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A Class of Abstract Property with Tracial Approximation

Houqing Fang[a,b*], Xiaochun Fang[c]
a. School of Economics and Management, Southeast University, Nanjing 210018, P. R. China
b. Department of Mathematics, Jiangsu University, Zhenjiang 212013, P. R. China C. Department of Mathematics, Tongji University, Shanghai 200092, P. R. China
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Abstract: In the quest to classify simple separable nuclear C^*-algebras, as suggested by G. A. Elliott, it has become necessary invoke some noncommutative topological dimension for example tracial topological rank. In this paper, we first propose some abstract properties (F), (TFA),(TFB), (TFC) and (TFD) on the category Q of a unital C^*-algebra. Then we prove that these properties (TFA)-(TFD) are equivalent for any object A in Q. The proposed properties generalized some noncommutative topological dimension, for example tracial topological rank, tracial rank zero, tracial stable rank one and so on.

Keywords: C^*-algebra; Tracial approximation; Property (F).

1 Notation and Definitions

In this section, we give some notations and definitions of properties (F), (TFA), (TFB), (TFC) and (TFD). Let Q be the category of a unital C^*-algebra. An element a in a C^*-algebra A is positive if $a \in A_{sa}$ and $sp(a) \subset R_+$. The set of all positive elements in A will be denoted by A_+(see Definition 1.4.4 in [8]). Let a and b be two positive elements in a C^*-algebra A and A^{**} the enveloping von Neumann algebra of A. We write $[a] \leq [b]$ (see Definition 3.5.2 in [8]), if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in Her(a)$, $v^*c \in A$, $cv \in A$, $vv^* = P_a$ and $v^*cv \in Her(b)$, where P_a is the range projection of a in A^{**}, here and in the sequel $Her(a) = \overline{aAa}$ and $Her(b) = \overline{bAb}$. We write $[a] = [b]$ if $v^*Her(a)v = Her(b)$. Let n be a positive integer. We write $n[a] \leq [b]$, if there exist n mutually orthogonal positive elements $b_1, b_2, \ldots, b_n \in Her(b)$ such that $[a] \leq [b_i]$, $i = 1, 2, \ldots, n$.

For any $0 < \sigma_1 < \sigma_2 \leq 1$, we define a real-valued continuous function $f_{\sigma_1}^{\sigma_2}(t)$ on the interval $[0, 1]$ as follows:

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & if\ t \geq \sigma_2, \\ \frac{t-\sigma_1}{\sigma_2-\sigma_1} & if\ \sigma_1 \leq t \leq \sigma_2, \\ 0 & if\ 0 < t \leq \sigma_1. \end{cases}$$

Now we give the following definitions.

Definition 1 *(F) An abstract property (F) on Q is hereditary if A has the property (F), then pAp also has the property (F) for any object A in Q and any projection p in A.*

For example, the properties of stable rank one, trace stable range one, and real rank zero have the hereditary property.

Definition 2 *(TFA) For any object A in Q, we say that A has the property (TFA) if for any $\varepsilon > 0$, $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, finite subset $F \subseteq A$ and nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with the hereditary abstract property (F) and $1_B = p$ such that*

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$, here and in the sequel $pxp \in_\varepsilon B$ means that there exists $y \in B$ such that $\|pxp - y\| < \varepsilon$,

(iii) $[f_{\sigma_1}^{\sigma_2}((1-p)b(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pbp)]$.

*Corresponding author. *E-mail address*: doudou@ujs.edu.cn

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Definition 3 (TFB) *For any object A in Q, we say that A has the property (TFB) if for any $\varepsilon > 0$, $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, finite subset $F \subseteq A$, nonzero element $a \geq 0$, positive integer n and finite subset $G = \{b_1,\ b_2,\ \ldots,\ b_m\}$ of A_+, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with the hereditary abstract property (F) and $1_B = p$ such that*

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$,

(iii) $n[f_{\sigma_1}^{\sigma_2}((1-p)b_j(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pb_jp)]$ for all $b_j \in G$.

Definition 4 (TFC) *For any object A in Q, we say that A has the property (TFC) if for any $\varepsilon > 0$, $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, finite subset $F \subseteq A$ and nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with the hereditary abstract property (F) and $1_B = p$ such that*

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$,

(iii) $n[1-p] \leq [p]$, $n[f_{\sigma_1}^{\sigma_2}((1-p)b(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pbp)]$.

Definition 5 (TFD) *For any object A in Q, we say that A has the property (TFD) if for any $\varepsilon > 0$, finite subset $F \subseteq A$, finite subset $H \subseteq A_+$ and $W = \{w_1,\ w_2, \ldots, w_l\} \subseteq \mathfrak{R}_0^4$, where $\mathfrak{R}_0^4$ is a set of quaternary real array $(d_1,\ d_2,\ d_3,\ d_4)$ such that for any $0 < d_4 < d_3 < d_2 < d_1 < 1$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with the hereditary abstract property (F) and $1_B = p$ such that*

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$,

(iii) $n[f_{d_1}^{d_2}((1-p)b(1-p))] \leq [f_{d_3}^{d_4}(pbp)]$ for any $b \in H$ and $w \in W$.

2 Introduction and main results

In the paper [3], Elliott first proposed the Elliott conjecture and solved the K-theoretical classification of AF-algebras. Since then many classes of C^*-algebras have been found to be classified by the Elliott invariant, for more details, see [1] and the reference therein. One of important classes is the class of simple unital AH-algebras [2]. Lin [7] gave a very important axiomatic version of the classification of AH-algebras without dimension growth, and he developed a certain abstract approximation property and introduced the class of tracially approximate interval algebras.

Following the notion of Lin's tracial approximation by interval algebras, Elliott and Niu in [4] considered simple unital C^*-algebras tracial approximation by certain C^*-algebras. Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebras in Ω, denoted by $TA\Omega$, is defined as follows. A simple unital C^*-algebra A is said to belong to the class $TA\Omega$ if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any a nonzero element $a \geq 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebras B of A with $1_B = p$ and $B \in \Omega$, such that (1) $\|xp - px\| < \varepsilon$ for all $x \in F$, (2) $pxp \in_\varepsilon B$ for all $x \in F$, (3) $[1-p] \leq [a]$, $1-p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

The question of which properties are inherited from a class Ω to the class $TA\Omega$ is interesting and sometimes important. In fact the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of the classification theorem in [4] by Elliott and Niu. In [5] and [6], the author show that certain K_0 and K_1 groups properties are inherited form the class Ω to the class $TA\Omega$.

Also motivated by Lin [7], the main purpose of this paper is twofold:

(i) To find several equivalent definitions of general C*-algebra of the tracial approximation, that is, the abstract property (F) on the category Q of a unital C^*-algebra, and the properties (TFA), (TFB), (TFC) and (TFD);

(ii) To prove that the properties (TFA), (TFB), (TFC) and (TFD) are equivalent for any object A in Q.

Our main results can be stated as follows:

Theorem 1 *Suppose that A is a C^*-algebra and satisfies the property (TFA). Then pAp satisfies the property (TFA) for any projection $p \in A$.*

Theorem 2 *Let A be a C^*-algebra. If A has the property (TFA), then for any $\varepsilon > 0$, $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, finite subset F in A and finite subset $H = \{b_1, b_2, \ldots, b_m\} \subset A_+$, there exist a non-zero projection $p \in A$ and C^*-subalgebra B of A with $1_B = p$ and the hereditary abstract property (F) such that*

(a) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(b) *$pxp \in_\varepsilon B$ for all $x \in F$,*
(c) *$[f_{\sigma_1}^{\sigma_2}((1-p)b_j(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(pb_jp)]$ for all $b_j \in H$.*

Theorem 3 *Let A be a unital C^*-algebra. Then the following are equivalent:*
(a) *A has the property (TFA);*
(b) *A has the property (TFB);*
(c) *A has the property (TFC);*
(d) *A has the property (TFD).*

In this paper, we have developed some new abstract properties, which generalized some noncommutative topological dimension, for example tracial topological rank, tracial rank zero, tracial stable rank one and so on. In fact, we obtain some equivalent definition on the class of C^*-algebras of tracial topological rank, tracial rank zero and tracial stable rank one based on these properties. Those equivalent definition maybe more useful to further study the properties of those classes of C^*-algebras, which is our focus in the future. For example we may apply those equivalent definition to investigate their properties of K_0-groups and K_0-semigroups of tracial topolgical rank C^*-algebras.

3 Proof of the main results

In this section, we are in a position to give the proofs of the main results. The following lemma plays an important role.

Lemma 3.1 *([9]) Let A be a C^*-algebra and $a, b \in A_+$. Then we have*
(1) *Suppose that $\|a\| \leq 1$ and $\|b\| \leq 1$. Then for any $0 < \delta_1 < \delta_2 < \sigma_1 < \sigma_2 < 1$, there is $\eta = \eta(\delta_1, \delta_2) > 0$ such that $\|a - b\| < \eta$ implies that $[f_{\sigma_1}^{\sigma_2}(a)] \leq [f_{\delta_1}^{\delta_2}(b)] \leq [b]$;*
(2) *$[f_{\sigma_1}^{\sigma_2}(a)] \leq [f_{\sigma_3}^{\sigma_4}(a)]$ for any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$;*
(3) *If $0 \leq a \leq b$, then $[f_{\sigma_1}^{\sigma_2}(a)] \leq [f_{\sigma_3}^{\sigma_4}(b)]$ for any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$;*
(4) *Let $0 \leq a \leq 1$ and p be a nonzero projection in A. Then for any $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, there is $\eta = \eta(\sigma_3, \sigma_4)$ such that $\|ap - pa\| < \eta$ implies*

$$[f_{\sigma_1}^{\sigma_2}(a)] \leq [f_{\sigma_3}^{\sigma_4}(pap)] + [f_{\sigma_3}^{\sigma_4}((1-p)a(1-p))],$$
$$[f_{\sigma_1}^{\sigma_2}(pap)] + [f_{\sigma_1}^{\sigma_2}((1-p)a(1-p))] \leq [f_{\sigma_3}^{\sigma_4}(a)].$$

Now we turn to prove Theorem 1.

Proof. For any finite subset $F \subset pAp$, we assume that F contains the non-zero positive element b. Without loss of generality, we assume F is in the unit ball of pAp. For any $0 < \sigma_3 < \sigma_4 < d_3 < d_4 < d_1 < d_2 < \sigma_1 < \sigma_2 < 1$ and $\eta > 0$, if A has the property (TFA), then there exist projection q and C^*-algebra C of A such that C has the property (F) with $1_C = q$ and for all $x \in H$ and $H = F \bigcup \{p\}$,

$$\|xq - qx\| < \varepsilon, \quad qxq \in_\eta C \quad \text{and} \quad [f_{d_1}^{d_2}((1-q)b(1-q))] \leq [f_{d_3}^{d_4}(qbq)]. \tag{3.1}$$

Moreover, if $\eta > 0$ is sufficiently small, we infer from Lemma 3.1 and the above condition of (1) that there exist two projects $p' \in pAp$ and $p'' \in C$ such that $\|pqp - p'\| < 6\eta$ and $\|qpq - p''\| < 6\eta$. From these we deduce that $\|p' - p''\| < 12\eta$. So, there exists a unitary element $u \in A$ such that

$$\|u - 1\| < 12\sqrt{2}\eta \quad \text{and} \quad up''u^* = p'.$$

Let $B = p'uCu^*p'$. Then B satisfies the property (F) with $1_B = p'$ such that for all $x \in F$, $\|xp' - xqpq\| < 6\eta$, $\|xqpq - qpqx\| < 2\eta$ and $\|qpqx - p'x\| < 6\eta$, and we deduce that

$$\|xp' - p'x\| < 14\eta, \quad \|p'xp' - qpqxqpq\| < 12\eta \quad \text{and} \quad qpqxqpq \in_{72\eta} B. \tag{3.2}$$

From (3.2), we see that the conclusions (i) and (ii) of the property (TFA) hold. Finally, we should prove the conclusion (iii) of the property (TFA). If $\eta > 0$ is small enough, for any $\epsilon > 0$, we conclude from the conditions of (3.1) and (3.2) that

$$\|(1-q)b(1-q) - (1-p')b(1-p')\| < \epsilon \quad \text{and} \quad \|qbq - p'bp'\| < \epsilon. \tag{3.3}$$

It follows from (3.3) that the conditions of Lemma 3.1 are satisfied. In view of Lemma 3.1, we see that

$$\begin{aligned} &[f_{\sigma_1}^{\sigma_2}((1-p')b(1-p'))] \leq [f_{d_1}^{d_2}((1-q)b(1-q))] \\ &\text{and} \quad [f_{d_3}^{d_4}(qbq)] \leq [f_{\sigma_3}^{\sigma_4}(p'bp')]. \end{aligned} \tag{3.4}$$

By the conclusions (3.1) and (3.4), we infer that

$$[f_{\sigma_1}^{\sigma_2}((1-p')b(1-p'))] \leq [f_{\sigma_3}^{\sigma_4}(p'bp')].$$

Hence, the conclusion (iii) of the property (TFA) in Definition 2 holds. This completes the proof. ■

Now, we are ready to prove Theorem 2.

Proof. Recall that $0 < \sigma_3 < \sigma_4 < d_3 < d_4 < d_1 < d_2 < \sigma_1 < \sigma_2 < 1$. We shall prove Theorem 2 by induction. It follows from the property (TFA) of Definition 2 that the case of $m = 1$ holds. Suppose that the conditions $(a) - (c)$ of Theorem 2 hold for $m = j$. Then there exists C^*-algebra B_j of A with the property (F) and $1_{B_j} = p_j$. Moreover, for all $x \in F$ and $b_m \in H (m = 1, 2, \ldots, j)$, one sees that

$$\begin{aligned} &\|xp_j - p_jx\| < \eta, \quad p_jxp_j \in_\eta B_j \quad \text{and} \\ &[f_{d_1}^{d_2}((1-p_j)b_m(1-p_j))] \leq [f_{d_3}^{d_4}(p_jb_mp_j)]. \end{aligned} \tag{3.5}$$

Now, we prove the conclusions $(a) - (c)$ of Theorem 2 for $m = j + 1$. By Theorem 1, we see that $(1 - p_j)A(1 - p_j)$ has the property (TFA). By the property (TFA) defined in Definition 2 with

$$F_j = (1-p_j)A(1-p_j) \cup \{(1-p_j)b_m(1-p_j) : m = 1, 2, \ldots, j\}$$

, we obtain a C^*-subalgebra C_{j+1} of $(1 - p_j)A(1 - p_j)$ with the property (F) and $1_{C_{j+1}} = q_{j+1}$. Moreover, for all $x \in F_j$, one also sees that

$$\begin{aligned} &\|xq_{j+1} - q_{j+1}x\| < \eta, \quad q_{j+1}xq_{j+1} \in_\eta C_{j+1} \quad \text{and} \\ &[f_{d_1}^{d_2}((1-p_j-q_{j+1})b_{j+1}(1-p_j-q_{j+1}))] \leq [f_{d_3}^{d_4}(q_{j+1}b_{j+1}q_{j+1})]. \end{aligned} \tag{3.6}$$

Set $B_{j+1} = B_j \oplus C_{j+1}$ and $p_{j+1} = p_j + q_{j+1}$. Then B_{j+1} satisfies the property (F) and $1_{B_{j+1}} = p_{j+1}$. Furthermore, one has that for all $x \in F$,

$$\|xp_{j+1} - p_{j+1}x\| < \eta \quad \text{and} \quad p_{j+1}xp_{j+1} \in_\eta B_{j+1}. \tag{3.7}$$

From (3.7), we see that the conclusions (a) and (b) of Theorem 2 hold for $m = j+1$. Finally, we prove that the conclusion (c) of Theorem 2 holds for $m = j + 1$. If $\eta \geq 0$ is sufficiently small, for any $\epsilon > 0$, we conclude from the conditions of (3.6) and (3.7) that

$$\begin{aligned} \|(1-p_{j+1})b_m(1-p_{j+1}) + (p_{j+1}-p_j)b_i(p_{j+1}-p_j) - (1-p_j)b_m(1-p_j)\| < \epsilon, \\ \|p_{j+1}b_{j+1}p_{j+1} - q_{j+1}b_{j+1}q_{j+1}\| < \epsilon \quad \text{and} \\ \|p_jb_mp_j - p_{j+1}b_mp_{j+1}\| < \epsilon. \end{aligned} \tag{3.8}$$

It follows from (3.5)-(3.8) and Lemma 3.1 that the conclusion (c) of Theorem 2 holds for $m = 1, 2, ..., j$ is given by

$$\begin{aligned} &[f_{\sigma_1}^{\sigma_2}((1-p_{j+1})b_m(1-p_{j+1}))] \\ &\quad \leq [f_{\sigma_1}^{\sigma_2}((1-p_{j+1})b_m(1-p_{j+1}))] + [f_{\sigma_1}^{\sigma_2}((p_{j+1}-p_j)b_j(p_{j+1}-p_j))] \\ &\quad \leq [f_{d_1}^{d_2}((1-p_j)b_m(1-p_j))] \leq [f_{d_3}^{d_4}(p_jb_mp_j)] \\ &\quad \leq [f_{\sigma_3}^{\sigma_4}(p_{j+1}b_mp_{j+1})]. \end{aligned}$$

In view of (3.8), we find that

$$\begin{aligned} &[f_{\sigma_1}^{\sigma_2}((1-p_{j+1})b_m(1-p_{j+1}))] \\ &\quad \leq [f_{d_1}^{d_2}((1-p_{j+1})b_m(1-p_{j+1}))] \\ &\quad \leq [f_{d_3}^{d_4}(q_{j+1}b_{j+1}q_{j+1})] \\ &\quad \leq [f_{\sigma_3}^{\sigma_4}(p_{j+1}b_{j+1}p_{j+1})], \end{aligned}$$

which implies that the conclusion (c) of Theorem 2 holds for $m = j + 1$ is given. The proof is complete. ■

Finally, we give the proof of Theorem 3.

Proof. It is clear that the conclusions $(b) \Rightarrow (c), (c) \Rightarrow (a)$ and $(d) \Rightarrow (c)$ hold. Here and in the sequel, $A \Rightarrow B$ means that A implies B. In the following we shall prove that the conclusions $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ hold.

(i) We first give the proof of the conclusion $(a) \Rightarrow (b)$. For any finite subset $F = \{x_1, x_2, \ldots, x_n\} \subset A$, finite subset $H = \{b_1, b_2, \ldots, b_m\} \subset A_+$, $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$ and integer $n > 0$, it follows from (a) that the conclusions $(i) - (iii)$ of the property (TFA) in the definition 2 hold. For convenience of notations, let $H \subset F$. We also use the method of induction. The case of $m = 1$ immediately follows from the property (TFA) in the definition 2 and Theorem 2. Now, suppose that the conclusions $(i) - (iii)$ of the property (TFA) hold for $m = j$. Let $\eta = \delta(d_3,\ d_4,\ 2)$ and let $0 < \sigma_3 < \sigma_4 < d_3 < d_4 < d_1 < d_2 < \sigma_1 < \sigma_2 < 1$. By induction, we know that there exists C^*-algebra B_j of A has the property (F) and $1_{B_j} = p_j$. Thus, it follows that for all $x \in F$ and $b_i \in H$ $(i = 1, 2, \ldots, j)$,

$$\begin{aligned} &\|xp_j - p_jx\| < \eta/4, \quad p_jxp_j \in_{\eta/4} B_j \quad \text{and} \\ &j[f_{d_3}^{d_4}((1-p_j)b_i(1-p_j))] \le [f_{\sigma_3}^{\sigma_4}(p_jb_ip_j)]. \end{aligned} \tag{3.9}$$

By Theorems 1 and 2, we obtain a C^*-subalgebra C_{j+1} of $(1-p_j)A(1-p_j)$ with the property (F) and $1_{C_{j+1}} = q_{j+1} \le 1 - p_j$ such that for all $x \in (1-p_j)F(1-p_j)$ and for all $b_i \in H$,

$$\begin{aligned} &\|xq_{j+1} - q_{j+1}x\| < \eta/4, \quad q_{j+1}xq_{j+1} \in_{\eta/4} C_{j+1} \quad \text{and} \\ &[f_{\sigma_1}^{\sigma_2}((1-p_j-q_{j+1})b_i(1-p_j-q_{j+1}))] \le [f_{d_1}^{d_2}(q_{j+1}b_iq_{j+1})], \end{aligned} \tag{3.10}$$

where $0 < \sigma_3 < \sigma_4 < d_3 < d_4 < d_1 < d_2 < \sigma_1 < \sigma_2 < 1$. Set $B_{j+1} = B_j \oplus C_{j+1}$ and $p_{j+1} = p_j + q_{j+1}$. Then we have B_{j+1} with the property (F) and $1_{B_{j+1}} = p_{j+1}$ such that for all $x \in F$,

$$\|xp_{j+1} - p_{j+1}x\| < \eta \quad \text{and} \quad p_{j+1}xp_{j+1} \in_\eta B_{j+1}. \tag{3.11}$$

In view of (3.11), we find that the conclusions of (i) and (ii) in Definition 3 hold for $m = j + 1$. Finally, we should prove the conclusion of (iii) in Definition 3 holds for $m = j + 1$. By (3.10) and Lemma 3.1, we have

$$\begin{aligned} &2[f_{\sigma_1}^{\sigma_2}((1-p_{j+1})b_i(1-p_{j+1}))] \\ &\quad \le [f_{d_1}^{d_2}((1-p_{j+1})b_i(1-p_{j+1}))] + [f_{d_1}^{d_2}((q_{j+1}b_iq_{j+1}))] \\ &\quad \le [f_{d_3}^{d_4}((1-p_j)b_i(1-p_j))]. \end{aligned} \tag{3.12}$$

By (3.9) and (3.12), we obtain

$$\begin{aligned} (j+1)[f_{\sigma_1}^{\sigma_2}((1-p_{j+1})b_i(1-p_{j+1}))] &\le 2j[f_{\sigma_1}^{\sigma_2}((1-p_{j+1})b_i(1-p_{j+1}))] \\ &\le j[f_{d_3}^{d_4}((1-p_j)b_i(1-p_j))] \\ &\le [f_{\sigma_3}^{\sigma_4}(p_{j+1}b_ip_{j+1})]. \end{aligned}$$

Hence, the conclusion (iii) of the property (TFB) in Definition 3 holds for $m = j + 1$. This ends the induction. Thus, $(a) \Rightarrow (b)$ holds.

(ii) We finally give the proof of the conclusion $(c) \Rightarrow (d)$. The following proof needs to be taken into consideration. Let $\Re_0^4$ be the set of quaternary real array with $0 < d_4 < d_3 < d_2 < d_1 < 1$. Now, we will show that for any $\varepsilon > 0$, positive integer j, finite subset $F = \{x_1, x_2, \ldots, x_n\} \subset A$, finite subset $H = \{b_1, b_2, \ldots, b_m\} \subset A_+$, and $W = \{w_1, w_2, \ldots, w_k\} \subset \Re_0^4$, there exist a non-zero projection $p \in A$ and C^*-subalgebra B of A with $1_B = p'$ and the property (F) such that

$$\|xp' - p'x\| < \varepsilon \quad \text{and} \quad p'xp' \in_\varepsilon B \quad \text{for all} \quad x \in F, \tag{3.13}$$

$$j[f_{d_2}^{d_1}((1-p')b(1-p'))] \le [f_{d_4}^{d_3}(p'bp')] \quad \text{for all} \quad b \in H \quad \text{and} \quad w \in W. \tag{3.14}$$

We use the method of induction. The case of $k = 1$ follows the property (TFC) in Definition 4. Suppose that the case of $k = l$ holds. Let $W^* = \{w_1^*,\ w_2^*,\ \ldots,\ w_{l+1}^*\} \subset \Re_0^4$. For any $w_i \in W$, $w_i^* \in W^*$, $w_i = (d_1,\ d_2,\ d_3,\ d_4)$ and $w_i^* = (c_1,\ c_2,\ c_3,\ c_4)$ such that $0 < d_4 < d_3 < c_4 < c_3 < c_2 < c_1 < d_2 < d_1 < 1$. Then for any $\delta > 0$, $j \in \mathbb{N}$, finite subset $F = \{x_1,\ x_2,\ \ldots,\ x_n\} \subset A$, $H = \{b_1,\ b_2,\ \ldots,\ b_m\} \subset A_+$ and $W = \{w_1,\ w_2,\ \ldots,\ w_l\} \subset \Re_0^4$, there exist non-zero projection $p \in A$ and C^*-algebra $B_1 \subset A$ which satisfies the property (F) and $1_{B_1} = p$ such that for all $x \in F \bigcup H$, $b \in H$ and $w_i^* = (c_1,\ c_2,\ c_3,\ c_4) \in W^*$, $i = 1, 2, \ldots, l$,

$$\begin{aligned} \|xp - px\| < min\{\varepsilon/4, \delta/2\}, \quad pxp \in_{\varepsilon/4} B_1 \quad \text{and} \\ j[f_{c_2}^{c_1}((1-p)b(1-p))] \le [f_{c_4}^{c_3}(pbp)]. \end{aligned} \tag{3.15}$$

It follows from Theorem 1 and the above proof of $(a) \Rightarrow (b)$ that $(1-p)A(1-p)$ has the property (TFC). Let $F_1 = \{(1-p)x_i(1-p) : i = 1, 2, \ldots, n\}$ and let $H_1 = \{(1-p)b_j(1-p) : j = 1, 2, \ldots, m\}$. Then there exist projection q of A with $q \le 1-p$ and C^*-algebra $B_2 \subset (1-p)A(1-p)$ with the property (F) and $1_{B_2} = q$ such that for all $x \in F_1 \bigcup H_1$, $b \in H_1$ and $w_{l+1}^* = (c_1,\ c_2,\ c_3,\ c_4) \in W^*$,

$$\begin{aligned} \|xq - qx\| < min\{\varepsilon/4, \delta/2\}, \quad qxq \in_{\varepsilon/4} B_2 \quad \text{and} \\ j[f_{c_2}^{c_1}((1-p-q)b(1-p-q))] \le [f_{c_4}^{c_3}(qbq)]. \end{aligned} \tag{3.16}$$

Set $B = B_1 \bigoplus B_2$. Then B satisfies the property (F) and $1_B = p + q$. So we have for all $x \in F$,

$$\|x(p+q) - (p+q)x\| < \varepsilon \quad \text{and} \quad (p+q)x(p+q) \in_{\varepsilon} B. \tag{3.17}$$

Let $p' = p + q$. From (3.17), we know that the conclusion (3.13) holds for $m = l + 1$. Finally, we should prove the conclusion (3.14) holds for $m = l+1$. For each $\delta > 0$ and $b \in H$, $\|pb - bp\| < \delta$ and $\|(1-p-q)b - b(1-p-q)\| < \delta$, we have

$$\begin{aligned} j[f_{d_2}^{d_1}((1-p-q)b(1-p-q))] &\le j[f_{c_2}^{c_1}((1-p)b(1-p))] \\ &\le [f_{c_4}^{c_3}(pbp)] \le [f_{d_4}^{d_3}((p+q)b(p+q))], \end{aligned}$$

for all $w_i = (d_1,\ d_2,\ d_3,\ d_4) \in \{w_1,\ w_2,\ \ldots,\ w_l\}$ and $w_i^* = (c_1,\ c_2,\ c_3,\ c_4) \in \{w_1^*,\ w_2^*,\ \ldots,\ w_l^*\}$.

Similarly, by $\|pb - bp\| < \delta$, we have

$$\begin{aligned} j[f_{d_2}^{d_1}((1-p-q)b(1-p-q))] &\le j[f_{c_2}^{c_1}((1-p-q)b(1-p-q))] \\ &\le [f_{c_4}^{c_3}(qbq)] \le [f_{d_4}^{d_3}((p+q)b(p+q))], \end{aligned}$$

where $w_{l+1}^* = (c_1,\ c_2,\ c_3,\ c_4)$ and $w_{l+1} = (d_1,\ d_2,\ d_3,\ d_4)$.

Hence, the conclusion (3.14) holds for $m = l + 1$ is given by

$$j[f_{d_2}^{d_1}((1-p-q)b(1-p-q))] \le [f_{d_4}^{d_3}((p+q)b(p+q))],$$

for all $w_i = (d_1,\ d_2,\ d_3,\ d_4) \in \{w_1,\ w_2,\ \ldots,\ w_{l+1}\}$ and $w_i^* = (c_1,\ c_2,\ c_3,\ c_4) \in \{w_1^*,\ w_2^*,\ \ldots,\ w_{l+1}^*\}$.

Thus we have proved that $(c) \Rightarrow (d)$. This completes the proof. ■

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