k,X))$ such that*

1. $\mathbb{E}\{z|\zeta_t\} = \mathbb{E}\{z\} + \int_0^t \varphi(s)d\omega(s),$
2. $\Pi_s^b z = \Gamma_s^b\mathbb{E}z + \int_s^b \Gamma_r^b\varphi(r)d\omega(r),$
3. $R(\lambda,\Pi_s^b)z = R(\lambda,\Gamma_s^b)\mathbb{E}\{z|\zeta_t\} + \int_s^b \Gamma_r^b\varphi(r)d\omega(r).$

Proof: *The proof is straightforward adaption of the proof of* $[11,\ Lemma\ 2.3]$ *with suitable substitutions.*

Theorem 10. *The control system* (1.3) *is approximately controllable on* $[0, b]$ *if and only if one of the following conditions holds.*

1. $\Pi_0^b > 0.$
2. $\lambda R(\lambda, \Pi_0^b)$ *converges to the zero operator as* $\lambda \to 0^+$ *in the strong operator topology.*
3. $\lambda R(\lambda, \Pi_0^b)$ *converges to the zero operator as* $\lambda \to 0^+$ *in the weak operator topology.*

Proof: *The proof is straightforward adaption of the proof of* $[10,\ Theorem\ 2]$ *with suitable substitutions.* □

Theorem 11. *Under assumption* (H_1)*, the fractional order linear system* (2.4) *is approximately controllable i.e.* $\overline{K_b(0)} = X$.

Proof: *The proof is straightforward adaption of the proof of* $[15,\ Theorem\ (4.1)]$.

□

Lemma 12. *Under the conditions* $(H_2), (H_4)$ *and* (H_5)*, the operator* f_p *has a fixed point* $m_0 \in M_0$ *for each realization* $z(t)$ *of the system* (1.3).

Proof:*From* (H_2) C_0 *semigroup* $T(t)$ *is compact so* $T_\alpha(t)$ *is compact (see lemma* $(3.4), [14]$*),the integral operator* K *is compact and hence* f_p *is compact for each p,(see [1]). Now let* $||m|| \le \tilde{r}$*. Then from the condition* (H_4) *and from the inequality* (3.2) *and* (3.6)*, we have*

$$\begin{aligned}
||f_p(m)||^2 \quad &\le \left|\left| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) n(s) ds \right|\right|^2 \\
&\le \int_0^b \left|\left| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) n(s) ds \right|\right|^2 dt \\
&\le \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 b^2(1+c)^2 \frac{b^{2\alpha-1}}{(2\alpha-1)} ||F(p+z+m||_Z^2 \\
&\le \frac{M^2 b^{2\alpha+1}(1+c)^2}{(\Gamma(1+\alpha))^2(2\alpha-1)} \{a_1 + b_1||p+z+m||_Z\}^2 \\
&\le \frac{M^2 b^{2\alpha+1}(1+c)^2}{(\Gamma(1+\alpha))^2(2\alpha-1)} \{a_1 + b_1||p+z|| + b_1\tilde{r}\}^2. \qquad (4.5)
\end{aligned}$$

Let $D^2 = \frac{M^2 b^{2\alpha+1}(1+c)^2}{(\Gamma(1+\alpha))^2(2\alpha-1)}$*, using Schauder's fixed point theorem, it is clear from the compactness of* f_p *and* (4.12) *that* f_p *has a fixed point in* M_0 *in a ball of radius* $\tilde{r} > 0$*, if*

$$\tilde{r} > \frac{D(a_1 + b_1||p+z||)}{1 - Db_1}.$$

Thus $f_p(m_0) = m_0$. □

The approximate controllability of the semilinear system (1.2) is proved in following manner using the above lemma.

Lemma 13. *For each realization* $z(t)$ *of the system* (1.3)*, the semilinear fractional control system* (1.2) *is approximate controllable under the conditions* $(H_1) - (H_4)$.
Proof: *From the equation* (3.6)*, we have*

$$F(p+z+m) = n + q$$

Operating K *on both the sides at* $m = m_0$ *(fixed point of* f_p*) and using* (3.5) *, we get*

$$\begin{aligned}
KF(p+z+m_0) &= Kn + Kq. \\
&= m_0 + Kq.
\end{aligned}$$

Adding p *on both sides, we get*

$$p + KF(p+z+m_0) = p + m_0 + Kq.$$

Let $p + m_0 = y^*$*, then the above equation is equivalent to*

$$p + KF(y^* + z) = y^* + Kq.$$

Since, from the equation (2.5)

$$p = S(t)\psi(0) + KBr.$$

we have

$$S(t)\psi(0) + KBr + KF(y^* + z) = y^* + Kq$$
$$S(t)\psi(0) + K(Br - q) + KF(y^* + z) = y^*.$$

Thus, it follows that $y^*(t)$ *is a solution of the following fractional system*

$$\begin{aligned} {}^cD_t^\alpha y^*(t) = Ay^*(t) + f(t, y^*(t) + z(t)) + Br(t) - q(t), \\ y^*(0) = y_0. \end{aligned} \quad (4.6)$$

with control $(Br - q)$.
Moreover, since $y^*(t) = p(t) + m_0(t)$*, it follows that*

$$y^*(b) = p(b) + m_0(b).$$

as $m_0(b) = 0$ *it follows that*

$$y^*(b) = p(b).$$

From the equations (4.5) *and* (4.6)*, it is clear that the reachable set of* (4.5) *is a superset of the reachable set of the system* (2.4)*, which is dense in* X.
Further $q \in \overline{R(B)}$ *implies that for any given* $\epsilon_1 > 0$*, there exists* $v_1 \in Y$ *such that* $||q - Bv_1|| \leq \epsilon_1$.
Now consider the equation

$$\begin{aligned} {}^cD_t^\alpha y(t) &= Ay(t) + f(t, y(t) + z(t)) + B(r(t) - v_1(t)), \\ y(0) &= \psi(0). \end{aligned}$$

Let $y(t)$ *be the solution of the system* (4.8)*, corresponding to control* $v = r - v_1$*. Then* $||y^*(b) - y(b)||$ *can be made arbitrary small by choosing a suitable* v_1*, which implies that the reachable set of the system* (4.8) *is dense in the reachable set of the system* (4.6)*, which in turn is dense in* X*. This proves that the system* (1.2) *is approximately controllable.*

□

5 Controllability results if the system has finite delay

Let we have the fractional semilinear stochastic control system with finite delay of the form:

$$\begin{aligned} {}^cD_t^\alpha x(t) &= [Ax(t) + Bu(t) + f(t, x_t)] + \frac{d\omega(t)}{dt}, \ t > 0 \\ x(t) &= \xi(t), \quad t \in [-h, 0]. \end{aligned} \quad (5.1)$$

Here $x_t \in L_2([-h, 0], X) = \mathbb{C}\ (say)$-valued stochastic processes and defined as $x_t(s) = \{x(t+s)| - h \leq s \leq 0|\}$; By splitting the system (5.1), we get the following pair of coupled systems

$$\begin{aligned} {}^cD_t^\alpha y(t) &= [Ay(t) + Bv(t) + f(t, (y+z)_t]; \quad 0 \leq t \leq b \\ y(t) &= \psi(t), \quad t \in [-h, 0]. \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} {}^{c}D_t^{\alpha} z(t) &= [Az(t) + Bw(t)] + \frac{d\omega(t)}{dt}; \quad 0 \le t \le b \\ z(t) &= \xi(t) - \psi(t), \quad t \in [-h, 0]. \end{aligned} \tag{5.3}$$

The mild solution of the systems (5.1) can be written as

$$x(t) = \begin{cases} S_\alpha(t)\xi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\{Bu(s) + f(s, x_s)\} ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) d\omega(s), \ t > 0 \\ \xi(t) \qquad -h \le t \le 0 \end{cases} \tag{5.4}$$

the mild solution of the semilinear system (5.2), can be written as

$$y(t) = \begin{cases} S_\alpha(t)\psi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\{Bv(s) + f(s, (y+z)_s\} ds, \ t > 0 \\ \psi(t) \qquad -h \le t \le 0 \end{cases} \tag{5.5}$$

and the mild solution of the linear stochastic system (5.3), can be written as

$$z(t) = \begin{cases} S_\alpha(t)(\xi(0) - \psi(0)) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bw(s) ds \\ + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) d\omega(s), \ t > 0 \\ \xi(t) - \psi(t) \qquad -h \le t \le 0 \end{cases} \tag{5.6}$$

In similar manner as above we can define some operators as follows
$K : Z \to Z$ defined by

$$(Kz)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) z(s) ds$$

Now, let M_0 be the subspace of Z_h (see[15]) such that

$$M_0 = \begin{cases} m \in Z_h : m(t) = (Kn)(t), \ n \in N_0(L) & 0 \le t \le b \\ m(t) = 0, & -h \le t \le 0 \end{cases}$$

It can be noted that $m(b) = 0$ for all $m \in M_0$.
For each solution $p(t)$ of the system (2.4) with control r and for each realization $z(t)$ of the system (1.3), define the random operator $f_p : \overline{M_0} \to M_0$ as

$$f_p = \begin{cases} Kn, & 0 < t < b \\ 0, & -h \le t \le 0 \end{cases} \tag{5.7}$$

where n is given by the unique decomposition

$$F(p + z + m) = n + q : n \in N_0(L), \ q \in \overline{R(B)}, \tag{5.8}$$

where $F : L_2([0, b], \mathbb{C}) \to X$ given by

$$(Fx)(t) = f(t, x_t(\cdot)); \qquad 0 \le t \le b$$

It is easy to see that F satisfies Lipschitz continuity (H_3) and linear growth conditions (H_4). One can easily show that the system (5.1) is approximately controllable using $(H_1) - (H_4)$ and lemma (5). □

6 Examples

Example 1. Consider the stochastic control system governed by the fractional semilinear heat equation

$$ {}^cD_t^\alpha y(t,x) = \left[\frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t,y(t,x))\right] + \frac{\partial\omega(t)}{\partial t} $$
$$ for \quad 0 \le t \le \tau, \quad 0 \le x \le \pi. $$
$$ with \quad conditions \quad y(t,0) = y(t,\pi) = 0, \quad 0 \le t \le \tau. \tag{6.1} $$

The system (6.1) can be written in the abstract form (1.1), by setting $X = L_2(0,\pi)$ and $A = \frac{d^2}{dx^2}$, with domain consisting of all $y \in X$ with $(\frac{d^2y}{dx^2}) \in X$ and $y(0) = 0 = y(\pi)$. Take $\phi_(x) = (2/\pi)^{1/2} sin(nx),\ 0 \le x \le \pi,\ n = 1,2,3,...,$ then $\{\phi_n(x)\}$ is an orthonormal basis for X and ϕ_n ia an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A, $n = 1,2,3,....$ Then the C_0-semigroup $T(t)$ generated by A has $e^{\lambda_n t}$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions.
Define an infinite dimensional space U by

$$ U = \left\{u : u = \sum_{n=2}^{\infty} u_n\phi_n \ with \ \sum_{n=2}^{\infty} u_n^2 < \infty\right\}. $$

The norm defined by

$$ ||u||_U = \left(\sum_{n=2}^{\infty} u_n^2\right)^{1/2}. $$

Let B be a continuous linear operator from U to X defined as

$$ Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n, \quad u = \sum_{n=2}^{\infty} u_n\phi_n \in U. $$

The nonlinear operator f is assumed to satisfy conditions (H_3) and (H_4).
The approximate controllability of the corresponding semilinear deterministic heat equation of (6.1) was considered by Naito $[4]$ and proved under the conditions $(H_1) - (H_4)$. Here approximately controllability of the stochastic semilinear heat control system (6.1) is considered.
The system (6.1), can be associated with two control systems under the initial and boundary conditions, as given below

$$ {}^cD_t^\alpha y(t,x) = \frac{\partial^2 y(t,x)}{\partial x^2}, $$
$$ +Bv(t,x) + f(t,y(t,x) + z(t,x))\ t \in [0,b]\ x \in [0,\pi], \tag{6.2} $$
$$ {}^cD_t^\alpha z(t,x) = \left[\frac{\partial^2 z(t,x)}{\partial x^2} + Bw(t)\right]\partial t + \frac{\partial\omega(t)}{dt}. \tag{6.3} $$

The system (6.3) is a fractional stochastic system and for each realization $z(t)$ of the system (6.3), the system (6.2) is a deterministic system.
From lemma (6) and using the conditions $(H_1) - (H_4)$, it is clear that for each realization $z(t)$ of the system (6.3), the system (6.2) is approximately controllable. □

Example 2 Consider the stochastic control system with finite delay governed by the fractional semilinear heat equation

$$ {}^cD_t^\alpha y(t,x) = \left[\frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t,y(t+v,x))\right] + \frac{\partial\omega(t)}{\partial t}, $$
$$ for \quad 0 < t < \tau;\ v \in [-h,0]; \quad 0 < x < \pi, $$
$$ with \quad conditions \quad y(t,0) = y(t,\pi) = 0, \quad 0 \le t \le \tau, $$
$$ y(t,x) = \xi(t,x), \qquad -h \le t \le 0,\ 0 \le x \le \pi. \tag{6.4} $$

one can easily solve it similar as example (1).

References

[1] A. Pazy. Semigroup of linear operators and application to partial differential equations. *Spinger Verlag, New Yok*. (1983).

[2] A.Shukla. Urvashi Arora and N.Sukavanam, Approximate controllability of retarded semilinear stochastic system with non local conditions. *J.Appl. Math. Comput.*

[3] A. Shukla et al. Complete controllability of semi-linear stochastic system with delay. Rendiconti del Circolo Matematico di Palermo. *Nonlinear Stud.* (2012): 308-312.

[4] A. Shukla et al. Controllability of Semilinear Stochastic System with Multiple Delays in Control. 3rd International Conference on Advances in Control and Optimization of Dynamical Systems. *Nonlinear Stud.* (2014): 306-312.

[5] A. Shukla et al. Approximate Controllability of Fractional Semilinear Stochastic System of Order $\alpha \in (1, 2]$. *Nonlinear Stud.* 22(1)(2015): 131-138.

[6] J. Klamka. Schuder fixed point theorem in nonlinear controllability problems. *Control Cybernet.* 29(1)(2000): 153-165.

[7] J. M. Jeong et al. Approximate controllability of semilinear retarded functional differential equations. *J.Dyn.Control Syst.* 5(1999): 329-346.

[8] K. Naito. Controllability of Semilinear Control Systems Dominated by the Linear Part. *SIAM Journal on Control and Optimization* 25(1987): 715-722.

[9] K. Naito and J. Y. Park. Approximate Controllability for Trajectories of a Delay Volterra Control System. *Journal of Optimization Theory and Applications* 61(2)(1989): 271-279.

[10] N. I. Mahmudov and A. denker. On controllability of linear stochastic systems. *Int. J.Control.* 72(2)(2000): 144-151.

[11] N. I. Mahmudov. Controllability of Linear Stochastic Systems in Hilbert Spaces. *Journal of Mathematical Analysis and Applications*, 259(1)(2001): 64-82.

[12] N. Sukavanam. Approximate Controllability of Semilinear Control Systems with Growing Nonlinearity, in Mathematical Theory of Control Proceedings of International Conference. *Marcel Dekker, New York* (1993): 353-357.

[13] N. Sukavanam and M. Kumar. S-controllability of an abstract first order semilinear control system, *Numer. Funct. Anal. Optim.* 31(2010): 1023–1034.

[14] R. F. Curtain and H. J. Zwart. An Introduction to Infinite Dimensional Linear Systems Theory. *Springer Verlag.* (1995).

[15] S. Kumar and N. Sukavanam. Approximate controllability of fractional order semilinear systems with bounded delay. *J. Differential Equations* 252(11)(2012): 6163–6174.

[16] R. Sakthivel et al. Approximate controllability of fractional stochastic evolution equations. *Comput. Math. Appl.* 63(3)(2012): 660–668.

[17] R. Sakthivel et al. Mahmudov, Approximate controllability of second-order stochastic differential equations with impulsive effects. *Modern Phys. Lett. B* 24(14)(2010).

[18] Y. Zhou and F. Jiao. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* 59(3)(2010): 1063–1077.