ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.20(2015) No.1,pp.52-60

Approximation by Integro B-spline with Eight Degree and Its Super Convergence

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(Received 10 September 2014, accepted 27 May 2015)

Abstract: In this paper, we use eighth B-spline functions to develop numerical methods for approximation to the given integral values of a univariate real-valued function. The convergence analysis of the methods has been discussed and shown that the given approximations are better than quartic and sextic B-spline functions [4,5]. Numerical examples are presented to illustrate the applications of the method, and to compare the computed results with other known methods.

Keywords: Integro interpolation; B-spline with eight degree; Integral value; Error analysis.

1 Introduction

In this paper, We assume that the function values $y_i = y(x_i)$ at the knots are not given, but the integral values I_i of $y(x)$ are known on the subintervals $[x_i, x_{i+1}], i = 0, 1, ..., n-1$. We want to determine an integro-interpolating b-spline function $S(x)$ such that

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, i = 0, ..., n-1.$$

This problem has many applications in science and engineering. B-spline functions, which are piecewise polynomials with certain smoothness at the knots, can also be used to deal with the new interpolation problems. Firstly behforooz [1,2] developed cubic and quintic b-splines and discuss the integro interpolation. These method need several additional boundary conditions. Moreover, the error estimations were also not given. In [3] Zhanlav and Mijiddorj discussed the local integro cubic splines and their approximation properties. Lang and Xu [4] developed the integro quartic spline interpolation by using quartic B-splines and this method possesses super convergence orders in approximating function values and second-order derivative values at the knots. Recently, Jinming Wu and Xiaolei Zhang [5] discussed the integro sextic spline interpolation by using sextic B-splines and this method possesses super convergence order (eighth, sixth and fourth order, respectively) in approximating function values, second-order derivative values and fourth-order derivative values at the knots is mainly proved. In this paper, we use the technique in [4,5] and discuss the integro interpolation by using B-splines with eight degree. The super convergence (Tenth, eighth, sixth and fourth order, respectively) in approximating function values, second-order derivative values and fourth-order derivative values at the knots is mainly proved. Some numerical experiments are considered and numerical results are compared with the method developed in [4,5].

2 Eight B-spline

In order to develop the numerical method for integro interpolation we introduce the set $\{x_i\}$ so that $x_i = a + ih, h = \frac{b-a}{n}, i = 0, 1 \ldots, n$ where $\Delta_i = [x_i, x_{i+1}], i = 0, 1, ..., n-1$. The eighth b-spline with respect to $\Delta = \{\Delta_i\}_{i=0}^{n-1}$ defined in the following form:

$$S_8^7[a, b] = \{S(x) \in C^7[a, b] | S_i(x) \in P_8(x), i = 0, 1, ..., n-1\},$$

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IJNS.2015.08.15/877

where $S_i(x)$ denotes the restriction of $S(x)$ over Δ_i, and $P_8(x)$ denotes the set of all univariate eighth polynomials. By using [6-8] and extend $[a, b]$ to $[a - 8h, b + 8h]$ with the equidistant knots $x_i = a + ih(i = -8, -7, ..., n + 8)$, we obtain the eighth uniform B-spline in the following form:

$$B_i(x) = \frac{1}{8!h}$$

$$\begin{cases} (x - x_{i-2})^8, & [x_{i-2}, x_{i-1}], \\ (x - x_{i-2})^8 - 9(x - x_{i-1})^8, & [x_{i-1}, x_i], \\ (x - x_{i-2})^8 - 9(x - x_{i-1})^8 + 36(x - x_i)^8, & [x_i, x_{i+1}], \\ (x - x_{i-2})^8 - 9(x - x_{i-1})^8 + 36(x - x_i)^8 - 84(x - x_{i+1})^8, & [x_{i+1}, x_{i+2}], \\ (x - x_{i-2})^8 - 9(x - x_{i-1})^8 + 36(x - x_i)^8 - 84(x - x_{i+1})^8 + 126(x - x_{i+2})^8, & [x_{i+2}, x_{i+3}], \\ (x - x_{i+7})^8 - 9(x - x_{i+6})^8 + 36(x - x_{i+5})^8 - 84(x - x_{i+4})^8, & [x_{i+3}, x_{i+4}], \\ (x - x_{i+7})^8 - 9(x - x_{i+6})^8 + 36(x - x_{i+5})^8, & [x_{i+4}, x_{i+5}], \\ (x - x_{i+7})^8 - 9(x - x_{i+6})^8, & [x_{i+5}, x_{i+6}], \\ (x - x_{i+7})^8, & [x_{i+6}, x_{i+7}], \\ 0, & \text{O.W..} \end{cases} \tag{1}$$

Some properties of $B_i(x)$ as follows.

- $B_i(x), i = -6, -5, ..., n + 1$, are linearly independent, and they form the basis splines of S_8^7.
- $B_i^{(k)} = B_{i+1}^{(k+1)}(x + h), i = -6, -5, ..., n, k = 0, 1, ..., 7$, the values of $B_i^{(k)}$ at the knots are given in table1.
- $\sum_{i=-6}^{n+1} B_i(x) \equiv 1, x \in [a, b]$.
- $B_i(x), i = -6, -5, ..., n + 1$, are non-negative and locally supported on $[x_{i-2}, x_{i+7}]$.

Thus, we have

$$\int_{x_{i-2}}^{x_{i-1}} B_i(x)dx = \int_{x_{i+6}}^{x_{i+7}} B_i(x)dx = \frac{h}{9!} = \frac{h}{362880},$$

$$\int_{x_{i-1}}^{x_i} B_i(x)dx = \int_{x_{i+5}}^{x_{i+6}} B_i(x)dx = \frac{502h}{9!} = \frac{251h}{181440},$$

$$\int_{x_i}^{x_{i+1}} B_i(x)dx = \int_{x_{i+4}}^{x_{i+5}} B_i(x)dx = \frac{14608h}{9!} = \frac{913h}{22680},$$

$$\int_{x_{i+1}}^{x_{i+2}} B_i(x)dx = \int_{x_{i+3}}^{x_{i+4}} B_i(x)dx = \frac{88234h}{9!} = \frac{88234h}{362880},$$

$$\int_{x_{i+2}}^{x_{i+3}} B_i(x)dx = \frac{156190h}{9!} = \frac{156190h}{362880},$$

$$\int_{x_i}^{x_{i+1}} B_i(x)dx = 0, (j \geq i + 7, j \leq i - 7),$$

Table 1: The values of $B_i^{(k)}(x), i = -6, ..., n + 1, k = 0, 1, ..., 7$, at the knots.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}	x_{i+7}
$B_i(x)$	0	$\frac{1}{8!}$	$\frac{247}{8!}$	$\frac{4293}{8!}$	$\frac{15619}{8!}$	$\frac{15619}{8!}$	$\frac{4293}{8!}$	$\frac{247}{8!}$	$\frac{1}{8!}$	0
$B_i'(x)$	0	$\frac{952}{8!h}$	$\frac{8568}{8!h}$	$\frac{9800}{8!h}$	$\frac{-9800}{8!h}$	$\frac{-8568}{8!h}$	$\frac{-925}{8!h}$	$\frac{-925}{8!h}$	$\frac{-8}{8!h}$	0
$B_i''(x)$	0	$\frac{56}{8!h^2}$	$\frac{3080}{8!h^2}$	$\frac{10584}{8!h^2}$	$\frac{-13720}{8!h^2}$	$\frac{-13720}{8!h^2}$	$\frac{10584}{8!h^2}$	$\frac{3080}{8!h^2}$	$\frac{56}{8!h^2}$	0
$B_i'''(x)$	0	$\frac{336}{8!h^3}$	$\frac{7728}{8!h^3}$	$\frac{-3024}{8!h^3}$	$\frac{-31920}{8!h^3}$	$\frac{31920}{8!h^3}$	$\frac{3024}{8!h^3}$	$\frac{-7728}{8!h^3}$	$\frac{-336}{8!h^3}$	0
$B_i^{(4)}(x)$	0	$\frac{1680}{8!h^4}$	$\frac{11760}{8!h^4}$	$\frac{-45360}{8!h^4}$	$\frac{31920}{8!h^4}$	$\frac{31920}{8!h^4}$	$\frac{-45360}{8!h^4}$	$\frac{11760}{8!h^4}$	$\frac{1680}{8!h^4}$	0
$B_i^{(5)}(x)$	0	$\frac{6720}{8!h^5}$	$\frac{-6720}{8!h^5}$	$\frac{-60480}{8!h^5}$	$\frac{168000}{8!h^5}$	$\frac{-168000}{8!h^5}$	$\frac{60480}{8!h^5}$	$\frac{6720}{8!h^5}$	$\frac{-6720}{8!h^5}$	0
$B_i^{(6)}(x)$	0	$\frac{20160}{8!h^6}$	$\frac{-100800}{8!h^6}$	$\frac{181440}{8!h^6}$	$\frac{-100800}{8!h^6}$	$\frac{-100800}{8!h^6}$	$\frac{181440}{8!h^6}$	$\frac{-100800}{8!h^6}$	$\frac{20160}{8!h^6}$	0
$B_i^{(7)}(x)$	0	$\frac{40320}{8!h^7}$	$\frac{-282240}{8!h^7}$	$\frac{846720}{8!h^7}$	$\frac{-1411200}{8!h^7}$	$\frac{1411200}{8!h^7}$	$\frac{-846720}{8!h^7}$	$\frac{282240}{8!h^7}$	$\frac{-40320}{8!h^7}$	0

3 Integro interpolation by using B-splines of degree eight

We assume that the integral values I_i of $y(x)$ on $[x_i, x_{i+1}], i = 0, 1, ..., n-1$, and eight boundary conditions $y_0 = y(x_0), y_0' = y'(x_0), y_0'' = y''(x_0), y_0''' = y'''(x_0)$, and $y_n = y(x_n), y_n' = y'(x_n), y_n'' = y''(x_n), y_n''' = y'''(x_n)$, to construct a eight b-spline $S(x)$ such that

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, i = 0, 1, ..., n-1, \tag{2}$$

with boundary conditions

$$\begin{cases} y_0 = y(x_0), y_0' = y'(x_0), y_0'' = y''(x_0), y_0''' = y'''(x_0), \\ y_n = y(x_n), y_n' = y'(x_n), y_n'' = y''(x_n), y_n''' = y'''(x_n). \end{cases} \tag{3}$$

The new integro eighth spline interpolations method $S(x)$, it can be uniquely expressed as

$$S(x) = \sum_{j=-6}^{n+1} c_j B_j(x),$$

and we get

$$\int_{x_i}^{x_{i+1}} S(x)dx = \int_{x_i}^{x_{i+1}} \sum_{j=-6}^{n+1} c_j B_j(x)dx =$$

$$\int_{x_i}^{x_{i+1}} \sum_{j=i-6}^{i+2} c_j B_j(x)dx = \sum_{j=i-6}^{i+2} c_j \int_{x_i}^{x_{i+1}} B_j(x)dx = I_i, \tag{4}$$

and also we have

$$I_i = \frac{h}{362880}(c_{i-6} + 502c_{i-5} + 14608c_{i-4} + 88234c_{i-3} +$$
$$156190c_{i-2} + 88234c_{i-1} + 14608c_i + 502c_{i+1} + c_{i+2}), \tag{5}$$

For discretization of boundary conditions we define:

$$\begin{cases} S(x_0) = y_0, & S(x_n) = y_n, & S'(x_0) = y_0', & S'(x_n) = y_n', \\ S(x_1) = y_1, & S(x_{n-1}) = y_{n-1}, & S''(x_0) = y_0'', & S''(x_n) = y_n'', \\ S(x_2) = y_2, & S(x_{n-2}) = y_{n-2}, & S'''(x_0) = y_0''', & S'''(x_n) = y_n''', \\ S(x_3) = y_3, & S(x_{n-3}) = y_{n-3}, \end{cases} \tag{6}$$

$$\begin{cases} c_{-6} + 247c_{-5} + 4293c_{-4} + 15619c_{-3} + 15619c_{-2} + 4293c_{-1} + 247c_0 + c_1 = 40320y_0, \\ c_{-5} + 247c_{-4} + 4293c_{-3} + 15619c_{-2} + 15619c_{-1} + 4293c_0 + 247c_1 + c_2 = 40320y_1, \\ c_{-4} + 247c_{-3} + 4293c_{-2} + 15619c_{-1} + 15619c_0 + 4293c_1 + 247c_2 + c_3 = 40320y_2, \\ c_{-3} + 247c_{-2} + 4293c_{-1} + 15619c_0 + 15619c_1 + 4293c_2 + 247c_3 + c_4 = 40320y_3, \end{cases} \tag{7}$$

$$\begin{cases} c_{n-6} + 247c_{n-5} + 4293c_{n-4} + 15619c_{n-3} + 15619c_{n-2} + 4293c_{n-1} + 247c_n + c_{n+1} = 40320y_n, \\ c_{n-7} + 247c_{n-6} + 4293c_{n-5} + 15619c_{n-4} + 15619c_{n-3} + 4293c_{n-2} + 247c_{n-1} + c_n = 40320y_{n-1}, \\ c_{n-8} + 247c_{n-7} + 4293c_{n-6} + 15619c_{n-5} + 15619c_{n-4} + 4293c_{n-3} + 247c_{n-2} + c_{n-1} = 40320y_{n-2}, \\ c_{n-9} + 247c_{n-8} + 4293c_{n-7} + 15619c_{n-6} + 15619c_{n-5} + 4293c_{n-4} + 247c_{n-3} + c_{n-2} = 40320y_{n-3}, \end{cases} \tag{8}$$

$$\begin{cases} -8c_{-6} - 952c_{-5} - 8568c_{-4} - 9800c_{-3} + 9800c_{-2} + 8568c_{-1} + 952c_0 + 8c_1 = 40320hy_0', \\ 56c_{-6} + 3080c_{-5} + 10584c_{-4} - 13720c_{-3} - 13720c_{-2} + 10548c_{-1} + 3080c_0 + 56c_1 = 40320h^2y_0'', \\ -336c_{-6} - 7728c_{-5} + 3024c_{-4} + 31920c_{-3} - 31920c_{-2} - 3024c_{-1} + 7728c_0 + 336c_1 = 40320h^3y_0''', \end{cases} \tag{9}$$

$$\begin{cases} -8c_{n-6} - 952c_{n-5} - 8568c_{n-4} - 9800c_{n-3} + 9800c_{n-2} + 8568c_{n-1} + 952c_n + 8c_{n+1} = 40320hy_n', \\ 56c_{n-6} + 3080c_{n-5} + 10584c_{n-4} - 13720c_{n-3} - 13720c_{n-2} + 10548c_{n-1} + 3080c_n + 56c_{n+1} = 40320h^2y_n'', \\ -336c_{n-6} - 7728c_{n-5} + 3024c_{n-4} + 31920c_{n-3} - 31920c_{n-2} - 3024c_{n-1} + 7728c_n + 336c_{n+1} = 40320h^3y_n'''. \end{cases} \tag{10}$$

By using (5),(7) and (8) we get a system of linear equations of order $(n+8)(n+8)$, and may be written in a matrix form as

$$AC = F, \tag{11}$$

where

$$C = (c_{-6}, c_{-5}, ..., c_{n+1})^T, \quad D = (40320y_0, 40320y_1, 40320y_2, 40320y_3, \frac{362880I_0}{h}, ..., \frac{362880I_{n-1}}{h}, 40320y_{n-3}, 40320y_{n-2}, 40320y_{n-1}, 40320y_n),$$

or

$$D = (40320y_0, 40320hy_0', 40320h^2y_0'', 40320h^3y_0''', \frac{362880I_0}{h}, ..., \frac{362880I_{n-1}}{h}, 40320h^3y_n''', 40320h^2y_n'', 40320hy_n', 40320y_n),$$

$A =$

$$\begin{pmatrix}
1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & & & & & \\
0 & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & & & & \\
0 & 0 & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & & & \\
0 & 0 & 0 & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & & \\
1 & 502 & 14608 & 88234 & 156190 & 88234 & 14608 & 502 & 1 & & & & & \\
 & 1 & 502 & 14608 & 88234 & 156190 & 88234 & 14608 & 502 & 1 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & 1 & 502 & 14608 & 88234 & 156190 & 88234 & 14608 & 502 & 1 & & \\
 & & & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & & \\
 & & & & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & \\
 & & & & & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & \\
 & & & & & & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1
\end{pmatrix}. \tag{12}$$

And also by using (5),(9) and (10), A is in the following form $A =$

$$\begin{pmatrix}
1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1 & & & \\
-8 & -952 & -8568 & -9800 & 9800 & 8568 & 952 & 8 & & & \\
56 & 3080 & 10584 & -13720 & -13720 & 10584 & 3080 & 56 & & & \\
-336 & -7728 & 3024 & 31920 & -31920 & -3024 & 7728 & 336 & & & \\
1 & 502 & 14608 & 88234 & 156190 & 88234 & 14608 & 502 & 1 & & \\
 & 1 & 502 & 14608 & 88234 & 156190 & 88234 & 14608 & 502 & 1 & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix} +$$

$$\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & 1 & 502 & 14608 & 88234 & 156190 & 88234 & 14608 & 502 & 1 & & \\
 & -336 & -7728 & 3024 & 31920 & -31920 & -3024 & 7728 & 336 & & & \\
 & & 56 & 3080 & 10584 & -13720 & -13720 & 10584 & 3080 & 56 & & \\
 & & & -8 & -952 & -8568 & -9800 & 9800 & 8568 & 952 & 8 & \\
 & & & & 1 & 247 & 4293 & 15619 & 15619 & 4293 & 247 & 1
\end{pmatrix}. \tag{13}$$

Theorem 1 *Integro interpolation by using B-splines of degree eight (2) and (3) is uniquely solvable.*

Proof. By using the Gauss elimination method for band matrix A, we have the determinant of coefficient matrix A, is nonzero.

By using Table 1, and after solving the system (11), we can obtain the integro b-spline of degree eight $S^{(k)}(x) = \sum_{i=-6}^{n+1} c_i B_i^{(k)}(x)$, to approximate $y^{(k)}(x), k = 0, ..., 7$, at the knots x_j. We have the following formulas

$$S_j = S(x_j) = \sum_{i=-6}^{n+1} c_i B_i(x_j) = \frac{1}{8!}(c_{j-6} + 247c_{j-5} +$$

$$4293c_{j-4} + 15619c_{j-3} + 15619c_{j-2} + 4293c_{j-1} + 247c_j + c_{j+1}), \tag{14}$$

$$m_j = S'(x_j) = \sum_{i=-6}^{n+1} c_i B_i'(x_j) = \frac{1}{h8!}(-8c_{j-6} - 952c_{j-5}-$$

$$8568c_{j-4} - 9800c_{j-3} + 9800c_{j-2} + 8568c_{j-1} + 952c_j + 8c_{j+1}), \tag{15}$$

$$M_j = S''(x_j) = \sum_{i=-6}^{n+1} c_i B_i''(x_j) = \frac{1}{h^2 8!}(56c_{j-6} + 3080c_{j-5}+$$

$$10584c_{j-4} - 13720c_{j-3} - 13720c_{j-2} + 10584c_{j-1} + 3080c_j + 56c_{j+1}), \tag{16}$$

$$T_j = S'''(x_j) = \sum_{i=-6}^{n+1} c_i B_i'''(x_j) = \frac{1}{h^3 8!}(-336c_{j-6} - 7728c_{j-5}+$$

$$3024c_{j-4} + 31920c_{j-3} - 31920c_{j-2} - 3024c_{j-1} + 7728c_j + 336c_{j+1}), \tag{17}$$

$$t_j = S^{(4)}(x_j) = \sum_{i=-6}^{n+1} c_i B_i^{(4)}(x_j) = \frac{1}{h^4 8!}(1680c_{j-6} + 11760c_{j-5} - 45360$$

$$c_{j-4} + 31920c_{j-3} + 31920c_{j-2} - 45360c_{j-1} + 11760c_j + 1680c_{j+1}), \tag{18}$$

$$N_j = S^{(5)}(x_j) = \sum_{i=-6}^{n+1} c_i B_i^{(5)}(x_j) = \frac{1}{h^5 8!}(-6720c_{j-6} + 6720c_{j-5} + 60480$$

$$c_{j-4} - 168000c_{j-3} + 168000c_{j-2} - 60480c_{j-1} - 6720c_j + 6720c_{j+1}), \tag{19}$$

$$n_j = S^{(6)}(x_j) = \sum_{i=-6}^{n+1} c_i B_i^{(6)}(x_j) = \frac{1}{h^6 8!}(20160c_{j-6} - 100800c_{j-5} + 181440$$

$$c_{j-4} - 100800c_{j-3} - 100800c_{j-2} + 181440c_{j-1} - 100800c_j + 20160c_{j+1}), \tag{20}$$

$$p_j = S^{(7)}(x_j) = \sum_{i=-6}^{n+1} c_i B_i^{(7)}(x_j) = \frac{1}{h^7 8!}(-40320c_{j-6} + 282240c_{j-5} - 846720$$

$$c_{j-4} + 1411200c_{j-3} - 1411200c_{j-2} + 846720c_{j-1} - 282240c_j + 40320c_{j+1}), \tag{21}$$

■

4 Error analysis

Table 2: Maximum absolute errors in solution of example 1.

n	$E(n)$	$E''(n)$	$E^{(4)}(n)$
10	2.62×10^{-12}	5.71×10^{-9}	2.06×10^{-5}
20	2.79×10^{-15}	2.62×10^{-11}	3.54×10^{-7}

Table 3: Maximum absolute errors in [4] of example 1.

n	$E(n)$	$E''(n)$
10	2.490×10^{-7}	2.391×10^{-3}
20	4.309×10^{-9}	1.506×10^{-4}

In order to discuss the errors analyze, we need some useful operators for a given step h and an infinitely differentiable $y(x)$, (m is positive integer),[4,5] in the following form:

$$\mathsf{E}y(x) = y(x+h), \mathsf{D}y(x) = y'(x), Iy(x) = y(x), \mathsf{E}y(x) = e^{h\mathsf{D}}, \mathsf{E}^m y(x) = e^{mh\mathsf{D}},$$

$$\mathsf{E}^m y(x) = y(x+mh), \mathsf{D}^m y(x) = y^{(m)}(x), I^m y(x) = y(x), I_i = \frac{\mathsf{E}-I}{\mathsf{D}} y_i.$$

Table 4: Maximum absolute errors in [5] of example 1.

n	$E(n)$	$E''(n)$	$E^{(4)}(n)$
10	9.47×10^{-10}	1.77×10^{-6}	6.18×10^{-3}
20	3.67×10^{-12}	2.80×10^{-8}	3.89×10^{-4}

Table 5: Maximum absolute errors in solution of example 2.

n	$E(n)$	$E''(n)$	$E^{(4)}(n)$
10	2.22×10^{-15}	2.19×10^{-12}	1.08×10^{-8}
20	2.00×10^{-15}	8.28×10^{-12}	1.88×10^{-7}

Theorem 2 *Let $y(x)$ be an infinitely differentiable function and $S(x)$ be the integro b-spline eight degree obtained by (2) and (3) for $i = 0, 1, ..., n$, we get*

$$S_i = y(x_i) + \frac{h^{10}}{4790016} y^{(10)}(x_i) + O(h^{12}), \tag{22}$$

$$M_i = y''(x_i) - \frac{h^8}{172800} y^{(10)}(x_i) + O(h^{10}), \tag{23}$$

$$t_i = y^{(4)}(x_i) + \frac{h^6}{6048} y^{(10)}(x_i) + O(h^8), \tag{24}$$

$$n_i = y^{(6)}(x_i) - \frac{h^4}{240} y^{(10)}(x_i) + O(h^6). \tag{25}$$

Proof. By using (5) and (14) we get

$$\frac{h}{9}(S_{j-6} + 502S_{j-5} + 14608S_{j-4} + 88234S_{j-3} + 156190S_{j-2} + 88234S_{j-1}$$
$$+14608S_j + 502S_{j+1} + S_{j+2}) = (I_{j-6} + 247I_{j-5} + 4293I_{j-4} + 15619I_{j-3}+$$
$$15619I_{j-2} + 4293I_{j-1} + 247I_j + I_{j+1}), \tag{26}$$

Using operator notations, we get

$$S_j = \frac{9}{h}[\frac{\mathsf{E}^{-6} + 247\mathsf{E}^{-5} + 4293\mathsf{E}^{-4} + 15619\mathsf{E}^{-3} + 15619\mathsf{E}^{-2} + 4293\mathsf{E}^{-1} + 247I + \mathsf{E}^1}{\mathsf{E}^{-6} + 502\mathsf{E}^{-5} + 14608\mathsf{E}^{-4} + 88234\mathsf{E}^{-3} + 156190\mathsf{E}^{-2} + 88234\mathsf{E}^{-1} + 14608I + 502\mathsf{E}^1 + \mathsf{E}^2}]I_j$$

By using $u = h\mathsf{D}, \mathsf{E} = e^{h\mathsf{D}}$ and $I_j = \frac{\mathsf{E}-I}{\mathsf{D}} y_j$ we have

$$S_j = \frac{9}{u}[\frac{-e^{-6u} + 247e^{-5u} - 4046e^{-4u} - 11326^{-3u} + 11326e^{-2u} + 4046I + 247I + 246^{u} + e^{2u}}{e^{-6u} + 502e^{-5u} + 14608e^{-4u} + 88234e^{-3u} + 156190e^{-2u} + 88234e^{-u} + 14608I + 502e^{u} + e^{2u}}]y_j,$$

by using taylor expansion, we get

$$S_j = (1 + \frac{u^{10}}{4790016} - \frac{691u^{12}}{5943974400} + \frac{u^{14}}{37324800} - ...)y_j, \tag{27}$$

$$S_j = (y_j + \frac{h^{10}}{4790016} y_j^{(10)} + O(h^{12})). \tag{28}$$

Then, we have

$$Max_{0\le j\le n}|S_j - y(x_j)| \equiv O(h^{10}). \tag{29}$$

Similarly, by using (5) and (14), we easily have

$$\frac{h^3}{9}(M_{j-6} + 502M_{j-5} + 14608M_{j-4} + 88234M_{j-3} + 156190M_{j-2} + 88234M_{j-1}$$
$$+14608M_j + 502M_{j+1} + M_{j+2}) = (56I_{j-6} + 3080I_{j-5} + 10584I_{j-4} - 13720I_{j-3}+$$
$$-13720I_{j-2} + 10584I_{j-1} + 3080I_j + 56I_{j+1}), \tag{30}$$

Using operator notations, we obtain

$$M_j=\frac{9}{h^3}[\frac{56\mathsf{E}^{-6}+3080\mathsf{E}^{-5}+10584\mathsf{E}^{-4}-13720\mathsf{E}^{-3}-13720\mathsf{E}^{-2}+10584\mathsf{E}^{-1}+3080I+56\mathsf{E}^{1}}{\mathsf{E}^{-6}+502\mathsf{E}^{-5}+14608\mathsf{E}^{-4}+88234\mathsf{E}^{-3}+156190\mathsf{E}^{-2}+88234\mathsf{E}^{-1}+14608I+502\mathsf{E}^{1}+\mathsf{E}^{2}}]I_j$$

By using $u=h\mathsf{D}$, $\mathsf{E}=e^{h\mathsf{D}}$ and $I_j=\frac{\mathsf{E}-I}{\mathsf{D}}y_j$ we have

$$M_j=\frac{9}{h^2u}[\frac{-56e^{-6u}-3024e^{-5u}-7504e^{-4u}+24308e^{-3u}-24304e^{-u}+7504I+30246^{u}+56e^{2u}}{e^{-6u}+502e^{-5u}+14608e^{-4u}+88234e^{-3u}+156190e^{-2u}+88234e^{-u}+14608I+502e^{u}+e^{2u}}]y_j$$

$$M_j=(D^2-\frac{D^{10}h^8}{172800}+\frac{17D^{12}h^{10}}{9580032}-...)y_j, \tag{31}$$

$$M_j=y_j''-\frac{h^8}{172800}y_j^{(10)}+O(h^{10}). \tag{32}$$

Hence, we get

$$Max_{0\leq j\leq n}|M_j-y(x_j)|\equiv O(h^8). \tag{33}$$

And also, by using (5) and (14), we have

$$\frac{h^5}{9}(S_{j-6}+502S_{j-5}+14608S_{j-4}+88234S_{j-3}+156190S_{j-2}+88234S_{j-1}$$
$$+14608S_j+502S_{j+1}+S_{j+2})=(1680I_{j-6}+11760I_{j-5}-45360I_{j-4}+31920I_{j-3}+$$
$$31920I_{j-2}-45360I_{j-1}+11760I_j+1680I_{j+1}), \tag{34}$$

Using operator notations, we obtain

$$t_j=\frac{9}{h^4u}[\frac{1680e^{-6u}+11760e^{-5u}-45360e^{-4u}+31920e^{-3u}+31920e^{-2u}-45360e^{-u}+11760I+1680e^{u}}{e^{-6u}+502e^{-5u}+14608e^{-4u}+88234e^{-3u}+156190e^{-2u}+88234e^{-u}+14608I+502e^{u}+e^{2u}}]$$
$$(\frac{e^u-I}{I}),$$

$$t_j=(D^4+\frac{D^{10}h^6}{6048}-\frac{17D^{12}h^8}{34560}-...)y_j, \tag{35}$$

$$t_j=y_j^{(4)}+\frac{h^6}{6048}y_j^{(10)}+O(h^8). \tag{36}$$

Hence, we get

$$Max_{0\leq j\leq n}|t_j-y(x_j)|\equiv O(h^6). \tag{37}$$

The following expression can be proved similarly and it is omitted here.

$$n_j=(D^6-\frac{D^{10}h^4}{240}+\frac{D^{12}h^6}{3024}-...)y_j, \tag{38}$$

$$n_j=y_j^{(6)}-\frac{h^4}{240}y_j^{(10)}+O(h^6). \tag{39}$$

Hence, we get

$$Max_{0\leq j\leq n}|n_j-y(x_j)|\equiv O(h^4). \tag{40}$$

■

Corollary 1. *Let $y(x)$ be an infinitely differentiable function and $S(x)$ be the integro b-spline with eight degree obtained by (5) and (14), we have*

$$\|S^{(k)}(x)-y^{(k)}(x)\|_\infty\equiv O(h^{10-k}), k=0,2,4,6. \tag{41}$$

5 Numerical Results

We applied our presented method to solve three examples with different value of $n = 10, 20$ and the observed maximum absolute errors are given in Tables. Numerical results are given to illustrate the efficiency of methods and compared with the methods in [4,5]. Suppose that $y(x) \in C^\infty[0, 1]$ and consider the following test functions

Example 1:

$$y(x) = cos(\pi x),$$

Example 2:

$$y(x) = e^x,$$

Example 3:

$$y(x) = \frac{1}{x+2}.$$

These maximum errors $E(n), E''(n)$ and $E^{(4)}(n)$ between the approximated function $y(x)$ and its corresponding integro b-spline with eight-degree are listed in Tables 2,5, and 8.

$$E(n) = Max_{0\leq j\leq n}|S_j - y(x_j)|,\ \ E''(n) = Max_{0\leq j\leq n}|M_j - y''(x_j)|,\ \ E^{(4)}(n) = Max_{0\leq j\leq n}|t_j - y^{(4)}(x_j)|.$$

Table 6: Maximum absolute errors in [5] of example 2.

n	$E(n)$	$E''(n)$	$E^{(4)}(n)$
10	1.72×10^{-12}	2.12×10^{-9}	2.99×10^{-5}
20	1.45×10^{-14}	7.39×10^{-11}	3.64×10^{-6}

Table 7: Maximum absolute errors in [4] of example 2.

n	$E(n)$	$E''(n)$
10	6.817×10^{-10}	6.407×10^{-6}
20	1.157×10^{-11}	4.136×10^{-7}

Table 8: Maximum absolute errors in solution of example 3.

n	$E(n)$	$E''(n)$	$E^{(4)}(n)$
10	8.12×10^{-14}	1.08×10^{-10}	1.80×10^{-7}
20	3.33×10^{-16}	1.10×10^{-12}	6.47×10^{-9}

Table 9: Maximum absolute errors in [5]of example 3.

n	$E(n)$	$E''(n)$	$E^{(4)}(n)$
10	1.01×10^{-11}	1.32×10^{-8}	2.39×10^{-4}
20	8.46×10^{-14}	4.07×10^{-10}	2.82×10^{-5}

6 Conclusions

We have developed B-splines with eight degree to construct an approximating function based on the integral values, rather than the usual function values at the knots. It is called integro eighth spline interpolation. The super convergence (Tenth, eighth, sixth and fourth order, respectively) in approximating function values, second-order, fourth-order, sixth-order, eighth-order and tenth-order derivative values at the knots is mainly proved. Our numerical results are better than those produced by others.

Table 10: Maximum absolute errors in [4] of example 3.

n	$E(n)$	$E''(n)$
10	3.483×10^{-7}	4.855×10^{-4}
20	8.108×10^{-9}	4.408×10^{-5}

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