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Rogue Wave Solution and the Homotopy Analysis Method for Fractional Hirota Equation

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Abstract: In this work, we apply the homotopy analysis method to derive the rogue wave solution for the well-known fractional Hirota equation. On the basis of the homotopy analysis method, we develop a scheme to obtain the approximate solution of the fractional Hirota equation with initial condition when the convergence control parameter h equals -1, which is introduced by replacing some integer-order time derivatives by fractional derivatives.

Keywords: rogue wave solutions; fractional Hirota equation; fractional partial differential equations(FPDEs); homotopy analysis method(HAM)

1 Introduction

In recent years, the study of fractional differential equations has drawn much attention both from the mathematical and physical points of view from researchers in nonlinear phenomena. The primary reason for interest is that the exact description of most phenomena in fluidmechanics, viscoelasticity, biology, physics, engineering, and other areas of science are governed by nonlinear equations involving fractional-order derivatives [7, 8].

On the other hand, recently, the rogue wave phenomenon becomes hot in the nonlinear science. Today, there is widespread consensus on the existence of rogue waves in the ocean [2]. A number of mechanisms have been proposed to explain their unexpected emergence. One of the essential elements in many of these explanations is the idea that rogue waves could be related to rational solutions of the underlying evolution equations [5]. Such solutions could, in principle, describe a large wave that appears from nowhere and disappears without a trace [6], a behavior that has been reported for many known rogue wave events. The rogue wave solutions can be derived by lots of different integrable system method, for instance: Darboux transformation [9–12], Hirota bilinear method[13] and Algebraic-geometry solution reduction method[14–16]. Evidently these methods cannot be generalized to solve the fractional differential equation, since there is no Lax pair and bilinear form for the fractional differential equation.

One of the recently developed techniques is the homotopy analysis method (HAM) [17–19], which is a combination of the classical perturbation technique and homotopy concept as used in topology, to derive both exact and approximate solutions for linear and nonlinear problems. One of the advantages of HAM is that it does not need linearization, weak nonlinearity assumptions, discretization, small parameter, and linear term in a differential equation. Compared to other numerical and non-numerical techniques, HAM provides a convenient way to control and adjust the convergence region of the solution series. For example, Liang and Jeffrey [24] illustrated that when analytic approximation given by the other analytic method is divergent in the whole domain, one can gain convergent series solution by choosing a proper value of h. The application of HAM has been applied to a variety of problems arising from science and engineering for all types of boundary and initial conditions governed by nonlinear equations [25–28]. Recently, the HAM is extended for the first time to the fractional Korteweg-de Vries (KdV)-Burgers-Kuramoto equation [29] and derived its approximate analytic solutions. Then the HAM was successfully applied to solve many nonlinear fractional differential equations see [30–32]. There are lots of works for soliton solutions for the fractional differential equation [30–32].

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As is pointed out by Liao [33], "...... the so-called rogue wave is a hot topic of nonlinear waves. Certainly, it is valuable to apply the HAM ......" However, to the best of our knowledge, there is few work about rogue wave solutions for fractional differential equation until now. In our work, we aim at deriving the rogue wave solution for fractional Hirota equation.

As mentioned by several researchers in fractional calculus, derivatives of noninteger order are very effective for the description of many physical phenomena such as rheology, damping laws, and diffusion process. Several method have been used to solve fractional partial differential equations, such as Laplace transform method, Fourier transform method, Adomain's decompositon method(ADM) and so on. Although there are a lot of studies for the classical nonlinear partial differential equations, it seems that detailed studies of the nonlinear fractional differential equation are only beginning.

The current paper is organized as following: we first give some basic definitions and properties of fractional operators in Sec.2 which are required for the remaining part of the article. In Sec.3 we will introduce the homotopy analysis method briefly and apply this technique to solve fractional partial differential equations. Then a scheme is developed to obtain the exact solution and approximate solution of the Hirota equation which are given in Secs.4 and 5, respectively. Finally some appendixes and references are given at the end of this paper.

2 Basic Definitions

In this section, we recall some certain definitions and properties of the fractional operators[34].

Definition 1. A real function $f(t), t > 0$, is said to be in the space $\mathbb{C}_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in \mathbb{C}(0,\infty)$, and it is said to be in the space $\mathbb{C}_\mu^n$, if and only if $f^{(n)} \in \mathbb{C}_\mu, n \in \mathbb{N}$[34].

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in \mathbb{C}_\lambda, \lambda \geq -1$, is defined by[34]

$$J^\alpha f(t) = D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, (\alpha > 0), \tag{2.1}$$

$$J^0 f(t) = f(t), \tag{2.2}$$

where Γ is the Gamma function. Some of the properties of the operator (J^α) are given in the following:

For $f \in \mathbb{C}_\lambda, \lambda \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$

(1) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
(2) $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) *of* $f(t)$ in the Caputo's sense is defined by[34]

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ \ (\alpha > 0) \tag{2.3}$$

where $n = [\alpha] + 1, t > 0, \ f \in \mathbb{C}_{-1}^n$. The following are two basic properties of the Caputo's fractional derivative[35]:

(1) Let $f \in \mathbb{C}_{-1}^n, n \in \mathbb{N}$. Then $D^\alpha f, \ 0 \leq \alpha \leq n$ is well defined and $D^\alpha f \in \mathbb{C}_{-1}^n$.
(2) Let $n-1 < \alpha < n \in \mathbb{N}$ and $f \in \mathbb{C}_\lambda^n, \ \lambda \geq -1$. Then

$$(J^\alpha D^\alpha) f(t) = f(t) - \sum_{k=0}^{n-1} f^k(0^+) \frac{t^k}{k!}, t > 0. \tag{2.4}$$

When $\alpha \in \mathbb{N}$, $D^\alpha f(t) = f^{(\alpha)}(t)$, is the ordinary derivative.
In this article only real and positive α will be considered. Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation[29, 36]

$$D^\alpha(\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t), \tag{2.5}$$

where λ, μ are constants, and satisfies the so-called Leibnitz rule

$$D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t), \tag{2.6}$$

if $f(\tau)$ is continuous in $[0,t]$ and $g(\tau)$ has $(n+1)$ continuous derivatives in $[0,t]$.
The Caputo's definition of fractional differentiation is used throughout this paper.

3 The Homotopy Analysis Method

In this article, we use the homotopy analysis method to solve the nonlinear fractional partial differential equation. This method proposed by a Chinese mathematician Liao[17]. We apply Liao's basic ideas to the nonlinear fractional partial differential equation. Let us consider a scalar partial differential equation(PDE) with two independent variables x and t having the form:

$$\mathcal{NFD}\,(u(x,t)) = 0, \tag{3.1}$$

where $\mathcal{NFD}$ is a nonlinear operator, $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the same way. On the basis of the constructed zero-order deformation equation by Liao[18], we give the following zero-order deformation equation in the similar way

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] = qh\mathcal{NFD}[v(x,t;q)], \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear noninteger order operator and it possesses the property $\mathcal{L}(C) = 0, u_0(x,t)$ is an initial guess of $u(x,t)$ satisfying the initial conditions, $v(x,t;q)$ is an unknown function on independent variables x,t,q. It is important to that one has great freedom to choose auxiliary parameter h in HAM. The $q = 0$ and $q = 1$, give respectively[20–23]

$$v(x,t;0) = u_0(x,t),\ \ v(x,t;1) = u(x,t). \tag{3.3}$$

Thus as q increases from 0 to 1, the solution $v(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $v(x,t;q)$ in a Taylor series with respect to q, one has

$$v(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{3.4}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m v(x,t;q)}{\partial q^m}\bigg|_{q=0} \tag{3.5}$$

If the auxiliary linear noninteger order operator, the initial guess, and the auxiliary parameter h are so properly chosen, the series Eq.(3.4) converges at $q = 1$. Hence we have[19, 37–39]

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{3.6}$$

which must be one of the solution of the original nonlinear equation, as proved by[18]. We substitute $h = -1$, then Eq.(3.2) becomes

$$(1-q)\mathcal{L}[v(x,t;q) - u_0(x,t)] + q\mathcal{NFD}v(x,t;q) = 0, \tag{3.7}$$

according to Eq.(3.4) the governing equation can be deduced from the zero-order deformation Eq.(3.2).
Define the vector[18]

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}. \tag{3.8}$$

Differentiating Eq.(3.2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation[18]

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathcal{NFR}(\vec{u}_{m-1}(x,t)), \tag{3.9}$$

where

$$\mathcal{NFR}(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{NFD}(v(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0}, \tag{3.10}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{3.11}$$

The mth-order deformation Eq.(3.9) is linear and thus can be easily solved, especially by means of a symbolic computation software such as Mathematica and Maple.

4 Integer-order Hirota equation

We first apply the HAM to the integrable Hirota equation. In the dimensionless form it is given by [40]

$$\psi_t - \frac{1}{2}i\psi_{xx} - i|\psi|^2\psi - \beta\psi_{xxx} - 6\beta|\psi|^2\psi_x = 0, \tag{4.1}$$

where x is the propagation variable and t is the transverse variable, β is a real coefficient, denotes the strength of higher-order linear and nonlinear effects, and ψ is a complex valued function of independent variables x and t. We consider Eq.(4.1) with the initial condition

$$\psi(x,0) = 1 - \frac{4}{1+4x^2}. \tag{4.2}$$

To study the Eq.(4.1) more conveniently, we introduce the gauge transformation $\psi(x,t) = e^{it}\phi(x,t)$, it follows that

$$\phi_t - \frac{1}{2}i\phi_{xx} - i|\phi|^2\phi - \beta\phi_{xxx} - 6\beta|\phi|^2\phi_x + i\phi = 0. \tag{4.3}$$

We define the coordinate transformation such that $\tau = t$ and $\kappa = x + 6\beta t$, then $\phi(x,t) \to \phi(\kappa,\tau)$ and Eq.(4.3) is equivalent to

$$\phi_\tau - \frac{1}{2}i\phi_{\kappa\kappa} - i|\phi|^2\phi - \beta\phi_{\kappa\kappa\kappa} - 6\beta|\phi|^2\phi_\kappa + i\phi + 6\beta\phi_\kappa = 0. \tag{4.4}$$

Below, we explain how to derive an exact rogue wave solution of Eq.(4.4). Let us assume the auxiliary linear operator be

$$\mathcal{L}[v(\kappa,\tau;q)] = D_\tau^1 v(\kappa,\tau;q) = v_\tau(\kappa,\tau;q). \tag{4.5}$$

We define the nonlinear operator from Eq.(4.4) as

$$\begin{aligned}\mathcal{NFD}v(\kappa,\tau;q) =& D_\tau^1 v(\kappa,\tau;q) - \frac{1}{2}iv_{\kappa\kappa}(\kappa,\tau;q) - i|v(\kappa,\tau;q)|^2 v(\kappa,\tau;q) - \beta v_{\kappa\kappa\kappa}(\kappa,\tau;q) \\ &- 6\beta|v(\kappa,\tau;q)|^2 v_\kappa(\kappa,\tau;q) + iv(\kappa,\tau;q) + 6\beta v_\kappa(\kappa,\tau;q).\end{aligned} \tag{4.6}$$

Following the procedure outlined in the above section, we now construct the so-called zeroth-order deformation equation

$$(1-q)\mathcal{L}[v(\kappa,\tau;q) - \phi_0(\kappa,\tau)] = qh\mathcal{NFD}v(\kappa,\tau;q). \tag{4.7}$$

Also note that, when $q = 0$ and $q = 1$ respectively, we obtain

$$v(\kappa,\tau;0) = \phi_0(\kappa,\tau) = \phi(\kappa,0),\ \ v(\kappa,\tau;1) = \phi(\kappa,\tau). \tag{4.8}$$

Eqs.(3.9)-(3.11) yield the mth-order deformation equation

$$\mathcal{L}[\phi_m(\kappa,\tau)-\chi_m\phi_{m-1}(\kappa,\tau)] = h\mathcal{NFR}(\vec{\phi}_{m-1}(\kappa,\tau)), \tag{4.9}$$

where

$$\begin{aligned}\mathcal{NFR}(\vec{\phi}_{m-1}(\kappa,\tau)) =& D_\tau^1\phi_{m-1} - \frac{1}{2}i(\phi_{m-1})_{\kappa\kappa} - i\sum_{j=0}^{m-1}\bar{\phi}_{m-1-j}\sum_{i=0}^{j}\phi_i\phi_{j-i} - \beta(\phi_{m-1})_{\kappa\kappa\kappa} \\ & - 6\beta\sum_{j=0}^{m-1}\bar{\phi}_{m-1-j}\sum_{i=0}^{j}\phi_i(\phi_{j-i})_\kappa + i\phi_{m-1} + 6\beta(\phi_{m-1})_\kappa,\end{aligned} \tag{4.10}$$

the overbar represents the complex conjugation and ϕ_i denotes $\phi_i(\kappa,\tau)$. Thus the solution of Eq.(4.9) for $m \geq 1$ becomes

$$\phi_m(\kappa,\tau) = \chi_m\phi_{m-1}(\kappa,\tau) + h\mathcal{L}^{-1}\mathcal{NFR}(\vec{\phi}_{m-1}(\kappa,\tau)). \tag{4.11}$$

It is obvious that $\phi_0(\kappa,\tau) = 1 - \frac{4}{1+4\kappa^2} \equiv f(\kappa)$, employing the above Eqs.(4.10) and (4.11) and using property (3.11), we now successively get

$$\phi_0(\kappa,\tau) = \phi(\kappa,0) = f(\kappa), \tag{4.12}$$

$$\phi_1(\kappa,\tau) = hD_\tau^{-1}\left[-\frac{1}{2}i\phi_{0\kappa\kappa} - i|\phi_0|^2\phi_0 - \beta\phi_{0\kappa\kappa\kappa} - 6\beta|\phi_0|^2\phi_{0\kappa} + i\phi_0 + 6\beta\phi_{0\kappa}\right] = hD_\tau^{-1}A_1, \tag{4.13}$$

where

$$A_1 = -\frac{1}{2}if_{\kappa\kappa} - if^3 - \beta f_{\kappa\kappa\kappa} - 6\beta f^2 f_\kappa + if + 6\beta f_\kappa.$$

Furthermore

$$\phi_2(\kappa,\tau) = h(h+1)D_\tau^{-1}A_1 + h^2D_\tau^{-2}A_2 = \frac{A_1h(h+1)}{\Gamma(2)}\tau + \frac{A_2h^2}{\Gamma(3)}\tau^2, \tag{4.14}$$

where

$$A_2 = -\frac{1}{2}iA_{1\kappa\kappa} - i\left(\bar{A}_1\phi_0^2 + 2\phi_0^2A_1\right) - \beta A_{1\kappa\kappa\kappa} - 6\beta\left(\bar{A}_1\phi_0\phi_{0\kappa} + \phi_0^2A_{1\kappa} + \phi_0A_1\phi_{0\kappa}\right) + iA_1 + 6\beta A_{1\kappa}.$$

In a similar way, we have

$$\phi_3(\kappa,\tau) = \frac{A_1h(h+1)^2}{\Gamma(2)}\tau + \frac{A_2h^2(h+1)}{\Gamma(3)}\tau^2 + \left(\frac{A_3h^3}{\Gamma(3)} + \frac{A_4h^3}{\Gamma^2(2)}\right)\frac{\Gamma(3)}{\Gamma(4)}\tau^3, \tag{4.15}$$

where

$$\begin{aligned}A_3 &= -\frac{1}{2}iA_{2\kappa\kappa} - i\left(\bar{A}_2\phi_0^2 + 2\phi_0^2A_2\right) - \beta A_{2\kappa\kappa\kappa} - 6\beta\left(\bar{A}_2\phi_0\phi_{0\kappa} + \phi_0^2A_{2\kappa} + \phi_0\phi_{0\kappa}A_2\right) + iA_2 + 6\beta A_{2\kappa}, \\ A_4 &= -i\left(2\bar{A}_1\phi_0A_1 + \phi_0A_1^2\right) - 6\beta\left(\bar{A}_1\phi_0A_{1\kappa} + \bar{A}_1A_1\phi_{0\kappa} + \phi_0A_1A_{1\kappa}\right).\end{aligned}$$

We can obtain higher-order series solution in a similar way, but we would like to point that the complexity of higher-order components grows fast with order increasing.

In the above terms we choose $h = -1$, then the Eqs.(4.12),(4.13),(4.14) and (4.15) become

$$\phi_0(\kappa,\tau) = f(\kappa), \quad \phi_1(\kappa,\tau) = -A_1\tau, \quad \phi_2(\kappa,\tau) = \frac{A_2}{2}\tau^2, \quad \phi_3(\kappa,\tau) = -\left(\frac{A_3}{6} + \frac{A_4}{3}\right)\tau^3, \tag{4.16}$$

and with the aid of Eq.(3.6), we have

$$\phi(\kappa,\tau) = f(\kappa) - A_1\tau + \frac{A_2}{2}\tau^2 - \left(\frac{A_3}{6} + \frac{A_4}{3}\right)\tau^3 + \cdots. \tag{4.17}$$

Starting with initial condition, we compute A_1, A_2, A_3 and A_4 by means of symbolic computation software Maple as follows:

$$A_1 = \frac{8i}{1+4\kappa^2}, A_2 = \frac{32}{(1+4\kappa^2)^2}, A_3 = -\frac{64(16i\kappa^4+40i\kappa^2-384\beta\kappa+9i)}{(1+4\kappa^2)^4}, A_4 = -\frac{64(16i\kappa^4-8i\kappa^2+192\beta\kappa-3i)}{(1+4\kappa^2)^4}.$$

Hence we have

$$\phi(\kappa,\tau) = 1 - \frac{4}{1+4\kappa^2} - \frac{8i}{1+4\kappa^2}\tau + \frac{16}{(1+4\kappa^2)^2}\tau^2 + \frac{32i}{(1+4\kappa^2)^2}\tau^3 + \cdots . \tag{4.18}$$

Using Taylor series expansion, the infinite series solution (4.18) yields the exact rogue wave solution

$$\phi(\kappa,\tau) = 1 - \frac{4(1+2i\tau)}{1+4\tau^2+4\kappa^2}. \tag{4.19}$$

We now successively obtain $\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4t^2+4(x+6\beta t)^2}\right]e^{it}$, which is also consistent with the rogue wave solution presented before in ref. [41]. From Fig 1(a) and 1(b) we can see that the obtained power series converges in a small time domain.

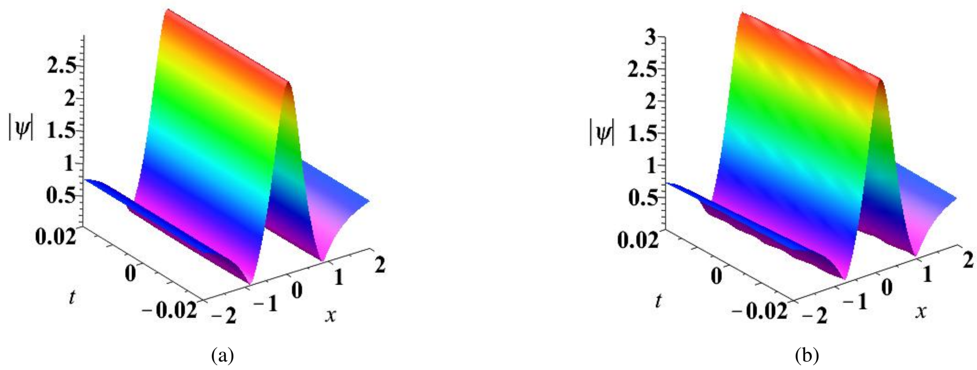

Figure 1: (color online) (a) The surface of the exact solution $\psi(x,t)$ to Eq.(4.1) with $\beta = 1$ on the (x,t) plane. (b) The surface of the 3rd-order approximate solution $\psi(x,t)$ to Eq.(4.1) with $\beta = 1$ on the (x,t) plane.

5 Fractional-order Hirota equation

Contrary to integrable Hirota equation, there is few studies of the fractional Hirota equation to the best of our knowledge. Thus we focus on obtaining approximate solution for fractional Hirota equation

$$D_t^\alpha\psi - \frac{1}{2}i\psi_{xx} - i|\psi|^2\psi - \beta\psi_{xxx} - 6\beta|\psi|^2\psi_x = 0, \tag{5.1}$$

where the variables x and t have the same physical meaning as in Eq.(4.1), $0 < \alpha < 1$. To solve the above problem with the HAM method we choose the linear noninteger-order operator

$$\mathcal{L}[u(x,t;q)] = D_t^\alpha u(x,t;q). \tag{5.2}$$

Furthermore, Eq.(5.1) approaches us to define the nonlinear operator as

$$\begin{aligned}\mathcal{NFD}[u(x,t;q)] =& D_t^\alpha u(x,t;q) - \frac{1}{2}i(u)_{xx}(x,t;q) - i|u(x,t;q)|^2u(x,t;q) \\ &- \beta(u)_{xxx}(x,t;q) - 6\beta|u(x,t;q)|^2(u)_x(x,t;q).\end{aligned} \tag{5.3}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[u(x,t;q)-\psi_0(x,t)] = qh\mathcal{NFD}u(x,t;q). \tag{5.4}$$

Obviously, when $q=0$ and $q=1$ respectively, we can write

$$u(x,t;0)=\psi_0(x,t)=\psi(x,0),\ \ u(x,t;1)=\psi(x,t). \tag{5.5}$$

Eqs.(3.9)-(3.11) yield the mth-order deformation equation

$$\mathcal{L}[\psi_m(x,t)-\chi_m\psi_{m-1}(x,t)] = h\mathcal{NFR}(\vec{\psi}_{m-1}(x,t)), \tag{5.6}$$

where

$$\begin{aligned}\mathcal{NFR}(\vec{\psi}_{m-1}(x,t)) =& D_t^{\alpha}\psi_{m-1}-\frac{1}{2}i(\psi_{m-1})_{xx}-i\sum_{j=0}^{m-1}\bar{\psi}_{m-1-j}\sum_{i=0}^{j}\psi_i\psi_{j-i}\\ &-\beta(\psi_{m-1})_{xxx}-6\beta\sum_{j=0}^{m-1}\bar{\psi}_{m-1-j}\sum_{i=0}^{j}\psi_i(\psi_{j-i})_x.\end{aligned} \tag{5.7}$$

Now the solution of Eq.(5.6) for $m\geq 1$ becomes

$$\psi_m(x,t)=\chi_m\psi_{m-1}(x,t)+h\mathcal{L}^{-1}\mathcal{NFR}(\vec{\psi}_{m-1}(x,t)). \tag{5.8}$$

Employing Eqs.(5.1) and (5.8) along with $\psi(x,0)=1-\frac{4}{1+4x^2}\equiv g(x)$, we now successively obtain

$$\psi_0(x,t)=\psi(x,0)=g(x), \tag{5.9}$$

$$\psi_1(x,t)=hD_t^{-\alpha}\left[-\frac{1}{2}i\psi_{0xx}-i|\psi_0|^2\psi_0-\beta\psi_{0xxx}-6\beta|\psi_0|^2\psi_{0x}\right]=hD_t^{-\alpha}B_1, \tag{5.10}$$

where

$$B_1=-\frac{1}{2}ig_{xx}-ig^3-\beta g_{xxx}-6\beta g^2 g_x.$$

Moreover

$$\psi_2(x,t)=h(h+1)D_t^{-\alpha}B_1+h^2D_t^{-2\alpha}B_2=\frac{B_1h(h+1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{B_2h^2}{\Gamma(2\alpha+1)}t^{2\alpha}, \tag{5.11}$$

where

$$B_2=-\frac{1}{2}iB_{1xx}-i\left(\bar{B}_1\psi_0^2+2\psi_0^2B_1\right)-\beta B_{1xxx}-6\beta\left(\bar{B}_1\psi_0\psi_{0x}+\psi_0^2B_{1x}+\psi_0B_1\psi_{0x}\right).$$

Similarly, we have

$$\psi_3(x,t)=\frac{B_1h(h+1)^2}{\Gamma(\alpha+1)}t^{\alpha}+\frac{B_2h^2(h+1)}{\Gamma(2\alpha+1)}t^{2\alpha}+\left(\frac{B_3h^3}{\Gamma(2\alpha+1)}+\frac{B_4h^3}{\Gamma^2(\alpha+1)}\right)\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}, \tag{5.12}$$

where

$$\begin{aligned}B_3&=-\frac{1}{2}iB_{2xx}-i\left(\bar{B}_2\psi_0^2+2\psi_0^2B_2\right)-\beta B_{2xxx}-6\beta\left(\bar{B}_2\psi_0\psi_{0x}+\psi_0^2B_{2x}+\psi_0\psi_{0x}B_2\right),\\ B_4&=-i\left(2\bar{B}_1\psi_0B_1+\psi_0B_1^2\right)-6\beta\left(\bar{B}_1\psi_0B_{1x}+\bar{B}_1A_1\psi_{0x}+\psi_0B_1B_{1x}\right).\end{aligned}$$

In the above terms we substitute $h=-1$, the dominant terms will be remained and the rest terms vanish, because they include factor of $h^m(h+1)^{n+1}, m,n\in\mathbb{N}$,

$$\begin{aligned}&\psi_0(x,t)=g(x),\psi_1(x,t)=-\frac{B_1}{\Gamma(\alpha+1)}t^{\alpha},\psi_2(x,t)=\frac{B_2}{\Gamma(2\alpha+1)}t^{2\alpha},\\ &\psi_3(x,t)=-\left(\frac{B_3}{\Gamma(2\alpha+1)}+\frac{B_4}{\Gamma^2(\alpha+1)}\right)\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha},\end{aligned} \tag{5.13}$$

Thus, the solution of Eq.(5.1) is show as follows:

$$\psi(x,t) = g(x) - \frac{B_1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{B_2}{\Gamma(2\alpha+1)}t^{2\alpha} - \left(\frac{B_3}{\Gamma(2\alpha+1)} + \frac{B_4}{\Gamma^2(\alpha+1)}\right)\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha} + \cdots. \tag{5.14}$$

The expressions of B_1, B_2, B_3 and B_4 can be computed using Maple, thus we successively obtain the 3rd-order approximate solution for Eq.(5.1). To make the paper more briefly, the B_1, B_2, B_3 and B_4 are moved to Appendix. The expressions of reported above depend on the real parameters α and β, we choose $\alpha = 0.8, \beta = 1$ for simplicity, then we have

$$\psi(x,t) = 1 - \frac{4}{1+4x^2} + \frac{C_1}{(1+4x^2)^2}t^{0.8} + \frac{C_2}{(1+4x^2)^3}t^{1.6} - \left(\frac{C_3}{(1+4x^2)^6} - \frac{C_4}{(1+4x^2)^6}\right)t^{2.4} + \cdots, \tag{5.15}$$

where the polynomials C_1, C_2, C_3 and C_4 are given in Appendix. Figure 2 represents $|\psi|$ in a small domain with $\alpha = 0.8, \beta = 1$ since the obtained series solution has a small convergence radius. We can see Fig 2 is similar to Fig 1(b). The obtained rouge wave solutions both possessing the height 3 when x and t equal to 0 and having two symmetrical troughs in two sides.

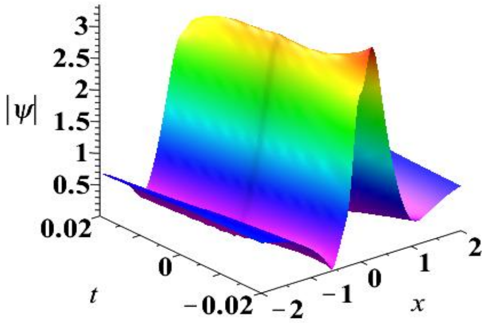

Figure 2: (color online) The surface of the 3rd-order approximate rogue wave solution $\psi(x,t)$ to Eq.(5.1) with $\alpha = 0.8, \beta = 1$ on the (x,t) plane.

6 Conclusions

In this article, we considered the Hirota equation with fractional and integer-order time derivatives respectively, and derived their exact or approximate analytic rogue wave solution using HAM. The illustrative examples suggest that HAM is an efficient and exact method for nonlinear problems in many areas of science and engineering.

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Appendix

$$B_1 = -\frac{16ix^4 - 40ix^2 + 192x\beta - 11i}{(4x^2+1)^2}, \tag{6.1}$$

$$B_2 = \frac{-64x^6 + 272x^4 + 4608i\beta x^3 + 276x^2 - 13824\beta^2x^2 + 1152i\beta x + 51 + 1152\beta^2}{(4x^2+1)^3}, \tag{6.2}$$

$$\begin{aligned} B_3 = \frac{1}{(4x^2+1)^6}[3(&4096ix^{12} + 5308416i\beta^2x^6 + 2654208i\beta^2x^8 - 7077888\beta^3x^7 + 212992\beta x^9 \\ &- 22528ix^{10} - 1437696i\beta^2x^4 - 15925248\beta^3x^5 + 475136\beta x^7 - 15616ix^8 \\ &- 24832ix^6 + 16957440\beta^3x^3 + 374784\beta x^5 - 3456i\beta^2 - 663552i\beta^2x^2 \\ &- 774144\beta^3x + 111616i\beta x^3 - 16528ix^4 - 4120ix^2 + 11072\beta x - 347i)], \end{aligned} \tag{6.3}$$

$$\begin{aligned} B_4 = -\frac{1}{(4x^2+1)^6}(&-22528ix^{10} + 4096ix^{12} - 49152\beta x^9 + 5308416i\beta^2x^6 - 2654208i\beta^2x^4 \\ &- 21233664\beta^3x^5 + 49152\beta x^7 + 7936ix^6 - 995328i\beta^2x^2 + 24772608\beta^3x^3 \\ &+ 202752\beta x^5 - 8848ix^4 + 29440ix^8 - 1327104\beta^3x + 89088\beta x^3 - 3608ix^2 \\ &+ 10560\beta x - 363i). \end{aligned} \tag{6.4}$$

$$C_1 = 1.073671274(16\,ix^4 - 40\,ix^2 - 11\,i + 192\,x), \tag{6.5}$$

$$C_2 = 0.6994843462(-64\,x^6 + 272\,x^4 - 13548\,x^2 + 4608\,ix^3 + 1203 + 1152\,ix), \tag{6.6}$$

$$\begin{aligned} C_3 = 1.006304016(&4096\,ix^{12} - 22528\,ix^{10} + 2638592\,ix^8 + 212992\,x^9 + 5283584\,ix^6 - 6602752\,x^7 \\ &- 1454224\,ix^4 - 15550464\,x^5 - 667672\,ix^2 + 17069056\,x^3 - 3803\,i - 763072\,x), \end{aligned} \tag{6.7}$$

$$\begin{aligned} C_4 = 0.5528058356(&4096\,ix^{12} - 22528\,ix^{10} + 29440\,ix^8 - 49152\,x^9 + 5316352\,ix^6 + 49152\,x^7 \\ &- 2663056\,ix^4 - 21030912\,x^5 - 998936\,ix^2 + 24861696\,x^3 - 363\,i - 1316544\,x). \end{aligned} \tag{6.8}$$