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New Similarity Reductions and Exact Solutions of Generalized Fifth Order KdV Equation with Variable Coefficients

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Abstract: In this work, complete symmetry analysis is presented for variable coefficients version of generalized fifth order KdV equation. A group classification is carried out by finding infinitesimal generators that give one-dimensional optimal system for symmetry algebras. The subalgebras in one-dimensional optimal system are further employed to obtain the form of nonlinear ordinary differential equations and then some exact solutions are yielded in a very systematic manner.

Keywords: Generalized Fifth Order KdV Equation with Time-Dependent Coefficients; Symmetry Analysis; Exact Solutions

1 Introduction

The Generalized fifth order KdV equation reads

$$u_t + auu_{xxx} + bu_xu_{xx} + cu^2u_x + u_{xxxxx} = 0, \tag{1}$$

where a, b and c are real constants. This includes the Lax [1], Swada-Kotera(SK) [2, 3], Kaup-Kupershmidt (KK) [4–6] and Ito equations [7].

As the constants a, b and c take different values, The properties of eq (1) drastically change. For instance, the lax equation with $a = 10$, $b = 20$ and $c = 30$, and The SK equation where $a = b = c = 5$, are completely integrable. These two equations have N-soliton solutions and an infinite set of conservation laws. The KK equation, with $a = 10, b = 25$ and $c = 20$, is also known to be integrable [5] and to have bilinear representations [8], but the explicit form of its N-soliton solution is apparently not known. A fourth equation in this class (1) is the Ito equation, with $a = 3, b = 6$ and $c = 2$, which is not integrable, but has a limited number of conservation laws [7].

The generalized fifth-order KdV (fKdV) equation is a model equation for plasma waves, capillary-gravity water waves and other dispersive phenomena when the cubic KdV-type dispersion is weak. The fKdV equation (1) describes the motion of long waves in shallow water under gravity and in one-dimensional nonlinear lattice. The nonlinear fifth-order KdV is one of the important mathematical model with wide range of applications in quantum mechanics. Li et al.[9], studied the generalized fifth-order KdV model equation with constant coefficients and obtained solitary wave and soliton solutions. Wazwaz [10], found soliton solutions for several forms of fifth-order KdV with constant coefficients by using tanh method. Wazwaz [11], derived soliton solutions of variable coefficients fifth order KdV by using wave ansätz method.

However, in multifarious real physical backgrounds, nonlinear partial differential equations with variable coefficients often provide more powerful and realistic models than their constant coefficient counterparts when the inhomogeneities of media is considered. So it is of great importance to find exact solutions of NLPDEs with variable coefficients and recently, many authors have researched in this direction [12-15]. In this paper, we have considered the variable coefficients version of generalized fifth order KdV equation as

$$u_t + \delta(t)uu_{xxx} + \sigma(t)u_xu_{xx} + \beta(t)u^2u_x + \rho(t)u_{xxxxx} = 0, \tag{2}$$

where $\delta(t), \sigma(t), \beta(t)$ and $\rho(t)$ are arbitrary functions.

In this direction, the layout of the paper is as follows: In Section 2, the classical Lie method is utilized to generate

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various symmetries of the (2) and admissible forms of various coefficients. A number of cases arise depending on the nature of the Lie symmetry generator. Further, the reduced ordinary differential equations (ODEs) corresponding to each member of the optimal system of subalgebras has studied with the aim of furnishing some new explicit solutions of variable coefficient version of general fifth order KdV equation in section 3. Finally, the concluding remarks are given in last section.

2 Lie Symmetries of Variable Coefficient Generalized Fifth Order KdV Equation

In this section, firstly we will present the general procedure for determining symmetries for any system of partial differential equations [16, 17].

2.1 Method of Lie Symmetries

Consider a n^{th} order system of partial differential equations (PDEs) of p independent variables and q dependent variables

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, 2, ...l, \tag{3}$$

involving $x = (x^1, x^2, ..., x^p)$, $u = (u^1, u^2, ..., u^q)$ and the derivatives of u with respect to x upto n, where $u^{(n)}$ represents all the derivatives of u of all the orders from 0 to n. We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system (3)

$$\begin{array}{l} \bar{u^j} = u^j + \epsilon\eta^j(x, u) + O(\epsilon^2), \;\; j = 1, ..., q \\ \bar{x^i} = x^i + \epsilon\xi^i(x, u) + O(\epsilon^2), \;\; i = 1, ..., p, \end{array} \tag{4}$$

where ϵ is the parameter of the transformation and ξ^i and η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator V associated with the above group of transformations can be written as

$$V = \sum_{i=1}^{p} \xi^i(x, u)\partial_{x^i} + \sum_{j=1}^{q} \eta^j(x, u)\partial_{u^j}. \tag{5}$$

The mathematical formulation of the invariance criterion for the system (3) under the infinitesimal transformations leads to the following conditions

$$Pr^{(n)}V[\Delta_\nu(x, u^{(n)})] = 0, \;\; \nu = 1, 2, ..., l, for \Delta_\nu(x, u^{(n)}) = 0, \tag{6}$$

where $Pr^{(n)}$ is called the nth- order prolongation of the infinitesimal generator given by

$$Pr^{(n)}V = V + \sum_{\alpha=1}^{q} \sum_{J} {\phi_\alpha}^J(x, u^{(n)})\partial_{u^\alpha_J}, \tag{7}$$

where $J = (j_1, j_2, ..., J_k)$, $1 \le j_k \le p$, $1 \le k \le n$. The second summation is extended over all derivatives of u^α up to order k. The higher prolongation elements are determined by the infinitesimals ξ^i and ϕ^α through

$${\phi_\alpha}^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}, \tag{8}$$

where $u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i}$ and $u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i}$.

Thus we obtain the infinitesimal generator V, which form a Lie algebra under the usual Lie bracket, by using the infinitesimals ξ^i and ϕ^j and then the original system (3) can be rewritten in terms of group invariants and thus the number of independent variables are reduced.

2.2 Lie Symmetries of Equation (2)

In this subsection, we apply infinitesimal transformations for the construction of solutions of partial differential equations. We will show that infinitesimal criterion for invariance of partial differential equations leads directly to an algorithm to determine infinitesimal generators admitted by given partial differential equations. Invariant surfaces of the corresponding Lie group of point transformations lead to invariant solutions.
As we have discussed in previous subsection, in order to find the Lie point symmetry group of equation (2), all the possible invariance of equation (2) are found under the transformation

$$\{x,t,u\} \longrightarrow \{x,t,u\} + \epsilon\{\xi,\tau,\phi\} \tag{9}$$

where ξ,τ and ϕ are functions of x,t and u and ϵ is an infinitesimal parameter. Substituting equation (9) into equation (2), expanding it to the first order of ϵ and equating the coefficients of the different derivatives of u, the classical symmetries of the variable coefficient fifth order KdV equation are obtained as follows:

$$\begin{aligned} &\xi = c_1x + c_2 \\ &\tau = \frac{\rho(t)}{\int(5\rho(t))dt + k_1}, \\ &\eta = 0, \end{aligned} \tag{10}$$

where c_1, c_2 and k_1 are arbitrary constants and the functions $\beta(t)$, $\sigma(t)$ and $\delta(t)$ are governed by following relations:

$$\begin{aligned} &-\tau\rho'(t)\beta(t) + \tau\beta'(t)\rho(t) + 4\beta(t)\rho(t)c_1 = 0, \\ &\tau\sigma'(t)\rho(t) - \tau\rho'(t)\sigma(t) + 2\sigma(t)\rho(t)c_1 = 0, \\ &\tau\delta'(t)\rho(t) - \tau\rho'(t)\delta(t) + 2\delta(t)\rho(t)c_1 = 0. \end{aligned} \tag{11}$$

The infinitesimal generators of the corresponding Lie algebra are given by

$$X_1 = x\frac{\partial}{\partial x} + \left(\frac{\int(5\rho(t))dt}{\rho(t)}\right)\frac{\partial}{\partial t},\ \ X_2 = \frac{1}{\rho(t)}\frac{\partial}{\partial t},\ \ X_3 = \frac{\partial}{\partial x}. \tag{12}$$

In order to perform symmetry reductions of Eq. (2) in a systematic manner, we need to obtain a classification of the subalgebras of the symmetry algebra into conjugacy classes under the adjoint action of the symmetry group. Because any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for a differential equation. Therefore, a way to determine which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. Because any transformation in the full symmetry group maps a solution to another solution, it suffices to find invariant solutions that are unrelated by transformations in the full symmetry group. This has led to the concept of an optimal system [17, 18]. To obtain the optimal system for Eq. (2), we present the following nonzero Lie brackets of the Lie algebra for (2):

$$[X_1, X_2] = -[X_2, X_1] = -5X_2,\ \ [X_1, X_3] = -[X_3, X_1] = -X_3. \tag{13}$$

A one-dimensional optimal system for Eq. (2) is given as follows:

$$\langle X_1,\ X_2 + X_3,\ X_2 - X_3,\ X_2,\ X_3\rangle. \tag{14}$$

Because the discrete symmetry $(x,t,u) \to (-x,t,u)$ will map $X_2 + X_3$ into $X_2 - X_3$, thus in the optimal system, reductions and exact solutions of remaining four essential vector fields of the optimal system are considered. It seems reasonable now to construct Lie ansätze and to seek exact solutions of the nonlinear Eq. (2). With this in mind, consider its Lie symmetry generated by the basic operators in the optimal system. According to the general procedure it is necessary to solve the characteristic equations

$$\frac{du}{\eta} = \frac{dx}{\xi} = \frac{dt}{\tau}. \tag{15}$$

On solving the Eq. (15) for the various operators in the optimal system, we obtain a set of non-equivalent Lie ansätze for the functions u. In Table (I), we have listed the Lie ansätze for all the essential fields comprising the optimal system and also the coefficient functions of the Eq. (2). Using the ansätze mentioned in Table (I), we reduced the nonlinear equation (2) to ordinary differential equation (ODE). Having exact solutions of ODE and using the relevant ansätze one obtains the solutions of the nonlinear equation (2). Let us now concentrate on each of the four essential fields as listed in Table (I).

3 Symmetry Reductions and Exact Solutions

In the following we consider, corresponding to each element in the optimal system of sub algebras, the reductions of Eq. (2) into ODE in terms of similarity variable ζ and the new dependent variables F obtained using the auxiliary equation (15) .

Table 1: Similarity Variables, Similarity Functions and the Coefficient Functions of Modified Kawahara Equation

Essential Vector Fields	Similarity Variables	Similarity Forms	Coefficient Functions
X_1	$x(\int 5\rho(t)dt)^{-\frac{1}{5}}$	$F(\zeta)$	$\beta(t)=\frac{\rho(t)k_2}{(\int 5\rho(t)dt)^{\frac{4}{5}}}, \sigma(t)=\frac{\rho(t)k_3}{(\int 5\rho(t)dt)^{\frac{2}{5}}}$ $\sigma(t)=\frac{\rho(t)k_4}{(\int 5\rho(t)dt)^{\frac{2}{5}}}$
X_2+X_3	$x-\int \rho(t)dt$	$F(\zeta)$	$\beta(t)=k_2\rho(t), \sigma(t)=k_3\rho(t),$ $\delta(t)=k_4\rho(t)$
X_2	x	$F(\zeta)$	$\beta(t)=k_2\rho(t), \sigma(t)=k_3\rho(t),$ $\delta(t)=k_4\rho(t)$
X_3	t	$F(\zeta)$	$\beta(t)=k_2\rho(t), \sigma(t)=k_3\rho(t),$ $\delta(t)=k_4\rho(t)$

where k_2, k_3 and k_4 are arbitrary constants and $\rho(t)$ is an arbitrary function of t.

(i)Vector Field X_1
Using the similarity variable, the forms of the similarity solution and the coefficient functions, Eq. (2) is reduced to the following ODE:

$$F''''' + k_4 F F''' + k_3 F' F'' + k_2 F^2 F' - \zeta F' = 0. \tag{16}$$

where prime $(')$ denotes the differentiation with respect to the variable ζ.

(ii)Vector Field X_2+X_3
Substituting the forms of the similarity solution and the coefficient functions, corresponding to this vector field, Eq. (4) yields the following ODE:

$$F''''' + k_4 F F''' + k_3 F' F'' + k_2 F^2 F' - F' = 0. \tag{17}$$

The tangent hyperbolic method is employed to obtain particular analytic solutions to Eq. (17). On balancing the highest order derivative terms with the nonlinear terms, one can easily find that the solution shall have the form as given by

$$F(\zeta) = a_0 + a_1 \tanh(\zeta) + a_2 \tanh^2(\zeta), \tag{18}$$

where a_0, a_1 and a_2 are constants to be determined. Substituting these forms in Eq. (17) and performing the algebraic calculations on the relations obtained among various parameters, we get

$$a_1 = 0,\ \ a_2 = -\frac{3}{2}a_0,\ \ k_4 = \frac{9}{a_0},\ \ k_3 = \frac{25}{a_0},\ \ k_3 = \frac{12}{{a_0}^2}$$

and

$$a_1 = 0,\ \ a_2 = -\frac{9}{8}a_0,\ \ k_4 = \frac{36}{a_0},\ \ k_3 = \frac{52}{3a_0},\ \ k_3 = \frac{192}{{a_0}^2}.$$

Then reverting back to the original variables, we obtain the following group-invariant solutions of Eq. (2):

$$u(x,t) = a_0 + a_2(\tanh(x - \int \rho(t)dt))^2, \tag{19}$$

where a_0 is arbitrary constant.

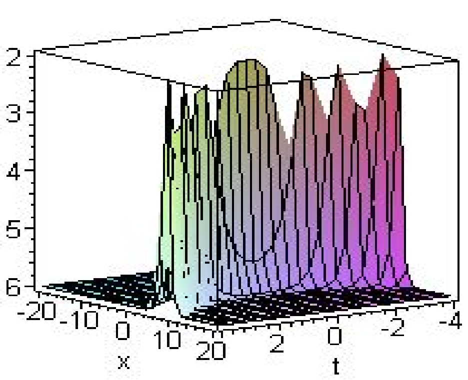

Figure 1: Graphical representation of solution (19), for $a_0 = 2, a_2 = 4$ and $\rho(t) = t$

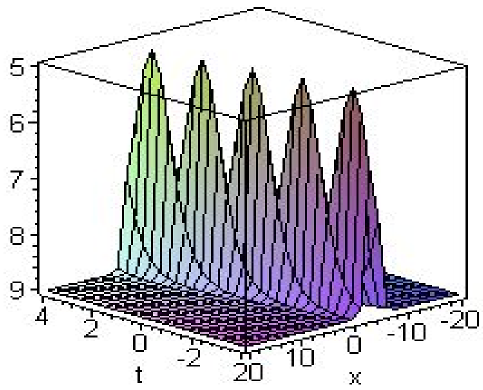

Figure 2: Graphical representation of solution (19), for $a_0 = 5, a_2 = 4$ and $\rho(t) = 1$

(iii)Vector Field X_2

For this vector field, Eq. (4) is transformed into the following fifth order nonlinear ODE:

$$F''''' + k_4 F F''' + k_3 F' F'' + k_2 F^2 F' = 0. \tag{20}$$

In order to find the solution of Eq. (20), a special solution in the form

$$F(\zeta) = a_0 + a_1 \tanh(\zeta) + a_2 \tanh^2(\zeta) \tag{21}$$

is considered. Substitution of these expressions for $F(\zeta)$ in Eq. (20) leads to the following relations among the various parameters involved.

$$a_1 = 0,\ \ k_4 = -\frac{2}{27}\frac{(k_2 a_2^2 + 167)}{a_2},\ \ k_3 = -\frac{1}{54}\frac{(k_2 a_2^2 + 1944)}{a_2}, \tag{22}$$

with a_2, k_2 are arbitrary constants.

Corresponding solution of Eq. (4) can be expressed as follows:

$$u(x,t) = -\frac{2}{3}a_2 + a_2 \tanh^2 x. \tag{23}$$

Also, we get other set of expressions as follow:

$$\begin{array}{l} a_1 = 0,\ \ k_4 = -\frac{8a_2^2}{a_0(a_0-a_2)(a_0+a_2)},\ \ k_3 = -\frac{4(-2a_2^3+15a_0^3-15a_0a_2^2)}{a_0a_2(a_0-a_2)(a_0+a_2)}, \\ k_2 = \frac{48a_2}{a_0(a_0-a_2)(a_0+a_2)}, \end{array}$$

with a_0, a_2 are arbitrary constants.

Then the solution of Eq. (4) becomes

$$u(x,t) = a_0 + a_2 \tanh^2 x. \tag{24}$$

(iv)Vector Field X_3

Corresponding to this vector field, we obtain the trivial solution of Eq. (4) is $u(x,t) = c$, where c is an arbitrary constant.

4 Concluding Remarks

In the present paper, generalized fifth order KdV equation has been studied with of obtaining some new exact solutions. After obtaining the point symmetries admitted by equation under consideration, a formal approach of identifying an optimal system of Lie sub algebras has been adopted with help of the adjoint action of the Lie algebra. The basic generators contained in the optimal system have then been exploited to achieve the desired reduction of PDEs to ODEs. The resulting ODEs have been examined subsequently for various types of exact solutions via some techniques which are essentially based on some special functions. We should notice that the obtained solutions in this paper play a crucial role in numerical simulation and the understanding of solitons dynamics of variants of the studied equation where they facilitate the verification of numerical solvers. For some special values of the parameters involved in the various solutions being reported, the graphs are plotted to render some geometrical appreciations of them.

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