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The Averaging of Linear Differential Inclusions with Variable Dimension on Finite Interval

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Abstract: In this paper, we consider the structure and properties of the solution set for linear differential inclusions with variable dimension and substantiate the Krylov-Bogolyubov averaging method.

Keywords: differential inclusion; small parameter; averaging

1 Introduction

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or the integrodifferential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov [10], and has been used extensively in engineering applications [3, 13, 20, 25]. Books that cover averaging theory for differential equations and inclusions include [3, 5, 9, 12–14, 18, 25]. In this paper, we consider the structure and properties of the solution set for linear differential inclusions with variable dimension and substantiate the Krylov-Bogolyubov averaging method.

2 Main definitions

Let $comp(\mathbb{R}^n)$ $(conv(\mathbb{R}^n))$ be the family of all nonempty compact (convex) subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(A,B)=\max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Let $I=[0,T]$, $n:I\to\mathbb{N}$ be a piecewise-constant function, right-continuous and bounded with constant $\overline{n}>0$: $n(t)\le\overline{n}$ for all $t\in I$.

Consider a linear differential inclusion

$$\dot{x}_i\in A_i(t)x_i+F_i(t),\ t\neq\tau_i,\ x_0(\tau_0)=x_0(0)=\overline{x}_0,\ i=\overline{0,m}, \tag{1}$$

$$x_i(\tau_i)=M_ix_{i-1}(\tau_i-0),\ i=\overline{1,m}, \tag{2}$$

where $x(t)\in\mathbb{R}^{n(t)}$ is a phase variable, $t\in I$ is time, $\tau_i\in I$, $i=\overline{1,m}$ are fixed moments of time enumerated in the increasing order $(\tau_i<\tau_{i+1})$ such that $n(\tau_i-0)\neq n(\tau_i)$, $A_i(t)$ is $(n(t)\times n(t))$-dimensional matrix-valued function, $F_i:[\tau_i,\tau_{i+1}]\to conv(\mathbb{R}^{n(t)})$ is a set-valued mapping, M_i is $(n(\tau_i)\times n(\tau_i-0))$ matrix, $\tau_{m+1}=T$.

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Example 1 *Consider the following linear system*

$$\dot{x} \in x + S_t(0),\ x(0) = 0, \tag{3}$$

where $t \in \mathbb{R}_+$, $n(t) = \left[1 + |\sqrt{2}sin(t)|\right]$, $M(t) = \begin{cases} E(n(t)), & n(t-0) - n(t) = 0 \\ m_{ij} = \frac{1}{n(t)}, & n(t-0) - n(t) \neq 0 \end{cases}$ *for* $t > 0$, $M(0) = E(n(0))$, $E(n(t))$ *is* $(n(t) \times n(t))$ *unit matrix,* $[\cdot]$ *- entier function.*

Obviously, $n(t) = \begin{cases} 1, & t \in [0, \frac{\pi}{4}), \\ 2, & t \in [\frac{\pi}{4} + \pi i, \frac{3\pi}{4} + \pi i), \\ 1, & t \in [\frac{3\pi}{4} + \pi i, \frac{5\pi}{4} + \pi i), \end{cases}$ $M(t) = \begin{cases} 1, & t \in (0, \frac{\pi}{4}) \cup \bigcup\limits_i (\frac{3\pi}{4} + \pi i, \frac{5\pi}{4} + \pi i), \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & t \in (\frac{\pi}{4} + \pi i, \frac{3\pi}{4} + \pi i), \\ (1,1), & t = \frac{3\pi}{4} + 2\pi i, \\ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, & t = \frac{\pi}{4} + 2\pi i, \end{cases}$,

$$S(t) = \begin{cases} [-t, t] \subset R, & t \in [0, \frac{\pi}{4}) \cup \bigcup\limits_i (\frac{3\pi}{4} + \pi i, \frac{5\pi}{4} + \pi i), \\ \{(s_1, s_2)^T \in R^2 \,:\, \sqrt{s_1^2 + s_2^2} \leq t\}, & t \in \bigcup\limits_i (\frac{\pi}{4} + \pi i, \frac{3\pi}{4} + \pi i), \end{cases} \quad i = 0, 1, \ldots .$$

Then $\quad X(t) = [t + 1 - e^t, e^t - t - 1],\ t \in \left[0, \frac{\pi}{4}\right)$,

$$X\left(\frac{\pi}{4}\right) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} X\left(\frac{\pi}{4} - 0\right) = \left\{(x_1(\alpha), x_2(\alpha))^T \,:\, x_i(\alpha) = \left(\frac{\pi}{8} + \frac{1}{2} - \frac{1}{2}e^{\frac{\pi}{4}}\right)\alpha,\ \alpha \in [-1, 1],\ i = 1, 2\right\},$$

$$X(t) = e^{t - \frac{\pi}{4}} X\left(\frac{\pi}{4}\right) + S_{e^{-\frac{\pi}{4}}(1+\frac{\pi}{4})e^t - t - 1}(0),\ t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right),$$

$$X\left(\frac{3\pi}{4}\right) = (1, 1) X\left(\frac{3\pi}{4} - 0\right) =$$

$$= \left[-\left(\frac{\pi}{8} + \frac{1}{2}\right)e^{\frac{\pi}{2}} + \frac{e^{\frac{3\pi}{4}}}{2} - \sqrt{2}\left(e^{\frac{\pi}{2}}\left(1 + \frac{\pi}{4}\right) - \frac{3\pi}{4} - 1\right), \left(\frac{\pi}{8} + \frac{1}{2}\right)e^{\frac{\pi}{2}} - \frac{e^{\frac{3\pi}{4}}}{2} + \sqrt{2}\left(e^{\frac{\pi}{2}}\left(1 + \frac{\pi}{4}\right) - \frac{3\pi}{4} - 1\right)\right].$$

et cetera.

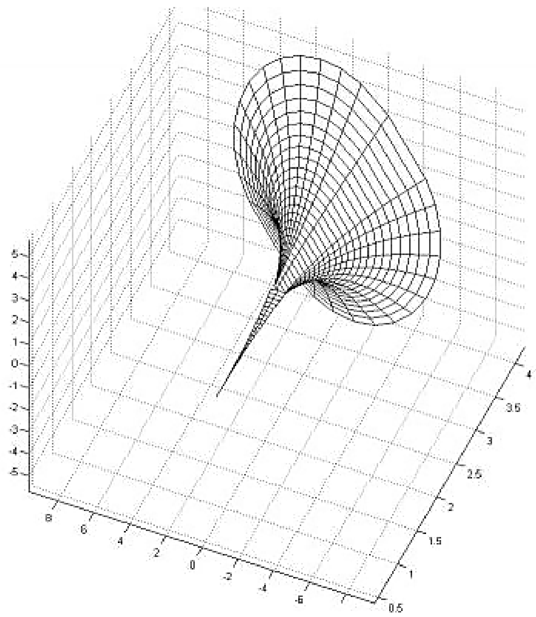

Figure 1: Graphical representations of the solution set of system (3) for $t \in [0, \frac{3\pi}{4})$.

We associate with system (1), (2) the following system

$$\dot{x} \in N(n(t))A(t)x + N(n(t))F(t),\ t \neq \tau_i,\ x(0) = x_0, \tag{4}$$

$$x(\tau_i) = M(n(\tau_i))x(\tau_i - 0),\ i = \overline{1, m}, \tag{5}$$

where $t \in I$ is time, $E(n(t))$ is $(n(t) \times n(t))$ unit matrix, $N(n(t))$ is $(\overline{n} \times \overline{n})$ -dimensional matrix-valued function such that

$$N(n(t)) = \begin{pmatrix} E(n(t)) & 0 \\ 0 & 0 \end{pmatrix},$$

$A(t)$ is $(\overline{n} \times \overline{n})$-dimensional matrix-valued function such that $N(n(t))A(t) \equiv P^T(n(t))A_i(t)P(n(t))$ for $t \in [\tau_i, \tau_{i+1})$, $i = \overline{0, m}$, $F : I \to conv(\mathbb{R}^{\overline{n}})$ is a set-valued mapping such that $N(n(t))F(t) \equiv P^T(n(t))F_i(t)$ for $t \in [\tau_i, \tau_{i+1}),\ i = \overline{0, m}$, $P(t)$ is $(n(t) \times \overline{n})$-dimensional matrix-valued function such that $P(n(t)) = (E(n(t))\, 0)$, $M(n(t))$ is $(\overline{n} \times \overline{n})$-dimensional matrix-valued function such that

$$M(n(t)) = M(n(\tau_i)) = \begin{pmatrix} M_i & 0 \\ 0 & E(\overline{n} - n(\tau_i)) \end{pmatrix},$$

for all $t \in [\tau_i, \tau_{i+1}),\ i = \overline{0, m-1},\ M_0 = E(\overline{n}),\ N(n(0))x_0 \equiv P^T(n(0))\overline{x}_0$.

Definition 1 *A vector-function $x : I \to (\mathbb{R}^{\overline{n}})$ is called a solution of system (4), (5), if it is absolutely continuous, satisfies inclusion (4) almost everywhere on intervals that do not contain τ_i, and satisfies condition (5) for $t = \tau_i$.*

It is obviously that, if differential inclusion (4) satisfies conditions of theorems of existence of the solutions on I (for example, see [1, 14, 18, 21]), then system (4), (5) will have a solution on I.

Let $X(I)$ denote the solution set of (4), (5) and $X(t) = \{x(t)\ :\ x(\cdot) \in X(I),\ t \in I\}$.

Remark 2 *If $n(t) \equiv n$, then system (4), (5) is an usual differential inclusion [1, 18, 21]. As a special case of system (1), (2) one can consider the process of origin and development of objects that are differentiated on the moment of creation [4, 8, 19] and control hybrid system [2, 6, 7, 11, 24]*

Now we consider linear differential inclusions

$$\dot{x} \in N(n(t))A_1(t)x + N(n(t))F_1(t),\ t \neq \tau_i,\ x(0) = x_0, \tag{6}$$

$$x(\tau_i) = M(n(\tau_i))x(\tau_i - 0), \tag{7}$$

$$\dot{y} \in N(n(t))A_2(t)y + N(n(t))F_2(t),\ t \neq \tau_i,\ y(0) = y_0, \tag{8}$$

$$y(\tau_i) = M(n(\tau_i))y(\tau_i - 0). \tag{9}$$

Theorem 3 *Let $A_1(t), A_2(t), F_1(t), F_2(t)$ satisfy the following conditions*

1) *$A_1(\cdot), A_2(\cdot)$ are $(\overline{n} \times \overline{n})$ -dimensional matrix-valued functions with continuous components on $I \setminus \{\tau_i\}$ and with right-continuous components at $\tau_i,\ i = \{1, 2, ..., m\}$;*

2) *$F_1(\cdot), F_2(\cdot) : I \to conv(\mathbb{R}^{\overline{n}})$ are right-continuous set-valued mappings on $I \setminus \{\tau_i\}$ and right-continuous set-valued mappings at $\tau_i,\ i = \{1, 2, ..., m\}$;*

3) *there exists $\eta > 0$ such that*

$$\|A_1(t) - A_2(t)\| \leq \eta, \quad h(F_1(t), F_2(t)) \leq \eta \tag{10}$$

for all $t \in I$;

4) *there exists $\lambda > 0$ such that*

$$\|M(n(t))\| \leq \lambda \tag{11}$$

for all $t \in I$.

Then

1) for any solution $x(\cdot)$ of system (6),(7) there exists a solution $y(\cdot)$ of system (8),(9) such that

$$\|x(t) - y(t)\| \le \lambda^{i(t)} e^{\sqrt{n} a_1 t} \delta_0 + C\eta; \tag{12}$$

2) for any solution $y(\cdot)$ of system (8),(9) there exists a solution $x(\cdot)$ of system (6),(7) such that inequality (12) holds.

Thereby,

$$h(X(t), Y(t)) \le \lambda^{i(t)} e^{\sqrt{n} a_1 t} \delta_0 + C\eta, \tag{13}$$

where $\|x_0 - y_0\| = \delta_0$, $a_1 = \max_{t\in I} \|A_1(t)\|$, C is a constant depending on λ, a_1 and the moments of the change of dimension τ_i, $i(t)$ is an amount of the moments of change of dimension τ_i on $[0, t] \subset I$.

Proof. Let us prove the first statement of the theorem and hence the validity of the inclusion

$$X(t) \subset Y(t) + S_\mu(0),$$

where $\mu = \lambda^{i(t)} e^{\sqrt{n} a_1 t} \delta_0 + C\eta$.

Let $\tau_0 = 0,\ \tau_{m+1} = T$ and choose any solution $x(\cdot) \in X$.

Then

$$x(t) = x(\tau_i + 0) + \int_{\tau_i}^{t} [N(n(s))A_1(s)x(s) + N(n(s))f_1(s)]ds \tag{14}$$

for all $t \in [\tau_i, \tau_{i+1})$, where $f_1(\cdot)$ is a measurable vector-function such that $f_1(t) \in F_1(t)$ a.e. $t \in I$.

We choose $f_2(t) = \min_{f\in F_2(t)} \|f_1(t) - f\|$. The vector f exists and is unique because of the compactness and convexity of the set $F_2(t)$ and the strong convexity of the function being minimized. Obviously, $f_2(\cdot)$ is a measurable vector function.

Let $y(\cdot)$ be such that

$$y(t) = y(\tau_i + 0) + \int_{\tau_i}^{t} [N(n(s))A_2(t)y(s) + N(n(s))f_2(s)]ds. \tag{15}$$

Hence $y(\cdot) \in Y$.

Denote by

$$\delta_i^- = \|x(\tau_i - 0) - y(\tau_i - 0)\|, \quad \delta_i^+ = \|x(\tau_i + 0) - y(\tau_i + 0)\|.$$

From (14) and (15), we get

$$\|x(t) - y(t)\| =$$

$$= \left\| x(\tau_i + 0) + \int_{\tau_i}^{t} [N(n(s))A_1(s)x(s) + N(n(s))f_1(s)]ds - y(\tau_i + 0) - \int_{\tau_i}^{t} [N(n(s))A_2(t)y(s) + N(n(s))f_2(s)]ds \right\|$$

$$\le \|x(\tau_i + 0) - y(\tau_i + 0)\|$$

$$+ \int_{\tau_i}^{t} \|[N(n(s))A_1(s)x(s) + N(n(s))f_1(s)] - [N(n(s))A_2(t)y(s) + N(n(s))f_2(s)]\| ds$$

$$\le \|x(\tau_i + 0) - y(\tau_i + 0)\| +$$

$$+ \int_{\tau_i}^{t} \|[N(n(s))A_1(s)x(s) - N(n(s))A_2(t)y(s)] + [N(n(s))f_1(s) - N(n(s))f_2(s)]\| ds$$

$$\le \|x(\tau_i + 0) - y(\tau_i + 0)\| + \int_{\tau_i}^{t} \|[N(n(s))A_1(s)x(s) - N(n(s))A_1(s)y(s)]\| ds$$

$$+\int\limits_{\tau_i}^{t}\|[N(n(s))A_1(s)y(s)-N(n(s))A_2(t)y(s)]\|ds+\int\limits_{\tau_i}^{t}\|[N(n(s))f_1(s)-N(n(s))f_2(s)]\|ds$$

$$\le \delta_i^+ + \int\limits_{\tau_i}^{t}\|N(n(s))\|\|A_1(s)\|\|x(s)-y(s)\|ds$$

$$+\int\limits_{\tau_i}^{t}\|N(n(s))\|\|A_1(s)y(s)-A_2(s)y(s)\|ds+\int\limits_{\tau_i}^{t}\|N(n(s))\|\|f_1(s)-f_2(s)\|ds$$

$$\le \delta_i^+ + \int\limits_{\tau_i}^{t}\|N(n(s))\|\|A_1(s)\|\|x(s)-y(s)\|ds$$

$$+\int\limits_{\tau_i}^{t}\|N(n(s))\|\|A_1(s)y(s)-A_2(s)y(s)\|ds+\int\limits_{\tau_i}^{t}\|N(n(s))\|h(F_1(s),F_2(s))ds$$

$$\le \delta_i^+ + \int\limits_{\tau_i}^{t}[\sqrt{n}\|A_1(s)\|\|x(s)-y(s)\|+2\sqrt{n}\eta]ds.$$

Using Gronwall-Bellmans inequality, we obtain

$$\|x(t)-y(t)\|\le\left(\delta_i^+ + \frac{2\eta}{a_1}\right)e^{\sqrt{n}a_1(t-\tau_i)}-\frac{2\eta}{a_1},\ t\in[\tau_i,\tau_{i+1}).$$

Hence

$$\delta_{i+1}^-\le\left(\delta_i^+ + \frac{2\eta}{a_1}\right)e^{\sqrt{n}a_1(\tau_{i+1}-\tau_i)}-\frac{2\eta}{a_1}.$$

Also we have

$$\delta_i^+=\|x(\tau_i+0)-y(\tau_i+0)\|=\|M(n(\tau_i))x(\tau_i-0)-M(n(\tau_i))y(\tau_i-0)\|$$

$$\le\lambda\|x(\tau_i-0)-y(\tau_i-0)\|=\lambda\delta_i^-.$$

As $\delta_0^-=\delta_0^+=\delta_0$, then we obtain

$$\|x(t)-y(t))\le\left(\delta_i^+ + \frac{2\eta}{a_1}\right)e^{\sqrt{n}a_1(t-\tau_i)}-\frac{2\eta}{a_1}\le\left(\lambda\delta_i^- + \frac{2\eta}{a_1}\right)e^{\sqrt{n}a_1(t-\tau_i)}-\frac{2\eta}{a_1}$$

$$\le\left(\lambda\left(\left(\delta_{i-1}^+ + \frac{2\eta}{a_1}\right)e^{a_1(\tau_i-\tau_{i-1})}-\frac{2\eta}{a_1}\right)+\frac{2\eta}{a_1}\right)e^{a_1(t-\tau_i)}-\frac{2\eta}{a_1}$$

$$\le\lambda\left(\left(\delta_{i-1}^+ + \frac{2\eta}{a_1}\right)e^{\sqrt{n}a_1(\tau_i-\tau_{i-1})}-\frac{2\eta}{a_1}\right)e^{\sqrt{n}a_1(t-\tau_i)}+2\eta\left(\frac{e^{\sqrt{n}a_1(\tau_{i+1}-\tau_i)}-1}{a_1}\right)$$

$$=\lambda\delta_{i-1}^+e^{\sqrt{n}a_1(t-\tau_{i-1})}+2\eta\left(\lambda\frac{e^{\sqrt{n}a_1(\tau_i-\tau_{i-1})}-1}{a_1}e^{\sqrt{n}a_1(\tau_{i+1}-\tau_i)}+\frac{e^{\sqrt{n}a_1(\tau_{i+1}-\tau_i)}-1}{a_1}\right)$$

$$\le\ldots\le\lambda^{i(t)}e^{\sqrt{n}a_1t}\delta_0+C\eta.$$

We have proved the first statement of the theorem. The proof of the second part of the theorem is similar to the proof of the first one. The theorem is proved. ■

Remark 4 *Now we consider some special cases of this theorem.*

Let $\eta = 0,\ \delta_0 \neq 0$. *Then*

$$h(X(t), Y(t)) \leq \lambda^{i(t)} e^{\sqrt{\overline{n}} a_1 t} \delta_0,$$

i.e.

$$h(X(t), Y(t)) \leq \lambda^{m} e^{\sqrt{\overline{n}} a_1 T} \delta_0 \tag{16}$$

for all $t \in I$.

By (16), for any $\varepsilon > 0$ *there exists* $\delta(\varepsilon) = \varepsilon \lambda^{-m} e^{-\sqrt{\overline{n}} a_1 T}$ *such that if* $\delta_0 = \|x_0 - y_0\| < \delta$ *then* $h(X(t),\ Y(t)) < \varepsilon$ *for all* $t \in I$. *Hence, the set of solutions of system (6) continuously depends on initial conditions.*

If $\delta_0 = 0$ *and* $\eta \neq 0$ *then* $h(X(t),\ Y(t)) \leq C\eta$ *for all* $t \in I$ *i.e. for any* $\varepsilon > 0$ *there exists* $\eta(\varepsilon) = \frac{\varepsilon}{C}$ *such that if inequalities (10), (26) hold then* $h(X(t),\ Y(t)) < \varepsilon$ *for all* $t \in I$. *This statement expresses a continuity property of the set of solutions of system in some function space of right-hand sides.*

3 The method of averaging

Now we consider a linear system with a small parameter

$$\dot{x} \in \varepsilon[N(n(t))A(t)x + N(n(t))F(t)],\ t \neq \tau_i,\ x(0) = x_0, \tag{17}$$

$$x(\tau_i) = M(n(\tau_i))x(\tau_i - 0), \tag{18}$$

where $\varepsilon > 0$ is a small parameter.

In this paper, we associate with system (17),(18) the following averaged system

$$\dot{y} \in \varepsilon[N(n(t))\overline{A}y + N(n(t))\overline{F}],\ t \neq \tau_i,\ y(0) = x_0, \tag{19}$$

$$y(\tau_i) = M(n(\tau_i))y(\tau_i - 0), \tag{20}$$

where

$$\overline{A} = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(t)dt, \quad \overline{F} = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t)dt. \tag{21}$$

We will rewrite systems (17),(18) and (19),(20) in a following aspect

$$\dot{x} \in \varepsilon[N(n(t))A(t)x + N(n(t))F(t)],\ t \neq \tau_i,\ x(0) = x_0, \tag{22}$$

$$\Delta x(\tau_i) = K(n(\tau_i))x(\tau_i - 0), \tag{23}$$

$$\dot{y} \in \varepsilon[N(n(t))\overline{A}y + N(n(t))\overline{F}],\ t \neq \tau_i,\ y(0) = x_0, \tag{24}$$

$$\Delta y(\tau_i) = K(n(\tau_i))y(\tau_i - 0), \tag{25}$$

where $K(n(t)) = M(n(t)) - E(n(t))$.

Theorem 5 *Let the following conditions hold:*

1) $A(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{\overline{n}\times\overline{n}}$ *is* $(\overline{n}\times\overline{n})$*-dimensional matrix-valued function with right-continuous components on* $\mathbb{R}_+\backslash\{\tau_i\}$ *and with right-continuous components at* $t = \tau_i$;

2) $F(\cdot) : R_+ \to conv(\mathbb{R}^{\overline{n}})$ *is a continuous set-valued mapping on* $\mathbb{R}_+\backslash\{\tau_i\}$ *and right-continuous set-valued mappings at* $t = \tau_i$;

3) *there exists* $M > 0$ *such that* $\|N(n(t))A(t)\| \leq M, \quad \|N(n(t))F(t)\| \leq M$ *for all* $t \geq 0$;

4) *there exists* $\lambda > 0$ *such that*

$$\|K(n(t))\| \leq \lambda \tag{26}$$

for all $t \geq 0$;

5) $\lim\limits_{T\to\infty} i(T) = M_0$;

6) $M_0\lambda < 1$.

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon_0(\eta, L) > 0$ *such that*

1) for any solution $x(\cdot)$ *of system (22),(23) there exists a solution* $y(\cdot)$ *of system (24),(25) such that*

$$\|x(t) - y(t)\| < \eta; \tag{27}$$

2) for any solution $y(\cdot)$ *of system (24),(24) there exists a solution* $x(\cdot)$ *of system (22),(23) such that inequality (27) holds.*

Thereby,

$$h(X(t), Y(t)) < \eta, \tag{28}$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$.

Proof. Let us prove the first statement of the theorem and hence the validity of the inclusion

$$X(t) \subset Y(t) + S_\eta(0).$$

We choose any solution $x(\cdot) \in X$. Then

$$x(t) = x_0 + \varepsilon \int\limits_0^t [N(n(s))A(s)x(s) + N(n(s))f(s)]ds + \sum_{0\le\tau_i<t} K(n(\tau_i))x(\tau_i - 0) \tag{29}$$

for all $t \in [0, L\varepsilon^{-1}]$, where $f(\cdot)$ is a measurable vector function such that $f(t) \in F(t)$ a.e. $t \in [0, L\varepsilon^{-1}]$.

We take $\overline{f}(t) = \min\limits_{f\in\overline{F}(t)} \|f(t) - f\|$. The vector f exists and is unique because of the compactness and convexity of the set $\overline{F}(t)$ and the strong convexity of the function being minimized. Obviously, $\overline{f}(\cdot)$ is a measurable vector function.

Let $y(\cdot)$ be such that

$$y(t) = x_0 + \varepsilon \int\limits_0^t [N(n(s))\overline{A}y(s) + N(n(s))\overline{f}]ds + \sum_{0\le\tau_i<t} K(n(\tau_i))y(\tau_i - 0). \tag{30}$$

Hence $y(\cdot) \in Y$.

By (29),(30) we have

$$\begin{aligned}
&\|x(t) - y(t)\| \\
&\le \Bigg\| x_0 + \varepsilon \int\limits_0^t [N(n(s))A(s)x(s) + N(n(s))f(s)]ds + \sum_{0\le\tau_i<t} K(n(\tau_i))x(\tau_i - 0) - \\
&\quad - x_0 - \varepsilon \int\limits_0^t [N(n(s))\overline{A}y(s) + N(n(s))\overline{f}]ds - \sum_{0\le\tau_i<t} K(n(\tau_i))y(\tau_i - 0)\Bigg\| \\
&\le \varepsilon \Bigg\| \int\limits_0^t [N(n(s))A(s)x(s) + N(n(s))f(s)]ds - \int\limits_0^t [N(n(s))\overline{A}y(s) + N(n(s))\overline{f}]ds \Bigg\| \\
&\quad + \Bigg\| \sum_{0\le\tau_i<t} K(n(\tau_i))x(\tau_i - 0) - \sum_{0\le\tau_i<t} K(n(\tau_i))y(\tau_i - 0) \Bigg\| \\
&\le \varepsilon \Bigg\| \int\limits_0^t N(n(s))A(s)x(s)ds - \int\limits_0^t N(n(s))\overline{A}y(s)ds \Bigg\|
\end{aligned}$$

$$+\varepsilon\left\|\int_0^t N(n(s))f(s)ds-\int_0^t N(n(s))\overline{f}ds\right\|+\left\|\sum_{0\le\tau_i<t}K(n(\tau_i))(x(\tau_i-0)-y(\tau_i-0))\right\|$$

$$\le\varepsilon\left\|\int_0^t N(n(s))A(s)x(s)ds-\int_0^t N(n(s))\overline{A}y(s)ds\right\|$$

$$+\varepsilon\left\|\int_0^t N(n(s))f(s)ds-\int_0^t N(n(s))\overline{f}ds\right\|+\lambda\sum_{0\le\tau_i<t}\|x(\tau_i-0)-y(\tau_i-0)\|$$

$$\le\varepsilon\left\|\int_0^t N(n(s))A(s)x(s)ds-\int_0^t N(n(s))\overline{A}y(s)ds\right\|$$

$$+\varepsilon\left\|\int_0^t N(n(s))f(s)ds-\int_0^t N(n(s))\overline{f}ds\right\|+M_0\lambda\sup_{0\le s\le t}\|x(s)-y(s)\|. \tag{31}$$

Divide the interval $[0,L\varepsilon^{-1}]$ on the partial intervals by the points $t_k=\frac{kL}{\varepsilon m}$, $k\in\{0,1,...,m-1\}$. Let $t\in[t_k,t_{k+1}]$, then

$$\left\|\int_0^t N(n(s))A\,(s)\,x(s)ds-\int_0^t N(n(s))\overline{A}\,y(s)ds\right\|$$

$$\le\left\|\int_0^t N(n(s))A\,(s)\,x(s)ds-\int_0^t N(n(s))\overline{A}\,x(s)ds\right\|+\left\|\int_0^t N(n(s))\overline{A}x(s)ds-\int_0^t N(n(s))\overline{A}\,y(s)ds\right\|$$

$$\le\left\|\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}N(n(s))A\,(s)\,x(s)ds+\int_{t_k}^{t}N(n(s))A\,(s)\,x(s)ds-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}N(n(s))\overline{A}\,x(s)ds-\int_{t_k}^{t}N(n(s))\overline{A}\,x(s)ds\right\|$$

$$+\left\|\int_0^t N(n(s))\overline{A}\,x(s)ds-\int_0^t N(n(s))\overline{A}\,y(s)ds\right\|\le\sum_{i=0}^{k-1}\left\|\int_{t_i}^{t_{i+1}}N(n(s))A\,(s)\,x(s)ds-\int_{t_i}^{t_{i+1}}N(n(s))\overline{A}\,x(s)ds\right\|$$

$$+\left\|\int_{t_k}^{t}N(n(s))A\,(s)\,x(s)ds-\int_{t_k}^{t}N(n(s))\overline{A}\,x(s)ds\right\|+\left\|\int_0^t N(n(s))\overline{A}\,x(s)ds-\int_0^t N(n(s))\overline{A}\,y(s)ds\right\|. \tag{32}$$

Let $x_i=x(t_i)$, we have

$$\left\|\int_{t_i}^{t_{i+1}}N(n(s))A(s)x(s)ds-\int_{t_i}^{t_{i+1}}N(n(s))\bar{A}x(s)ds\right\|$$

$$\le\left\|\int_{t_i}^{t_{i+1}}N(n(s))A(s)x(s)ds-\int_{t_i}^{t_{i+1}}N(n(s))A(s)x_ids\right\|+\left\|\int_{t_i}^{t_{i+1}}N(n(s))A(s)x_ids-\int_{t_i}^{t_{i+1}}N(n(s))\bar{A}x_ids\right\|$$

$$+\left\|\int_{t_i}^{t_{i+1}}N(n(s))\bar{A}x_ids-\int_{t_i}^{t_{i+1}}N(n(s))\bar{A}x(s)ds\right\|. \tag{33}$$

From (21) it follows that there exists an increasing function $\theta(t)$ such that

1) $\lim\limits_{t\to\infty}\theta(t)=0$;

2)

$$\left\| \int_0^t [N(n(s))A(s) - N(n(s))\overline{A}]ds \right\| \le t\theta(t), \quad \left\| \int_0^t N(n(s))F(s)\,ds - \int_0^t N(n(s))\overline{F}ds, \right\| \le t\theta(t). \tag{34}$$

Then we obtain

$$\left\| \int_{t_i}^{t_{i+1}} N(n(s))A(s)\,x_i\,ds - \int_{t_i}^{t_{i+1}} N(n(s))\overline{A}x_i\,ds \right\| = \left\| \int_{t_i}^{t_{i+1}} N(n(s))A(s)\,ds \cdot x_i - \int_{t_i}^{t_{i+1}} N(n(s))\overline{A}ds \cdot x_i \right\|$$

$$\le \|x_i\| \cdot \left\| \int_{t_i}^{t_{i+1}} [N(n(s))A(s) - N(n(s))\overline{A}]ds \right\|$$

$$\le \|x_i\| \cdot \left(\left\| \int_0^{t_{i+1}} [N(n(s))A(s) - N(n(s))\overline{A}]ds \right\| + \left\| \int_0^{t_i} [N(n(s))A(s) - N(n(s))\overline{A}]ds \right\| \right)$$

$$\le \|x_i\| \cdot (t_{i+1}\theta(t_{i+1}) + t_i\theta(t_i)) \le 2\Psi(\varepsilon)\|x_i\| \le 2\Psi(\varepsilon)\left(\|x_0\| + ML\right)e^{ML}, \tag{35}$$

where $\varepsilon\Psi(\varepsilon) = \sup\limits_{\tau\in[0,L]} \tau\theta\left(\frac{\tau}{\varepsilon}\right) \to 0$ by $\varepsilon \to 0$.

Now we have

$$\left\| \int_{t_i}^{t_{i+1}} N(n(s))A(s)\,x(s)ds - \int_{t_i}^{t_{i+1}} N(n(s))A(s)\,x_i\,ds \right\|$$

$$= \varepsilon \left\| \int_{t_i}^{t_{i+1}} N(n(s))A(s) \int_{t_i}^{s} (N(n(\tau))A(\tau)\,x(\tau) + N(n(\tau))f(\tau))d\tau ds \right\|$$

$$\le \varepsilon \left(\int_{t_i}^{t_{i+1}} M \int_{t_i}^{s} (M\|x(\tau)\| + M)d\tau ds \right) \le \frac{L^2M^2}{2\varepsilon m^2}\left((\|x_0\| + ML)e^{ML} + 1\right). \tag{36}$$

So we obtain

$$\left\| \int_{t_i}^{t_{i+1}} \overline{N}(n(s))Ax_i ds - \int_{t_i}^{t_{i+1}} N(n(s))\overline{A}\,x(s)ds \right\| \le \frac{L^2M^2}{2\varepsilon m^2}\left((\|x_0\| + ML)e^{ML} + 1\right). \tag{37}$$

Combining (35) - (37) and (33), we have

$$\left\| \int_{t_i}^{t_{i+1}} N(n(s))A(s)\,x(s)ds - \int_{t_i}^{t_{i+1}} N(n(s))\overline{A}\,x(s)ds \right\| \le 2\Psi(\varepsilon)\left(\|x_0\| + ML\right)e^{ML}$$

$$+\frac{L^2M^2}{\varepsilon m^2}\left((\|x_0\| + ML)e^{ML} + 1\right). \tag{38}$$

Now we will estimate the second summand in (32):

$$\left\| \int_{t_k}^{t} N(n(s))A(s)\,x(s)ds - \int_{t_k}^{t} N(n(s))\overline{A}\,x(s)ds \right\| \le \int_{t_k}^{t} \left\| N(n(s))A(s)\,x(s) - N(n(s))\overline{A}\,x(s) \right\| ds$$

$$\leq \int_{t_k}^{t} \|x(s)\|\|N(n(s))\| \left\|A(s) - \overline{A}\right\| ds \leq 2M(\|x_0\| + ML)\, e^{ML} \frac{L}{\varepsilon m}. \tag{39}$$

Let $\delta(t) = \max\limits_{0 \leq s \leq t} \|x(s) - y(s)\|$, then

$$\left\| \int_0^t N(n(s))\bar{A}x(s)ds - \int_0^t N(n(s))\bar{A}y(s)ds \right\| \leq \int_0^t \left\|N(n(s))\bar{A}x(s) - N(n(s))\bar{A}y(s)\right\| ds$$

$$\leq \left\|\bar{A}\right\| \int_0^t \|x(s) - y(s)\|\, ds \leq M \int_0^t \delta(s)ds. \tag{40}$$

From (38) - (40), we obtain

$$\left\| \int_0^t N(n(s))A(s)x(s)ds - \int_0^t N(n(s))\bar{A}y(s)ds \right\|$$

$$\leq \sum_{i=0}^{k-1} \left[2\Psi(\varepsilon)(\|x_0\| + ML)\, e^{ML} + \frac{L^2M^2}{\varepsilon m^2}\left((\|x_0\| + ML)e^{ML} + 1\right) \right] + \frac{2LM}{\varepsilon m}(\|x_0\| + ML)\, e^{ML} + M\int_0^t \delta(s)\, ds$$

$$\leq 2m\Psi(\varepsilon)(\|x_0\| + ML)\, e^{ML} + \frac{L^2M^2}{\varepsilon m}\left((\|x_0\| + ML)e^{ML} + 1\right) + \frac{2LM}{\varepsilon m}(\|x_0\| + ML)\, e^{ML} + M\int_0^t \delta(s)\, ds. \tag{41}$$

From (31), (34) and (41), we have:

$$\delta(t) \leq 2m\varepsilon\Psi(\varepsilon)(\|x_0\| + ML)\, e^{ML} + \frac{L^2M^2}{m}\left((\|x_0\| + ML)e^{ML} + 1\right)$$

$$+ \frac{2LM}{m}(\|x_0\| + ML)\, e^{ML} + \varepsilon M \int_0^t \delta(s)\, ds + \varepsilon\Psi(\varepsilon) + M_0\lambda\delta(t)$$

$$= M_0\lambda\delta(t) + \left(2m(\|x_0\| + ML)\, e^{ML} + 1\right)\varepsilon\Psi(\varepsilon) + \frac{LM}{m}\left((LM+2)(\|x_0\| + ML)e^{ML} + LM\right) + \varepsilon M\int_0^t \delta(s)\, ds.$$

Using Gronwall-Bellmans inequality, we obtain

$$\delta(t) \leq \frac{1}{1 - M_0\lambda}\left(\left(2m(\|x_0\| + ML)\, e^{ML} + 1\right)\varepsilon\Psi(\varepsilon) + \frac{LM}{m}\left((LM+2)(\|x_0\| + ML)e^{ML} + LM\right) \right) e^{ML}. \tag{42}$$

Choosing m_0 and ε_0 such that

$$\frac{LM}{m_0(1 - M_0\lambda)}\left((LM+2)(\|x_0\| + ML)e^{ML} + LM\right) e^{ML} < \frac{\eta}{2}, \tag{43}$$

$$\frac{1}{1 - M_0\lambda} e^{ML}\left(2m_0(\|x_0\| + ML)\, e^{ML} + 1\right)\varepsilon_0\Psi(\varepsilon_0) < \frac{\eta}{2}. \tag{44}$$

By (43), (44) we have $\|x(t) - y(t)\| < \eta$.

We have proved the first statement of the theorem. The proof of the second part of the theorem is similar to the proof of the first one. The theorem is proved. ■

Remark 6 *If* $\lim\limits_{T \to \infty} i(T) = \infty$ *and*

1) $A(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{\overline{n}\times\overline{n}}$ *is* $(\overline{n}\times\overline{n})$*-dimensional matrix-valued function with right-continuous components on* $\mathbb{R}_+\backslash\{\tau_i\}$ *and with right-continuous components at* $t = \tau_i$*;*

2) $F(\cdot) : R_+ \to conv(\mathbb{R}^{\overline{n}})$ *is a continuous set-valued mapping on* $\mathbb{R}_+\backslash\{\tau_i\}$ *and right-continuous set-valued mappings at* $t = \tau_i$*;*

3) there exists $M > 0$ *such that* $\|N(n(t))A(t)\| \le M, \quad \|N(n(t))F(t)\| \le M$ *for all* $t \ge 0$*;*

4) $i(T) < \infty$ *for every* $0 \le T < \infty$*;*

5) matrix-valued function $K(n(t))$ *:* $\sum\limits_{i=1}^{\infty} \|K(n(\tau_i))\| = \vartheta < 1.$

Then for any $\eta > 0$ *and* $L > 0$ *there exists* $\varepsilon_0(\eta, L) > 0$ *such that*

1) for any solution $x(\cdot)$ *of system (22),(23) there exists a solution* $y(\cdot)$ *of system (24),(25) such that*

$$\|x(t) - y(t)\| < \eta; \tag{45}$$

2) for any solution $y(\cdot)$ *of system (24),(24) there exists a solution* $x(\cdot)$ *of system (22),(23) such that inequality (45) holds.*

Thereby, $h(X(t), Y(t)) < \eta$*, for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$.
The proof is hold similarly with little changes in formulas (31), (42), (43), (44):

$$\begin{aligned} \|x(t) - y(t)\| \le \varepsilon \| \int\limits_0^t N(n(s))A(s)x(s)ds - \int\limits_0^t N(n(s))\overline{A}y(s)ds\| + \\ + \varepsilon \| \int\limits_0^t N(n(s))f(s)ds - \int\limits_0^t N(n(s))\overline{f}ds\| + \vartheta \sup_{0\le s\le t} \|x(s) - y(s)\|, \end{aligned} \tag{46}$$

$$\delta(t) \le \frac{1}{1-\vartheta}\left(\left(2m\left(\|x_0\| + ML\right)e^{ML} + 1\right)\varepsilon\Psi(\varepsilon) + \frac{LM}{m}\left((LM+2)(\|x_0\| + ML)e^{ML} + LM\right)\right)e^{ML}, \tag{47}$$

$$\frac{LM}{m_0(1-\vartheta)}\left((LM+2)(\|x_0\| + ML)e^{ML} + LM\right)e^{ML} < \frac{\eta}{2}, \tag{48}$$

$$\frac{1}{1-\vartheta}e^{ML}\left(2m_0\left(\|x_0\| + ML\right)e^{ML} + 1\right)\varepsilon_0\Psi(\varepsilon_0) < \frac{\eta}{2}. \tag{49}$$

4 Conclusions

Similarly it is possible to obtain a substantiation of an averaging method for fuzzy differential inclusions with variable dimension and therefore to generalize results received in [15–17, 22, 23] in linear case.

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