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Classification of Weak Solutions of Stationary Hartree-type Equations

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Abstract: This paper is concerned with the stationary Hartree-type equation

$$(-\Delta)^{\frac{\alpha}{2}}u = pu^{p-1}(|x|^{\alpha-n} * u^p), \quad u > 0 \; in \; R^n,$$

where $n \geq 1$, $p > 1$ and $\alpha \in (0, n)$. We obtain a classification result for weak solutions in $H^\alpha(R^n)$: $p = \frac{n+\alpha}{n-\alpha}$, $\alpha \in (0, n/2)$ and $u = v = c(\frac{t}{t^2+|x-x_0|^2})^{\frac{n-\alpha}{2}}$ with positive constants c, t and $x_0 \in R^n$. Moreover, if u is a weak solution in $\mathcal{D}^{\alpha,2}(R^n)$, the classification result above still holds except for $\alpha \in (0, n/2)$.

Keywords: Hartree-type equation; classification of positive solutions; fractional order Laplacian

1 Introduction

Recently, many authors are devoted to study the stationary Hartree-type equation

$$(-\Delta)^{\frac{\alpha}{2}}u = pu^{p-1}(|x|^{\alpha-n} * u^p), \quad u > 0 \; in \; R^n, \tag{1}$$

where $n \geq 1$, $p > 1$ and $\alpha \in (0, n)$.

Here the fractional order Laplacian operator is defined by the ideas in [12]. If $u \in H^\alpha(R^n)$, $(-\Delta)^{\frac{\alpha}{2}}u := \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u))$, where $\mathcal{F}(u)$ is the Fourier transformation of u, and $\mathcal{F}^{-1}(u)$ is the Fourier anti-transformation of u.

A positive function $u \in H^\alpha(R^n)$ is called a weak solution of system (1), if the following equality makes sense (cf. [4])

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}}u(x)(-\Delta)^{\frac{\alpha}{4}}\varphi(x)dx = p\int_{R^n} u^{p-1}(x)(|x|^{\alpha-n} * u^p)\varphi dx \tag{2}$$

for all $\varphi \in C_0^\infty(R^n)$. Here,

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}}u(x)(-\Delta)^{\frac{\alpha}{4}}\varphi(x)dx := Re\int_{R^n} |\xi|^\alpha \mathcal{F}(u)\overline{\mathcal{F}(\varphi)}d\xi.$$

When $\alpha = 2$, (1) arises in the Hartree-Fock theory of the Schrödinger problems (cf. [10]). It is also helpful to well understand the blowing up or the global existence and scattering of the solutions of the dynamic Hartree equation (cf. [5], [9] and [11]). Afterwards, Lei [7] studied the decay rates of positive solutions and the relation between integrable solutions and finite energy solutions. In view of the convolution term, a related integral system plays an important role there. Similarly, [13] also introduced an integral system

$$\begin{cases} u(x) = \sqrt{p}\int_{R^n} \dfrac{u^{p-1}(y)v(y)}{|x-y|^{n-\alpha}}dy, & u > 0 \quad in \quad R^n, \\ v(x) = \sqrt{p}\int_{R^n} \dfrac{u^p(y)}{|x-y|^{n-\alpha}}dy, & v > 0 \quad in \quad R^n \end{cases} \tag{3}$$

to study further the positive solutions of (1), and announced the following conclusion.

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Proposition 1.1 *Let u, v be positive solutions of (3) with $\alpha \in (1, n)$. Then the following items are equivalent:*
(I1) $u, v \in L^{\infty}_{loc}(R^n)$ and $p = \frac{n+\alpha}{n-\alpha}$;
(I2) $u(x) \equiv v(x) = c(\frac{t}{t^2+|x-x_0|^2})^{\frac{n-\alpha}{2}}$ and $\alpha \in (0, \frac{n}{2})$ with $c > 0$, $t > 0$ and $x_0 \in R^n$;
(I3) $u \in L^{n(p-1)/\alpha}(R^n)$.

According to [6], $\alpha \in (1, n)$ implies that the solution u of (3) has better regularity, i.e. $u \in C^1(R^n)$. It is essential for applying the Pohozaev identity to deduce (I1) from (I3) in [13]. Naturally, we are concerned about the case of $\alpha \in (0, 1]$. Indeed, according to Dirac's theory, (1) with $\alpha = 1$ is more meaningful in quantum mechanics.

In this paper, we consider the classification of weak solutions of (1) with $\alpha \in (0, n)$. In the definition of weak solutions, the Fourier transformation of u comes into play instead of the differentiability of u. Therefore, we can prove (I1) without the constraint of $\alpha > 1$.

Theorem 1.1 *Let $u \in H^{\alpha}(R^n)$ be a positive weak solution of (1), then we have three following conclusions which are also equivalent to each other:*
(i) $u^p v \in L^1(R^n)$;
(ii) $p = \frac{n+\alpha}{n-\alpha}$;
(iii) $u(x) \equiv v(x) = c(\frac{t}{t^2+|x-x_0|^2})^{\frac{n-\alpha}{2}}$ and $\alpha \in (0, \frac{n}{2})$ with $c > 0$, $t > 0$ and $x_0 \in R^n$.

Remark 1.1 *Here, we assume the solution $u \in H^{\alpha}(R^n) \subset L^2(R^n)$ in order that we can apply the Fourier transformation, which leads to a 'particular' result $\alpha \in (0, n/2)$ by Theorem 1.1. In fact, $H^{\alpha}(R^n)$ is not appropriate to investigate the decay properties. To apply the Fourier transformation, we assume u belongs to the dual of the Schwartz space $\mathcal{S}'$ (instead of $L^2(R^n)$) such that $|\xi|^{\alpha}\mathcal{F}(u) \in \mathcal{S}'$, and define*

$$\langle(-\Delta)^{\alpha/2}u, \phi\rangle = \langle\mathcal{F}^{-1}(|\xi|^{\alpha}\mathcal{F}(u)), \phi\rangle = \langle|\xi|^{\alpha}\mathcal{F}(u), \mathcal{F}^{-1}(\phi)\rangle$$

for the test function $\phi \in \mathcal{S}$. Thus, we introduce another weak solution of (1) in a homogeneous Sobolev space $\mathcal{D}^{\alpha,2}(R^n)$ (which is subspace of $\mathcal{S}'$), if replacing $\varphi \in C_0^{\infty}(R^n)$ by $\varphi \in \mathcal{S}$ in (2). This space is appropriate to describe the decay properties, and we can also define weak solution by the Fourier transformation.

2 The proof of Theorem 1.1

Theorem 2.1 *The weak solution $u \in H^{\alpha}(R^n)$ of (1) satisfies the following integral system*

$$\begin{cases} u(x) = \int_{R^n} \frac{u^{p-1}(y)v(y)}{|x-y|^{n-\alpha}}dy, & u > 0 \quad in \quad R^n, \\ v(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy, & v > 0 \quad in \quad R^n. \end{cases} \tag{4}$$

Proof. The idea in [4] is used here. For an arbitrary function $\phi \in C_0^{\infty}(R^n)$, set

$$\psi(x) = \int_{R^n} \frac{\phi(y)dy}{|x-y|^{n-\alpha}}.$$

By the Fourier transformation, we have

$$|\xi|^{\alpha}\mathcal{F}(\psi)(\xi) = \mathcal{F}(\phi)(\xi). \tag{5}$$

So ψ belongs to the homogeneous fractional order Sobolev space $\mathcal{D}^{\alpha,2}(R^n)$.

Since $C_0^{\infty}(R^n)$ is dense in $\mathcal{D}^{\alpha,2}(R^n)$, we can take $\varphi = \psi$ in (2). Thus,

$$Re\int_{R^n} |\xi|^{\alpha}\mathcal{F}(u)(\xi)\overline{\mathcal{F}(\psi)}(\xi)d\xi = \sqrt{p}\int_{R^n} u^{p-1}(x)v(x)\psi(x)dx, \tag{6}$$

where $v(x) = \sqrt{p}(|x|^{\alpha-n} * u^p(x))$.

In view of (5), the left hand side of (6)

$$Re\int_{R^n} |\xi|^{\alpha}\mathcal{F}(u)(\xi)\overline{\mathcal{F}(\psi)}(\xi)d\xi = \int_{R^n} \mathcal{F}(u)(\xi)\overline{\mathcal{F}(\phi)}(\xi)d\xi = \int_{R^n} u(x)\phi(x)dx.$$

On the other hand, the right hand side of (6)

$$\begin{aligned}\int_{R^n} u^{p-1}(x)v(x)\psi(x)dx &= \int_{R^n} u^{p-1}(x)v(x)\int_{R^n}\frac{\phi(y)dy}{|x-y|^{n-\alpha}}dx\\ &= \int_{R^n}\phi(y)\int_{R^n}\frac{u^{p-1}(x)v(x)}{|x-y|^{n-\alpha}}dxdy.\end{aligned}$$

Substituting these results into (6) and noting that ϕ is an arbitrary C_0^∞ function, we conclude that u, v solve the system (3) a.e. in R^n. Therefore, C_1u and C_2v also solve (4) for suitable positive constants C_1 and C_2. Since the integrability and the boundedness of the solutions are independent of the constants C_1 and C_2, we still denote C_1u and C_2v by u and v. ■

Remark 2.1 *Let u, v solve (4). Assume $u \in \mathcal{D}^{\alpha,2}(R^n)$ (or $H^\alpha(R^n)$), then it is a weak solution of (1) if we omit a constant. In fact, taking a Fourier transformation on both sides of (4), we have*

$$\mathcal{F}(u) = \mathcal{F}[|x|^{\alpha-n} * (u^{p-1}v)] = c|\xi|^{-\alpha}\mathcal{F}(u^{p-1}v).$$

Thus, for all $\phi \in \mathcal{S}$ (or $C_0^\infty(R^n)$), there holds by the Parseval equality,

$$\int_{R^n}|\xi|^\alpha\mathcal{F}(u)(\xi)\overline{\mathcal{F}(\phi)}(\xi)d\xi = c\int_{R^n}\mathcal{F}(u^{p-1}v)(\xi)\overline{\mathcal{F}(\phi)}(\xi)dx = c\int_{R^n}u^{p-1}v\phi dx.$$

This shows that u is a weak solution of (1).

Theorem 2.2 *If (4) has positive solutions, then $p > \frac{n}{n-\alpha}$.*

Proof. The idea in [1] is used here and an analogous result can be seen in [2]. Clearly,

$$u(x) \geq c\int_{B_R(0)}\frac{u^{p-1}(y)v(y)dy}{|x-y|^{n-\alpha}} \geq \frac{c}{(R+|x|)^{n-\alpha}}\int_{B_R(0)}u^{p-1}(y)v(y)dy. \tag{7}$$

Therefore,

$$\begin{aligned}\int_{B_R(0)}u^p(x)dx &\geq c\int_{B_R(0)}\frac{dx}{(R+|x|)^{p(n-\alpha)}}(\int_{B_R(0)}u^{p-1}(y)v(y)dy)^p\\ &\geq \frac{c}{R^{p(n-\alpha)-n}}(\int_{B_R(0)}u^{p-1}(y)v(y)dy)^p.\end{aligned} \tag{8}$$

Here c is independent of R. Similarly, from

$$v(x) \geq \frac{c}{(R+|x|)^{n-\alpha}}\int_{B_R(0)}u^p(y)dy, \tag{9}$$

and (7), (8), we also deduce

$$\begin{aligned}\int_{B_R(0)}u^{p-1}(x)v(x)dx &\geq \int_{B_R(0)}\frac{cu^{p-1}(x)dx}{(R+|x|)^{n-\alpha}}\int_{B_R(0)}u^p(y)dy\\ &\geq \frac{c}{R^{2[p(n-\alpha)-n]}}(\int_{B_R(0)}u^{p-1}(y)v(y)dy)^p,\end{aligned}$$

which implies

$$\int_{B_R(0)}u^{p-1}(x)v(x)dx \leq CR^{2[p(n-\alpha)-n]/(p-1)}. \tag{10}$$

If $1 < p < \frac{n}{n-\alpha}$, (10) with $R \to \infty$ leads to $\|u^{p-1}v\|_{L^1(R^n)} = 0$. This contradicts with $u^{p-1}v > 0$.

If $p = \frac{n}{n-\alpha}$, (10) implies $u^{p-1}v \in L^1(R^n)$ if we let $R \to \infty$. Multiplying (9) by u^{p-1} and integrating on $A_R := B_R(0) \setminus B_{R/2}(0)$, we still have

$$\int_{A_R}u^{p-1}(x)v(x)dx \geq c(\int_{B_R(0)}u^{p-1}(y)v(y)dy)^p.$$

Letting $R \to \infty$ and noting $u^{p-1}v \in L^1(R^n)$, we obtain $\|u^{p-1}v\|_{L^1(R^n)} = 0$. It is also impossible. ■

Proof of Theorem 1.1 Now, we split the proof of Theorem 1.1 into four steps.

Proof. *Step 1.* According to the definition of weak solutions (2), we can take $\varphi = u$ since $C_0^\infty(R^n)$ is dense in $H^\alpha(R^n)$. Thus,

$$\sqrt{p}\int_{R^n} u^p v dx = \int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u(x)(-\Delta)^{\frac{\alpha}{4}} u(x) dx. \tag{11}$$

In view of $u \in H^\alpha(R^n)$, it follows

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u(x)(-\Delta)^{\frac{\alpha}{4}} u(x) dx = \int_{R^n} |\xi|^\alpha |\mathcal{F}(u)|^2 d\xi < \infty.$$

Therefore, we obtain the conclusion $u^p v \in L^1(R^n)$.

Step 2. We claim that $u^p v \in L^1(R^n)$ implies $p = \frac{n+\alpha}{n-\alpha}$.

The idea in [8] is used here. Noting $u \in H^\alpha(R^n)$ and $u^p v \in L^1(R^n)$, we could consider the following functional

$$E(u) = \frac{1}{2}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(x)|^2 - \frac{1}{2\sqrt{p}}\int_{R^n} u^p(x)v(x)dx.$$

For $\mu > 0$, by scaling we have

$$E(u(\frac{x}{\mu})) = \frac{1}{2}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(\frac{x}{\mu})|^2 dx - \frac{1}{2\sqrt{p}}\int_{R^n} u^p(\frac{x}{\mu})v(\frac{x}{\mu})dx := E_1 - E_2.$$

Direct calculation yields

$$\begin{aligned} & E_1(u(\tfrac{x}{\mu})) \\ &= \frac{1}{2}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(\frac{x}{\mu})|^2 dx = \frac{1}{2}\int_{R^n} \mathcal{F}[(-\Delta)^{\frac{\alpha}{4}} u(\frac{x}{\mu})]\overline{\mathcal{F}[(-\Delta)^{\frac{\alpha}{4}} u(\frac{x}{\mu})]}dx \\ &= \frac{1}{2}\int_{R^n} \|\frac{\xi}{\mu}|^{\frac{\alpha}{2}} \mathcal{F}[u(\frac{x}{\mu})]|^2 d\xi = \frac{1}{2}\int_{R^n} \mu^{n-\alpha}\||\xi|^{\frac{\alpha}{2}}\mathcal{F}[u(x)]|^2 d\xi \\ &= \frac{1}{2}\mu^{n-\alpha}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u|^2 dx \end{aligned}$$

and

$$E_2(u(\frac{x}{\mu})) = \frac{1}{2\sqrt{p}}\mu^{n+\alpha}\int_{R^n} u^p(x)v(x)dx.$$

Thus,

$$E(u(\frac{x}{\mu})) = \frac{1}{2}\mu^{n-\alpha}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(x)|^2 dx - \frac{1}{2\sqrt{p}}\mu^{n+\alpha}\int_{R^n} u^p(x)v(x)dx. \tag{12}$$

Since u is a weak solution of (1), it is also the critical point of the functional $E(u)$. Thus, we have

$$\frac{d}{d\mu}E(u(\frac{x}{\mu}))|_{\mu=1} = 0.$$

From (12), we deduce

$$\frac{n-\alpha}{2}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(x)|^2 dx = \frac{n+\alpha}{2\sqrt{p}}\int_{R^n} u^p(x)v(x)dx. \tag{13}$$

Combining with (11), we obtain

$$p\frac{n-\alpha}{n+\alpha}\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(x)|^2 dx = \int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u(x)(-\Delta)^{\frac{\alpha}{4}} u(x) dx. \tag{14}$$

By using of Parseval identity, we get

$$\begin{aligned} &\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u(x)(-\Delta)^{\frac{\alpha}{4}} u(x) dx = \int_{R^n} |\xi|^\alpha \mathcal{F}(u)\overline{\mathcal{F}(u)}dx \\ &= \int_{R^n} (|\xi|^{\frac{\alpha}{2}}\mathcal{F}(u))(|\xi|^{\frac{\alpha}{2}}\overline{\mathcal{F}(u)})dx = \int_{R^n} \mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u]\overline{\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u]}dx \\ &= \int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} u(x)|^2 dx. \end{aligned} \tag{15}$$

Inserting (15) into (14), we can see $p = \frac{n+\alpha}{n-\alpha}$.

Step 3. We claim that $p = \frac{n+\alpha}{n-\alpha}$ implies $u(x) \equiv v(x) = c(\frac{t}{t^2+|x-x_0|^2})^{\frac{n-\alpha}{2}}$ and $\alpha \in (0, \frac{n}{2})$, where c, t are positive constant and $x_0 \in R^n$.

Clearly, $u \in H^\alpha(R^n)$ implies $u \in L^{\frac{2n}{n-\alpha}}(R^n)$. In view of $p = \frac{n+\alpha}{n-\alpha}$, we have

$$u \in L^{\frac{n(p-1)}{\alpha}}(R^n).$$

By Theorem 2.1, u, v solve (4). So we can apply Lemmas 2.3 and 2.4 in [13] to deduce that both u and v are bounded. According to Theorem 5 in [3], by $p = \frac{n+\alpha}{n-\alpha}$ we see at once the classification result

$$u(x) \equiv v(x) = c(\frac{t}{t^2 + |x - x_0|^2})^{\frac{n-\alpha}{2}} \tag{16}$$

with $c > 0, t > 0$ and $x_0 \in R^n$.

Next, we prove $\alpha \in (0, \frac{n}{2})$. Otherwise, $n - 2(n - \alpha) > 0$. By (16),

$$\int_{R^n} u^2 dx \geq C \int_R^\infty r^{n-2(n-\alpha)} \frac{dr}{r} = \infty.$$

It is a contradiction with $u \in H^\alpha(R^n) \subset L^2(R^n)$.

Step 4. Let $u(x) \equiv v(x) = c(\frac{t}{t^2+|x-x_0|^2})^{\frac{n-\alpha}{2}}$ be the radially symmetric solution of (1) with constant $c = c(n) > 0$ and $t > 0$, then $u^p v \in L^1(R^n)$.

In view of (16), $u, v \in L^\infty(R^n)$. In addition, there exists a large constant $R > 0$ such that

$$u(x) \leq C|x|^{\alpha-n}, \quad as \quad |x| > R.$$

Therefore,

$$\begin{aligned} \int_{R^n} u^p v dx &= \int_{B_R(0)} u^p v dx + \int_{R^n \setminus B_R(0)} u^p v dx \\ &\leq C + C \int_R^\infty r^{n-(p+1)(n-\alpha)} \frac{dr}{r}. \end{aligned}$$

In addition, by Theorems 2.1 and 2.2, we have $p > \frac{n}{n-\alpha}$, and hence

$$n - (p + 1)(n - \alpha) < 0.$$

Thus, $u^p v \in L^1(R^n)$. ■

Remark 2.2 *For the weak solutions in $\mathcal{D}^{\alpha,2}(R^n)$, we can still obtain Theorem 1.1 without the particular constraint $\alpha \in (0, n/2)$. In fact, we can also take the test function in $\mathcal{D}^{\alpha,2}(R^n)$ if noting that the subspace $C_0^\infty(R^n)$ of $\mathcal{S}$ is dense in such a homogeneous Sobolev space. In addition, by the Sobolev inequality, $u \in \mathcal{D}^{\alpha,2}(R^n)$ also leads to $u \in L^{\frac{2n}{n-\alpha}}(R^n)$. So the argument above still works except for the proof of $\alpha < n/2$.*

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