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On the Nonexistence of Pure Multi-Solitons for the Benjamin-Bona-Mohony Equation

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Abstract: In this paper, we consider the nonexistence of pure multi-solitons for the Benjamin-Bona-Mohony (BBM) equation. In stead of describing the solution for all time, we use the conservation laws to control the speeds at $+\infty$ for ingoing multi-solitons. By estimated the tail of the solution $u(t,x)$ for large t and large x, we get a contradiction in this regime. Indeed, we prove the nonexistence of pure multi-solitons for the BBM equation.

Keywords: BBM equation; multi-soliton; collision; solitary wave.

1 Introduction

This paper concerns the BBM equation, which is:

$$(1-\partial_x^2)u_t+\partial_x(u+u^2)=0, \quad t,x\in\mathbb{R}, \tag{1.1}$$

for the function $u(t,x)$ of time t and a single spatial variable x. The BBM equation was introduced by Peregrine[1] and Benjamin, Bona and Mahony[2] as an alternative to the KdV equation

$$\partial_t u+\partial_x(\partial_x^2 u+u^2)=0, \tag{KdV}$$

for the description of the unidirectional propagation of long waves of small amplitude in water.

1.1 Review on the collision problem for BBM type equations

We briefly review some results concerning the problem of solitons for BBM type models. Collision problems for gKdV and BBM have been studied since 60's from different points of view [3–8].

Mizumachi [7] studied the asymptotic stability of solitary wave solutions to the regularized long-wave equation (RLW) in $H^1(R)$

$$(1-\partial_x^2)u_t+\partial_x\left(u+\frac{1}{2}u^2\right)=0, \quad t,x\in\mathbb{R}. \tag{RLW}$$

He extended the method of Miller and Weinstein [6] to the RLW equation.

Assumed $\alpha := inf_{y\in\mathbb{R}}|u_0-Q_{c_0}(\cdot+y)|_{H^1}$, then there exists a $c_*>1$ and an $\alpha_0>0$ satisfying the following cases:
Case 1: Suppose $c_0\in(1,c_*)$ and $\alpha\in(0,\alpha_0)$. Then, there exists a $c_+>1$ and a C^1-function $x(t)$ such that

$$u(t,\cdot+x(t))\rightharpoonup Q_{c_+} \quad in \;\; H^1(\mathbb{R}),$$

and $x_t\to c_+$ as $t\to\infty$.
Case 2: The conclusion of case 1 holds for the speed $c_0\in[c_*,\infty)$, except for an exceptional set of values that have no finite accumulation point.

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El Dika and Martel [9] combined a generalization of argument of [4]-[5] with some tools developed by El Dika [3] for the BBM equation to consider the generalized BBM equation:

$$(1-\partial_x^2)u_t + \partial_x(u+u^p) = 0, \qquad (gBBM)$$

for $p \geq 2$ integer.

For $c > 1$, they proved that the N solitary waves $\varphi_{c_j}(x - c_j^0 t - x_j^0)$ of gBBM are stable and their sum is also stable. Moreover, in an appropriate sense, they provided that the solitary waves are sufficiently decoupled. They also stated another result for gBBM equation: Let $N \geq 1$, given $1 < c_1^0 < \ldots < c_N^0$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$, then there exists a unique solution $U(t)$ of gBBM such that

$$\lim_{t\to+\infty} \left\| U(t) - \sum_{j=1}^{N} \varphi_{c_j^0}(x - x_j^0 - c_j^0 t) \right\|_{H^1} = 0.$$

Martel extended the result of [9] to the collision of two solitons of the BBM equation in the following two cases:

(a) Solitons of different speeds [10]: they considered two solitary waves $\varphi_{c_1}(x - c_1 t)$, $\varphi_{c_2}(x - c_2 t)$ in the case where $1 < c_2 < c_1$ and c_2 is close to c_1, which means that the function φ_{c_2} is small in H^1. It was proved that collision of two solitary waves of BBM equation is inelastic but almost elastic, implied that there exists no pure 2-soliton solution corresponding to the speeds c_1, c_2 in the case of $0 < c_2 - 1 < \epsilon$, for $\epsilon > 0$.

(b) Solitons with almost equal speeds[11]: they considered the BBM equation, for $\lambda \in [0, 1)$,

$$(1-\lambda\partial_x^2)u_t + \partial_x(\partial_x^2 u - u + u^2) = 0. \qquad (BBM)$$

Also, they got the same result of [10] that there exists no pure 2-soliton solution here.

1.2 Main result

In the present paper, we extend the method introduced in [12] to the BBM equation. The main point of this context is to prove nonexistence of pure multi-soliton of BBM equation without trying to describe the solution for all time, and we do not need to compute the main order of the solution for all t, x.

Recall that the cauchy problem for (1.1) is globally well-posed in H^1 [2], and any H^1 solutions $u(t)$ of (1.1) satisfy the following conservation laws:

$$\frac{1}{2}\int (u_x^2 + u^2)dx = M(u(t)) = M(u(0)) \quad (mass) \qquad (1.2)$$

$$\int \left(\frac{1}{2}u^2 + \frac{1}{3}u^3\right)dx = E(u(t)) = E(u(0)) \quad (energy) \qquad (1.3)$$

The quantity $\int u(t)$ is also formally conserved. It is also well-known that the BBM equation has soliton solutions: for $c > 1$, set

$$\varphi_c(x) = (c-1)Q\left(\sqrt{\frac{c-1}{c}}x\right),$$

where

$$Q(x) = \frac{3}{2}sech^2\left(\frac{x}{2}\right) \qquad satisfies \quad Q'' + Q^2 = Q.$$

We call outgoing multi-soliton a solution $u(t)$ of (1.1) such that:

$$\lim_{t\to+\infty} \left\| u(t) - \sum_{j=1}^{N} \varphi_{c_j}(x - c_j t - \triangle_j) \right\|_{H^1} = 0, \qquad (1.4)$$

for $N \geq 2$, $1 < c_N < \ldots < c_1$, and $\triangle_j \in \mathbb{R}$.

Similarly, we call ingoing multi-soliton a solution $u(t)$ of (1.1) such that:

$$\lim_{t\to-\infty} \left\| u(t) - \sum_{j=1}^{N} \varphi_{c_j}(x - c_j t - \triangle_j) \right\|_{H^1} = 0. \qquad (1.5)$$

However, the existence and uniqueness of an ingoing multi-soliton was proved in [9] and [10]. Moreover, $u(t)$ is such that, for any $s \geq 1, t \geq 0$,

$$\left\| u(t) - \sum_{j=1}^{N} \varphi_{c_j}(x - c_j t - \triangle_j) \right\|_{H^s} \leq A_s e^{-\gamma t},$$

where $\gamma > 0$, and $A_s > 0$.

The following is the main result of this paper.

Theorem 1.1 *Let $N \geq 2$, $u(t)$ be an ingoing multi-soliton of (1.1), with parameters $\triangle_1, \ldots, \triangle_N \in \mathbb{R}$, $1 < c_N < \ldots < c_1$,*

$$\lim_{t \to -\infty} \left\| u(t) - \sum_{j=1}^{N} \varphi_{c_j}(x - c_j t - \triangle_j) \right\|_{H^1} = 0.$$

Then, $u(t)$ is not an outgoing multi-soliton at $+\infty$.

In particular, Theorem 1.1 implies that there exists no pure multi-soliton of (1.1) with speeds $c_1, c_2, \ldots, c_N$ at $+\infty$ or at $-\infty$.

Remark 1 *In the case $N = 2$, Theorem 1.1 proves the nonexistence of pure 2-soliton under the condition $1 < c_2 < c_1$.*

2 Proof the nonexistence of pure N-solitons

In this section, we reduce the problem to its simpler form and use the conservation laws to control the speeds at $+\infty$ for an ingoing multi-soliton. Then, we prove theorem 1.1.

2.1 Conservation laws on outgoing N-solitons

Let

$$c_1 > 1 \quad \text{and} \quad \lambda = \frac{c_1 - 1}{c_1} \in (0, 1). \tag{2.1}$$

We introduce the following change of variable(see [7])

$$x' = \lambda^{\frac{1}{2}} \left(x - \frac{t}{1 - \lambda} \right), \quad t' = \frac{\lambda^{\frac{3}{2}}}{1 - \lambda} t, \quad z(t', x') = \frac{1 - \lambda}{\lambda} u(t, x). \tag{2.2}$$

If $u(t, x)$ is a solution of (1.1), then $z(t', x')$ satisfies

$$(1 - \lambda \partial_{x'}^2)\partial_{t'} z + \partial_{x'}(\partial_{x'}^2 z - z + z^2) = 0. \tag{2.3}$$

Lemma 2.1 *(i) Let $c > 1$. By the change of variable (2.2), a solitary wave solution $\varphi_c(x - ct)$ to (1.1) is transformed into $\widetilde{Q}_\sigma(y_\sigma)$ which is a solution of (2.3) where*

$$\widetilde{Q}_\sigma(x) = \sigma \theta_\sigma Q(\sqrt{\sigma} x), \quad Q(x) = \frac{3}{2} sech^2(\frac{x}{2}),$$

$$\sigma = \frac{c - 1}{c\lambda}, \quad \theta_\sigma = \frac{1 - \lambda}{1 - \lambda\sigma}, \quad \mu_\sigma = \frac{1 - \sigma}{1 - \lambda\sigma}, \quad y_\sigma = x' + \mu_\sigma t'.$$

Especially, if $c = c_1$, then $\mu_\sigma = 0, y_\sigma = x'$ and $\widetilde{Q}_\sigma(y_\sigma) = Q(x')$

$$Q'' + Q^2 = Q, \;\; (Q')^2 + \frac{2}{3} Q^3 = Q^2 \qquad on \;\; \mathbb{R}. \tag{2.4}$$

(ii) Moreover, $\widetilde{Q}_\sigma$ satisfies the following

$$\widetilde{Q}_\sigma'' + \frac{1}{\theta_\sigma} \widetilde{Q}_\sigma^2 = \sigma \widetilde{Q}_\sigma, \quad (\widetilde{Q}_\sigma')^2 + \frac{2}{3\theta_\sigma} \widetilde{Q}_\sigma^3 = \sigma \widetilde{Q}_\sigma^2.$$

For $\sigma > 0$ small,

$$\|\widetilde{Q}_\sigma\|_{L^\infty(\mathrm{R})} \sim (1 - \lambda)\sigma \|Q\|_{L^\infty(\mathrm{R})}, \quad \|\widetilde{Q}_\sigma\|_{L^2(\mathrm{R})} \sim (1 - \lambda)\sigma^{\frac{3}{4}} \|Q\|_{L^2(\mathrm{R})}.$$

Proof. The proof of Lemma 2.1 is similar to Claim 2.1 in [10], so it is omitted. ■

Lemma 2.2 *Let $N \geq 2$, $1 < c_N < \ldots < c_1$ and $\triangle_1, \ldots, \triangle_N \in \mathbb{R}$, let $u(t)$ be the solution of (1.1), which satisfies*

$$\lim_{t\to-\infty}\left\|u(t)-\sum_{j=1}^{N}\varphi_{c_j}(\cdot-c_jt-\triangle_j)\right\|_{H^1}=0.$$

Then, for all t,

$$\int \widetilde{Q}_\sigma^2(x)=\theta_\sigma^2\sigma^{\frac{3}{2}}\int Q^2,$$

$$\int \widetilde{Q}_\sigma^3(x)=\theta_\sigma^3\sigma^{\frac{5}{2}}\int Q^3,$$

$$\int \widetilde{Q}_\sigma(x)=\theta_\sigma\sigma^{\frac{1}{2}}\int Q.$$

Proof. The proof of Lemma 2.2 can be obtained by explicit computations. ■

Using conservation laws mass (1.2), energy (1.3), integral $\int u(t)$ and Lemma 2.2,

$$M(\widetilde{Q}_\sigma)=\frac{1}{2}\int(\widetilde{Q}_{\sigma x}^2+\widetilde{Q}_\sigma^2)dx=\frac{1}{2}\theta_\sigma^2\sigma^{\frac{5}{2}}\int Q_x^2+\frac{1}{2}\theta_\sigma^2\sigma^{\frac{3}{2}}\int Q^2,$$

$$E(\widetilde{Q}_\sigma)=\int\left(\frac{1}{2}\widetilde{Q}_\sigma^2+\frac{1}{3}\widetilde{Q}_\sigma^3\right)dx=\frac{1}{2}\theta_\sigma^2\sigma^{\frac{3}{2}}\int Q^2+\frac{1}{3}\theta_\sigma^3\sigma^{\frac{5}{2}}\int Q^3,$$

$$\int \widetilde{Q}_\sigma(x)=\theta_\sigma\sigma^{\frac{1}{2}}\int Q.$$

We can get the following identities:

$$\sum_{j=1}^{N}\left(\frac{c_j-1}{c_j\lambda}\right)^{\frac{1}{2}}=\sum_{j=1}^{N^-}\left(\frac{c_j^--1}{c_j^-\lambda}\right)^{\frac{1}{2}},\tag{2.5}$$

$$\sum_{j=1}^{N}\left(\frac{c_j-1}{c_j\lambda}\right)^{\frac{3}{2}}=\sum_{j=1}^{N^-}\left(\frac{c_j^--1}{c_j^-\lambda}\right)^{\frac{3}{2}},\tag{2.6}$$

$$\sum_{j=1}^{N}\left(\frac{c_j-1}{c_j\lambda}\right)^{\frac{5}{2}}=\sum_{j=1}^{N^-}\left(\frac{c_j^--1}{c_j^-\lambda}\right)^{\frac{5}{2}}.\tag{2.7}$$

Set $y_j=\frac{c_j-1}{c_j\lambda}$, $y_j^-=\frac{c_j^--1}{c_j^-\lambda}$, then (2.5)-(2.7) turn out to be

$$\sum_{j=1}^{N}y_j^{\frac{1}{2}}=\sum_{j=1}^{N^-}(y_j^-)^{\frac{1}{2}},\quad\sum_{j=1}^{N}y_j^{\frac{3}{2}}=\sum_{j=1}^{N^-}(y_j^-)^{\frac{3}{2}},\quad\sum_{j=1}^{N}y_j^{\frac{5}{2}}=\sum_{j=1}^{N^-}(y_j^-)^{\frac{5}{2}}.\tag{2.8}$$

2.2 Rigidity result for two solitons

Proposition 2.1 *Let $c > 1$, $u(t)$ is the ingoing 2-soliton of (1.1), which satisfies*

$$\lim_{t\to-\infty}\left\|u(t)-\sum_{j=1,2}\varphi_{c_j}(\cdot-c_jt)\right\|_{H^1}=0.\tag{2.9}$$

Assume that $u(t)$ is an outgoing multi-soliton, then there exist $1 < c_N < \ldots < c_1$ and $\triangle_1, \ldots, \triangle_N$ such that

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{j=1}^{N}\varphi_{c_j}(\cdot-c_jt-\triangle_j)\right\|_{H^1}=0.$$

Then $u(t)$ is a pure 2-soliton,

$$N = 2, \ \ 0 < y < 1, \ \ y_1 = 1 \ \ and \ \ y_2 = y.$$

Using Proposition 2.1, we can easily obtain Theorem 1.1, so our main task is to prove Proposition 2.1. By Lemma 2.1 and (2.8), we have

$$1 + y^{\frac{1}{2}} = \sum_{j=1}^{N} y_j^{\frac{1}{2}}, \quad 1 + y^{\frac{3}{2}} = \sum_{j=1}^{N} y_j^{\frac{3}{2}} \quad and \ \ 1 + y^{\frac{5}{2}} = \sum_{j=1}^{N} y_j^{\frac{5}{2}}.$$

Setting $a_j = y_j^{\frac{1}{2}}$ and $x = y^{\frac{1}{2}}$, then Proposition 2.1 turns out to prove the following Lemma.

Lemma 2.3 *Let $0 < x < 1$, $N \geq 2$ and $0 < a_N < \ldots < a_1$ be such that*

$$1 + x = \sum_{j=1}^{N} a_j, \quad 1 + x^3 = \sum_{j=1}^{N} a_j^3, \quad 1 + x^5 = \sum_{j=1}^{N} a_j^5. \tag{2.10}$$

Then $N = 2$, $a_1 = 1$ and $a_2 = x$.

Proof. (i)The case of $N = 2$:

Let $a_1 = a$ and $a_2 = b$, $0 < b < a \leq 1$ be such that

$$a + b = 1 + x,$$
$$a^3 + b^3 = 1 + x^3 = 1 + (a + b - 1)^3,$$
$$a^5 + b^5 = 1 + x^5 = 1 + (a + b - 1)^5.$$

Of course, $a = 1, \ b = x$ is a solution. Assuming that $0 < a < 1 + x, \ a \neq 1$, set $f(b) = a^3 + b^3 - 1 - (a + b - 1)^3$. We see that $f(1) = 0$ and $f'(b) = 3b^2 - 3(a + b - 1)^2 = 3[b^2 - (a + b - 1)^2]$. If $f'(b) = 0$, then $b^2 = (a + b - 1)^2$, $a = 1$, which is in contradiction with $a \neq 1$. So $f'(b) \neq 0$ on $(0, 1)$. That is to say, $f(b) > f(1)$ or $f(b) < f(1)$ on $(0, 1)$, and thus $f(b)$ has no zero on $(0, 1)$. So $a = 1, \ b = x$ is the unique solution for $N = 2$.

(ii) The case of $N \geq 3$:

We define the bounded set

$$\Omega = \left\{ (a_1, \ldots, a_N) \in (\mathbb{R}_+^*)^N \mid \sum_{j=1}^{N} a_j = 1 + x, \ \ \sum_{j=1}^{N} a_j^5 = 1 + x^5 \right\}.$$

Define function F:

$$F(a_1, \ldots, a_N) = \frac{1}{\sum_{j=1}^{N} a_j^3},$$

and look for the minimum on Ω of F.

If $(a_1, \ldots, a_N)$ is a point of Ω where the gradients of functions $\sum_{j=1}^{N} a_j$ and $\sum_{j=1}^{N} a_j^5$ are colinear, then $a_j = a$ for all $j \in 1, \ldots, N$. Thus,

$$Na = 1 + x, \quad Na^5 = 1 + x^5.$$

We see $a = \frac{1+x}{N}$, for $x \in (0, 1)$, we have $\frac{1}{N} < a < \frac{2}{N}$ and $a^4 = \frac{1+x^5}{Na} = \frac{1+x^5}{1+x} > \frac{1}{1+x} > \frac{1}{2}$ which imply $(\frac{2}{N})^4 > \frac{1}{2}$ and so $N < 2^{1+\frac{1}{4}}$. This is in contradiction with $N \geq 3$ and so no such point exist on Ω.

Therefore, we apply the method of Lagrange multipliers to characterize extrema of F on Ω. For a critical point $(a_1, \ldots, a_N) \in \Omega$ of F, there exist $\lambda, \ \mu \in \mathbb{R}$ such that

$$\frac{1}{a_j^3} = \lambda a_j + \mu a_j^5, \quad \forall j = 1, \ldots, N.$$

Define $\alpha = a_j^4$, $g(\alpha) = \mu\alpha^2 + \lambda\alpha - 1$, we can see that $g' = 2\mu\alpha + \lambda$ has at most one root on $[0, +\infty)$, thus $g(\alpha)$ has at most two roots on $[0, +\infty)$. However, we have already observe that Ω contains no point of the form $(a, \ldots, a)$.

Hence, the (a_j) take exactly two different values: there exist $0 < b < a \leq 1$ and $1 \leq k < N$ such that

$$ka + (N - k)b = 1 + x, \quad ka^5 + (N - k)b^5 = 1 + x^5.$$

1) For $k = 1$, then

$$a - 1 + (N-1)b = x, \quad a^5 - 1 + (N-1)b^5 = x^5.$$

Since

$$|a^3 - 1| = |a-1|(a^2 + a + 1) \geq |a-1|,$$

We have $a^3 - 1 \leq a - 1$, then

$$a^3 - 1 + (N-1)b^3 < a - 1 + (N-1)b^3 = x + (N-1)(b^3 - b).$$

Since $3b < a + 2b \leq 1 + x < 2$, we have $b < \frac{2}{3}$. Define $h(b) = b^3 - b$, then h is monotone increasing on $[\frac{\sqrt{3}}{3}, \frac{2}{3})$ and so $h(b) < h(\frac{2}{3})$. Note that

$$a^3 - 1 + (N-1)b^3 < x + (3-1)((\frac{2}{3})^3 - \frac{2}{3}) = x - \frac{20}{27}.$$

Observe that the function $h(x) = x^3 - x$ reaches it minimum at $x = \frac{\sqrt{3}}{3}$. While $h(\frac{\sqrt{3}}{3}) = -\frac{2\sqrt{3}}{9} > -\frac{20}{27}$, we have $x^3 - x > -\frac{20}{27}$, so

$$a^3 + (N-1)b^3 < 1 + x^3.$$

It follows that at such a critical point, F is strictly greater than $\frac{1}{1+x^3}$.

2) For $2 \leq k \leq N-2$, $N \geq 4$, then

$$ka + (N-k)b = 1 + x, \quad ka^5 + (N-k)b^5 = 1 + x^5.$$

By using the similar method of 1), we can get

$$k(a^3 - 1) + (N-k)b^3 < k(a-1) + (N-k)b^3 = x + 1 - k + (N-k)(b^3 - b) < x + 1 - k - \frac{10}{27}(N-k)$$

$$\leq x + 1 - k + \tfrac{10}{27}(N - 2 - N) = x + 1 - k - \tfrac{20}{27} < x^3 + 1 - k.$$

So

$$\frac{1}{ka^3 + (N-k)b^3} > \frac{1}{1+x^3}.$$

3) For $k = N-1$, $N \geq 4$, then

$$(N-1)a + b = 1 + x, \quad (N-1)a^5 + b^5 = 1 + x^5.$$

By using the similar method of 1), we also get

$$b^3 - 1 + (N-1)a^3 < b - 1 + (N-1)a^3 = x + (N-1)(a^3 - a).$$

Since $3a < 3a + b \leq 1 + x < 2$, we have $a < \frac{2}{3}$. So

$$b^3 - 1 + (N-1)a^3 < x + (4-1)((\frac{2}{3})^3 - \frac{2}{3}) = x - \frac{30}{27} < x^3.$$

Again, at such a critical point, F is strictly greater than $\frac{1}{1+x^3}$.

4) For $N = 3$, then there are two cases:

- $$a + 2b = 1 + x, \quad a^5 + 2b^5 = 1 + x^5.$$

  By direct computation, we obtain

  $$a^3 - 1 + 2b^3 < a - 1 + 2b^3 = x + 2(b^3 - b) < x + 2((\frac{2}{3})^3 - \frac{2}{3}) = x - \frac{20}{27} < x^3.$$

- $$2a + b = 1 + x, \quad 2a^5 + b^5 = 1 + x^5.$$

  In this case, we assume $0 < b < a < \frac{2}{3}$, then

  $$2a^3 - 1 + b^3 < b - 1 + 2a^3 = x + 2(a^3 - a) < x + 2((\frac{2}{3})^3 - \frac{2}{3}) = x - \frac{20}{27} < x^3.$$

Again, F is strictly greater than $\frac{1}{1+x^3}$, which means no such critical point exists in above cases. Therefore, we complete the proof of Lemma 2.3. ■

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