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Exact Solutions of Nonlinear Evolution Equations by Using Modified Simple Equation Method

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Abstract: In this study, the modified simple equation method (MSE) is used to construct exact solutions of the Foam Drainage and Klein-Gordon-Zakharov (KGZ) equations in mathematical physics. The exact solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive. Also we can see that when the parameters are assigned special values, solitary wave solutions can be obtained from the exact solutions. All calculations in this paper have been made with the aid of the Maple packet program.

Keywords: exact solutions; modified simple equation method; Foam Drainage equation; Klein-Gordon-Zakharov (KGZ) equations

1 Introduction

In the area of nonlinear science, nonlinear evolution equations (NLEEs) are often introduced to define the motion of isolated waves, localized in a small part of space, in many fields such as hydrodynamics, nonlinear optics, plasma physics, optical fibers, biology, chemical kinematics, solid state physics, chemical physics, etc. Especially, obtaining their explicit solutions is even more difficult. Therefore, the researchers realized a huge amount of research work to discover the exact traveling wave solutions of nonlinear physical phenomena. They have developed many practical methods and techniques, such as homogeneous balance method [15], Hirota's bilinear transformation method [20], inverse scattering method [2, 23, 30], tanh-function method [26, 32], extended tanh method [16, 34], Exp-function method [18, 36], sine-cosine method [5, 33], functional variable method [9, 37], $(\frac{G'}{G})$-expansion method [1, 6, 27, 31], modified simple equation method [22, 24, 35], first integral method [3, 14, 17, 21], extended Jacobi's elliptic function method [10], sub-equation method [19], modified extended direct algebraic method [28], differential transform method [8], and so on [4].

The aim of this paper is to present new solutions of some nonlinear evolution equations using modified simple equation method (MSE). In section 2, we describe the proposed method. In section 3, the exact solutions of Foam Drainage equation and Klein-Gordon-Zakharov (KGZ) equations are presented. In Section 4, result and discussion are given.

2 The modified simple equation method

Step 1. Take a general nonlinear PDE in the form

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0. \tag{1}$$

Employing a wave variable $\xi = x - ct$, Eq.(1) can be rewritten as nonlinear ODE

$$Q(u, u', u'', u''', ...) = 0. \tag{2}$$

,where the prime denotes the derivation with respect to ξ. Eq.(2) is then integrated as many times as possible and setting the constant of integration to be zero.

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Step 2. The solution of Eq.(2) can be expressed by a polynomial in $(\frac{\psi^{'}(\xi)}{\psi(\xi)})$ i.e.

$$u(\xi) = \sum_{k=0}^{m} a_k \left[\frac{\psi^{'}(\xi)}{\psi(\xi)}\right]^k \tag{3}$$

where $a_k\,(k = 0, 1, 2, ..., m)$ are arbitrary constants to be determined such that $a_m \neq 0$, and $\psi(\xi)$ is an unknown function to be determined later. In the tanh-function method, $(\frac{G'}{G})$-expansion method, Exp-function method, etc., the solution is represented in the terms of some pre-defined functions, but in the modified simple method , ψ is not pre-defined or not a solution of any pre-defined equation. Therefore, some fresh solutions may be found by this method. This is the difference of the this method [24, 25, 29].

Step 3. The positive integer m can be determined by considering the homogeneous balance between the highest order derivative term with the highest order nonlinear term appearing in Eq.(2).

Step 4. Substituting Eq.(3) into Eq.(2) As a result of this substitution, a polynomial of $\psi^{-j}(\xi)$ is verified with the derivatives of $\psi(\xi)$. We equate all the coefficients of $\psi^{-j}(\xi)$ to zero, where $j \geq 0$. This operation yields a system which can be solved to find $a_k\,(k = 0, 1, 2, ..., m)$ and $\psi(\xi)$. Substituting the values of a_k and $\psi(\xi)$ into Eq.(3) completes the determination of the solution of Eq.(1)

3 Applications

In this section, two examples are presented to illustrate the applicability of the modified simple equation method to solve nonlinear evolution equations.

3.1 Foam Drainage Equation

We first consider the Foam Drainage equation [12]:

$$u_t + \left(u^2 - \frac{\sqrt{u}}{2} u_x\right)_x = 0. \tag{4}$$

Using the wave variable $\xi = x - ct$ in Eq.(4), we obtain the following ODE:

$$-cu^{'} + \left(u^2 - \frac{\sqrt{u}}{2} u^{'}\right)^{'} = 0. \tag{5}$$

Integrating Eq.(5) once, with respect to ξ yields:

$$-cu + \left(u^2 - \frac{\sqrt{u}}{2} u^{'}\right) = 0. \tag{6}$$

Using the transformation

$$u(\xi) = v^2(\xi), \tag{7}$$

we get:

$$-cv^2 + v^4 - v^2 v^{'} = 0 \tag{8}$$

or

$$-c + v^2 - v^{'} = 0, \tag{9}$$

where the prime denotes differentiation with respect to ξ. Balancing the highest order derivative term $v^{'}$ with with the nonlinear term v^2 in (9) gives $m = 1$. Therefore, the solution (3) takes the form,

$$v(\xi) = a_0 + a_1 \left(\frac{\psi^{'}}{\psi}\right), \tag{10}$$

where a_0 and a_1 are constants but $a_1 \neq 0$, and $\psi(\xi)$ is an unknown function to be determined. Now, it is easy to find,

$$v^{'} = a_1 \left[\frac{\psi^{''}}{\psi} - \left(\frac{\psi^{'}}{\psi}\right)^2\right] \tag{11}$$

$$v^2 = a_0^2 + 2a_0a_1\left(\frac{\psi^{'}}{\psi}\right) + a_1^2\left(\frac{\psi^{'}}{\psi}\right)^2. \tag{12}$$

Substituting Eqs.(11)-(12) into Eq.(9) and then equating the coefficients of $\psi^0, \psi^{-1}, \psi^{-2}$ to zero, we separately obtain:

$$\psi^0 : -c + a_0^2 = 0, \tag{13}$$

$$\psi^{-1} : 2a_0a_1\psi^{'} - a_1\psi^{''} = 0, \tag{14}$$

$$\psi^{-2} : a_1^2\left(\psi^{'}\right)^2 + a_1\left(\psi^{'}\right)^2 = 0. \tag{15}$$

From (13) ,we obtain

$$a_0 = \pm\sqrt{c} \tag{16}$$

and from Eq.(15)

$$a_1 = -1, \text{ since } a_1 \neq 0. \tag{17}$$

is verified. From (17) Eq.(14) yields

$$2a_0\psi^{'} = \psi^{''}. \tag{18}$$

When we solved (18) , we find

$$\psi(\xi) = c_1 + c_2e^{2a_0\xi}. \tag{19}$$

Case 1: When $a_0 = \sqrt{c}$,

$$v(\xi) = \sqrt{c} - \frac{2c_2\sqrt{c}e^{2\sqrt{c}\xi}}{c_1 + c_2e^{2\sqrt{c}\xi}}, \tag{20}$$

where $\xi = x - ct$. Hence, the exact solution of Eq.(4) ;

$$u(\xi) = \left(\sqrt{c} - \frac{2c_2\sqrt{c}e^{2\sqrt{c}\xi}}{c_1 + c_2e^{2\sqrt{c}\xi}}\right)^2, \tag{21}$$

where $\xi = x - ct$.

Case 2: When $a_0 = -\sqrt{c}$;

$$u(\xi) = \left(-\sqrt{c} + \frac{2c_2\sqrt{c}e^{-2\sqrt{c}\xi}}{c_1 + c_2e^{-2\sqrt{c}\xi}}\right)^2, \tag{22}$$

where $\xi = x - ct$.

Note that these solutions are quite different from the travelling wave solutions found in [7, 12].

3.2 Klein–Gordon-Zakharov (KGZ) Equations

Let us secondly consider the KGZ equations

$$\begin{array}{c} u_{tt} - u_{xx} + u + uv + |u|^2 u = 0 \\ v_{tt} - v_{xx} = |u|_{xx}. \end{array} \tag{23}$$

This system describes interaction between Langmuir waves and ion sound waves. These are apparently coupled equations by two functions $u(x,t)$ and $v(x,t)$ where the function $u(x,t)$ is complex and denotes the fast time scale component of electric field raised by electrons and the function $v(x,t)$ is real and denotes the derivation of ion density from its equilibrium [11]. Applying the transformation

$$u(x,t) = u(\xi),\ \xi = x - ct \tag{24}$$

and substituting the Eqs.(24) into Eqs.(23) yields

$$\begin{array}{l} (c^2-1)u'' + u + uv + |u|^2 u = 0 \\ (c^2-1)v'' = (|u|^2)'' \end{array} \tag{25}$$

,where the prime denotes the derivation with respect to ξ. Integrating the second equation and for simplicity taking integration constant to zero, we obtain

$$v = \frac{|u|^2}{c^2-1}. \tag{26}$$

Then we substitute Eq.(26) in the first equation of Eq.(25), we provide

$$(c^2-1)^2u'' + (c^2-1)u + c^2u^3 = 0. \tag{27}$$

From Eq.(27) balancing term is $m = 1$. Therefore, the solution (3) takes the form,

$$u = a_0 + a_1(\frac{\psi'}{\psi}). \tag{28}$$

Here a_0, a_1 are constants such that $a_1 \neq 0$, and $\psi(\xi)$ is an unknown function to be determined. It is simple to compute that,

$$u'' = a_1\frac{\psi'''}{\psi} - 3a_1\frac{\psi''\psi'}{\psi^2} + 2a_1\left(\frac{\psi'}{\psi}\right)^3 \tag{29}$$

$$u^3 = a_1^3\left(\frac{\psi'}{\psi}\right)^3 + 3a_1^2a_0\left(\frac{\psi'}{\psi}\right)^2 + 3a_1a_0^2\left(\frac{\psi'}{\psi}\right) + a_0^3. \tag{30}$$

Substituting the values of u, u'', u^3 into Eq.(26) and then equating the coefficients of $\psi^0, \psi^{-1}, \psi^{-2}, \psi^{-3}$ to zero, we separately obtain:

$$\psi^0 : c^2a_0(a_0^2+1) - a_0 = 0 \tag{31}$$

$$\psi^{-1} : a_1(1+c^4-2c^2)\psi''' + a_1(c^2-1+3c^2a_0^2)\psi' = 0 \tag{32}$$

$$\psi^{-2} : \psi''\psi'(6c^2a_1 - 3c^4a_1 - 3a_1) = 0 \tag{33}$$

$$\psi^{-3} : (\psi')^3(2a_1 - 4c^2 + 2c^4a_1 + c^2a_1^3) = 0. \tag{34}$$

From Eq.(31) , we get

$$a_0 = 0, \pm\frac{\sqrt{-c^2+1}}{c}. \tag{35}$$

From Eq.(34) , we get

$$a_1 = \pm\frac{\sqrt{2}(c^2-1)i}{c}, a_1 \neq 0. \tag{36}$$

When solved ODE system (32)-(33), we have

$$\psi(\xi) = c_1 + c_2e^{\frac{\sqrt{-c^2+1}\sqrt{2}\xi i}{c^2-1}}. \tag{37}$$

Case 1: When $a_0 = \frac{\sqrt{-c^2+1}}{c}$ and $a_1 = \frac{\sqrt{2}(c^2-1)i}{c}$, the solution of Eq.(23) is

$$u(\xi) = \frac{\sqrt{-c^2+1}}{c} - \frac{2c_2(c^2-1)\sqrt{-c^2+1}e^{\frac{\sqrt{-c^2+1}\sqrt{2}\xi i}{c^2-1}}}{c(c^2-1)(c_1+c_2e^{\frac{\sqrt{-c^2+1}\sqrt{2}\xi i}{c^2-1}})} \tag{38}$$

and

$$v(\xi) = \frac{(-c_1+c_2e^{\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})(-c_1+c_2e^{-\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})}{c^2(c_1+c_2e^{\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})(c_1+c_2e^{-\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})}, \tag{39}$$

where $\xi = x - ct$.

Case 2: When $a_0 = -\frac{\sqrt{-c^2+1}}{c}$ and $a_1 = -\frac{\sqrt{2}(c^2-1)i}{c}$, the solution of Eq.(23) is

$$u(\xi) = -\frac{\sqrt{-c^2+1}}{c} + \frac{2c_2(c^2-1)\sqrt{-c^2+1}e^{\frac{\sqrt{-c^2+1}\sqrt{2}\xi i}{c^2-1}}}{c(c^2-1)(c_1+c_2e^{\frac{\sqrt{-c^2+1}\sqrt{2}\xi i}{c^2-1}})} \tag{40}$$

and

$$v(\xi)=-\frac{(-c_1+c_2e^{\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})(-c_1+c_2e^{-\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})}{c^2(c_1+c_2e^{\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})(c_1+c_2e^{-\frac{\sqrt{2}\xi i}{\sqrt{-c^2+1}}})}, \tag{41}$$

where $\xi = x - ct$.

Note that our solutions are different from the given ones in [7, 11, 13].

Since our solutions contains arbitrary constants, they are more general than in [7]. For different choices of c_1 and c_2, similiar forms of traveling wave solutions of the KGZ equations can also be obtained [11]. Moreover, while our solutions contains exponential functions, Ebadi's solutions contains Jacobi elliptic functions. As $m \to 0$ or $m \to 1$ the solutions are recovered from in Eqs.$(40) - (41)$

Since the obtained solutions are consist of exponential functions, they can be transformed to periodic, hyberbolic and solitary solutions. These solutions may be important of significance for the explanation of some practical physical problems.

4 Result and discussion

The modified simple equation method (MSE) has been succesfully used to set up new solutions. This method is reliable and effective and gives more solutions. We foresee that our results can be found potentially useful for applications in mathematical physics and engineering. So, we dealed with method can be extended to solve many systems of nonlinear partial differential equations which are arising in the theory of solitons and other areas such as physics, biology, chemistry, engineering. This is our task in the future.

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