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Inclusion Relationships Results for Certain Subclasses of Meromorphic Functions Associated with Linear Operator

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Abstract: In this paper we investigate a family of integral operators defined on the space of univalent meromorphic functions. We introduce and investigate several new subclasses of meromorphic starlike functions of order γ, meromorphic convex functions of order γ, meromorphic close-to-convex functions of order η and type γ and meromorphic quasi-convex functions of order η and type γ. Several interesting integral-preserving properties are also considered.

Keywords: meromorphic functions; hadamard product; generalized hypergeometric function; differential subordination

1 Introduction

Let Σ be the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. For functions $f(z) \in \Sigma$ given by (1.1) and $g(z) \in \Sigma$ defined by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \sum$ is said to be in the class $MS^*(\gamma)$ of meromorphic starlike functions of order γ in U if and only if (see Clunie [4], Aouf [1,2], Pommerenke [13] and Juneja and Reddy [9])

$$\Re e \left(\frac{zf'(z)}{f(z)} \right) < -\gamma \quad (z \in U;\ 0 \leq \gamma < 1)\,. \tag{1.4}$$

Also, a function $f \in \sum$ is said to be in the class $MC(\gamma)$ of meromorphic convex functions of order γ in U if and only if (see Miller [11])

$$\Re e \left(1 + \frac{zf''(z)}{f'(z)} \right) < -\gamma \quad (z \in U;\ 0 \leq \gamma < 1)\,. \tag{1.5}$$

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It is easy to observe from (1.4) and (1.5) that

$$f(z) \in MC(\gamma) \Leftrightarrow -zf^{'}(z) \in MS^{*}(\gamma). \tag{1.6}$$

For a function $f \in \sum$, we say that $f \in MK(\eta,\gamma)$ if there exists a function $g \in MS^{*}(\gamma)$ such that

$$\Re e\left(\frac{zf^{'}(z)}{g(z)}\right) < -\eta \quad (z \in U;\ 0 \leq \gamma,\eta < 1)\,. \tag{1.7}$$

Functions in the class $MK(\eta,\gamma)$ are called meromorphic close-to-convex functions of order η and type γ. We also say that a function $f \in \sum$ is in the class $MK^{*}(\eta,\gamma)$ of meromorphic quasi-convex functions of order η and type γ if there exists a function $g \in MC(\gamma)$ such that

$$\Re e\left(\frac{\left(zf^{'}(z)\right)^{'}}{g^{'}(z)}\right) < -\eta \quad (z \in U;\ 0 \leq \gamma,\eta < 1)\,. \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$f(z) \in MK^{*}(\eta,\gamma) \Leftrightarrow -zf^{'}(z) \in MK(\eta,\gamma). \tag{1.9}$$

For a function $f \in \sum$, we define a linear operator $I^{m}(\lambda,\ell)$ as follows:

$$I^{m}(\lambda,\ell)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\left[\frac{\ell+\lambda(k+1)}{\ell}\right]^{m} a_k z^k$$

$$(\lambda \geq 0; \ell > 0; m \in \mathbb{N}_0; z \in U^{*}). \tag{1.10}$$

The operator $I^{m}(\lambda,\ell)$ was introduced and studied by El-Ashwah [5, with $p = 1$]. By setting

$$I_{\lambda}^{m,\ell}(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\left[\frac{\ell+\lambda(k+1)}{\ell}\right]^{m} z^k, \tag{1.11}$$

we define a new operator $I_{q,s,\lambda}^{m,\ell}(\alpha_1)$ in terms of the Hadamard product (or convolution) as follows:

$$I_{\lambda}^{m,\ell}(z) * I_{q,s,\lambda}^{m,\ell}(\alpha_1)(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\Gamma_{k+1}(\alpha_1)z^k \quad (\lambda \geq 0; \ell > 0; m \in \mathbb{N}_0; q \leq s+1; q,s \in \mathbb{N}_0$$

$$\alpha_i \in \mathbb{C},\ i = 1,2,...q;\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^{-} = \{0,-1,-2,...\},\ j = 1,2,...s;\ z \in U^{*}\big)\,, \tag{1.12}$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{1}{k!}. \tag{1.13}$$

and $(\theta)_{\nu}$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_{\nu} = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^{*} = \mathbb{C}\backslash\{0\}) \\ \theta(\theta+1)...(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{1.14}$$

Then

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z) = I_{q,s,\lambda}^{m,\ell}(\alpha_1)(z) * f(z). \tag{1.15}$$

By using (1.11),(1.12) and (1.15), we have

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+1)}\right]^{m}\Gamma_{k+1}(\alpha_1)a_k z^k \ \ (z \in U^{*})\,. \tag{1.16}$$

It is readily verified from (1.16) that

$$z(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z))^{'} = \alpha_1 I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z) - (\alpha_1+1)I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z), \tag{1.17}$$

and

$$\lambda z\left(I_{q,s,\lambda}^{m+1,\ell}(\alpha_1)f(z)\right)^{'}=\ell I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)-(\ell+\lambda)\,I_{q,s,\lambda}^{m+1,\ell}(\alpha_1)f(z). \tag{1.18}$$

By specializing the parameters $\lambda,\ \ell$ and putting $s=1,\ q=2$ and $\alpha_1=\alpha_2=\beta_1=1$, we obtain the following operators:

(i) $I_{2,1,\lambda}^{m,\ell}(1)f(z)=\mathcal{L}^m(\lambda,\ell)f(z)$ (see El-Ashwah [6] and El-Ashwah [7, with $p=1$]);

(ii) $I_{2,1,1}^{m,1}(1)f(z)=P^m(\lambda,\ell)f(z)$ (see Aqlan et al.[3, with $p=1$]);

(iii) $I_{2,1,1}^{m,\ell}(1)f(z)=P_\ell^m f(z)$ (see Lashin [10]).

Also we note that:

(i) $I_{q,s,1}^{m,\ell}(\alpha_1)f(z)=I^{m,\ell}(q,s,\alpha_1)f(z)$, where the operator $I^{m,\ell}(q,s,\alpha_1)f(z)$ is defined by

$$I^{m,\ell}(q,s,\alpha_1)f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\left[\frac{\ell}{k+\ell+1}\right]^m\Gamma_{k+1}(\alpha_1)a_k z^k\ (\ell>0;m\in\mathbb{N}_0;z\in U^*);$$

(ii) $I_{q,s,\lambda}^{m,1}(\alpha_1)f(z)=I_\lambda^m(q,s,\alpha_1)f(z)$, where the operator $I_\lambda^m(q,s,\alpha_1)f(z)$ is defined by

$$I_\lambda^m(q,s,\alpha_1)f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\left[1+\lambda(k+1)\right]^{-m}\Gamma_{k+1}(\alpha_1)a_k z^k\ (\lambda\geq 0;m\in\mathbb{N}_0;z\in U^*);$$

(iii) $I_{q,s,1}^{m,1}(\alpha_1)f(z)=I^m(q,s,\alpha_1)f(z)$, where the operator $I^m(q,s,\alpha_1)f(z)$ is defined by

$$I^m(q,s,\alpha_1)f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}(k+2)^{-m}\,\Gamma_{k+1}(\alpha_1)a_k z^k\ (m\in\mathbb{N}_0;z\in U^*).$$

We now define the following subclasses of the meromorphic function class $\sum$ by means of the linear operator $I_{q,s,\lambda}^{m,\ell}(\alpha_1)$ given by (1.16).

Definition 1 *In conjunction with* (1.4) *and* (1.16),

$$MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)=\left\{f(z)\in\sum:\ I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\in MS^*(\gamma),\ 0\leq\gamma<1\right\},$$

Definition 2 *In conjunction with* (1.5) *and* (1.16),

$$MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)=\left\{f(z)\in\sum:\ I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\in MC(\gamma),\ 0\leq\gamma<1\right\},$$

Definition 3 *In conjunction with* (1.7) *and* (1.16),

$$MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma)=\left\{f(z)\in\sum:\ I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\in MK(\eta,\gamma),\ 0\leq\gamma,\eta<1\right\},$$

Definition 4 *In conjunction with* (1.8) *and* (1.16),

$$MK_{q,s,\alpha_1}^{*,m,\ell,\lambda}(\eta,\gamma)=\left\{f(z)\in\sum:\ I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\in MK^*(\eta,\gamma),\ 0\leq\gamma,\eta<1\right\},$$

Obviously, we know that

$$f(z)\in MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)\Leftrightarrow -zf^{'}(z)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma), \tag{1.19}$$

and

$$f(z)\in MK_{q,s,\alpha_1}^{*,m,\ell,\lambda}(\eta,\gamma)\Leftrightarrow -zf^{'}(z)\in MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma). \tag{1.20}$$

Lemma 1 (12) *Let* $\Phi(u,v)$ *be complex valued function,* $\Phi:D\to\mathbb{C},\ D\subset\mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ *is the complex plane*) *and let* $u=u_1+iu_2,\ v=v_1+iv_2$. *Suppose that* $\Phi(u,v)$ *satisfies the following conditions:*

(i) $\Phi(u,v)$ *is continuous in* D;

(ii) $(1,0)\in D$ *and* $\Re e\left\{\Phi(1,0)\right\}>0$;

(iii) $\Re e\left\{\Phi(iu_2,v_1)\right\}\leq 0$ *for all* $(iu_2,v_1)\in D$ *and such that* $v_1\leq-\dfrac{(1+u_2^2)}{2}$.

Let

$$q(z) = 1 + q_1 z + q_2 z^2 + ... \tag{1.21}$$

be regular in the unit disc U *such that* $(q(z), zq^{'}(z)) \in D$ *for all* $z \in U$*. If*

$$\Re e \left\{ \Phi(q(z), zq^{'}(z)) \right\} > 0 \quad (z \in U),$$

then

$$\Re e \left\{ q(z) \right\} > 0 \quad (z \in U).$$

Lemma 2 *(Jack [8])Suppose* $w(z)$ *be a nonconstant analytic function in* U *with* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value at a point* $z_0 \in U$ *on the circle* $|z| = r < 1$*, then* $z_0 w^{'}(z_0) = \zeta w(z_0)$*, where* $\zeta, \zeta \geq 1$ *is some real number.*

2 The main inclusion relationships

Unless otherwise mentioned, we assume throughout this paper that :
$\lambda \geq 0;\ \ell > 0\ q, s \in \mathbb{N},\ q \leq s + 1$ and $m \in \mathbb{N}_0$.

In this section, we give several inclusion relationships for meromorphic function classes, which are associated with the linear operator $I^{m,\ell}_{q,s,\lambda}(\alpha_1)$.

Theorem 3 *Let* $0 \leq \gamma < 1$*. Then*

$$MS^{m,\ell,\lambda}_{q,s,\alpha_1+1}(\gamma) \subset MS^{m,\ell,\lambda}_{q,s,\alpha_1}(\gamma) \subset MS^{m+1,\ell,\lambda}_{q,s,\alpha_1}(\gamma). \tag{2.1}$$

Proof. We first prove that

$$MS^{m,\ell,\lambda}_{q,s,\alpha_{1+1}}(\gamma) \subset MS^{m,\ell,\lambda}_{q,s,\alpha_1}(\gamma). \tag{2.2}$$

Let $f(z) \in MS^{m,\ell,\lambda}_{q,s,\alpha_1+1}(\gamma)$ and set

$$\frac{z\left(I^{m,\ell}_{q,s,\lambda}(\alpha_1)f(z)\right)^{'}}{I^{m,\ell}_{q,s,\lambda}(\alpha_1)f(z)} = -\gamma - (1-\gamma)q(z), \tag{2.3}$$

where $q(z)$ is given by (1.21). By using identity (1.17), we obtain

$$\alpha_1 \frac{I^{m,\ell}_{q,s,\lambda}(\alpha_1+1)f(z)}{I^{m,\ell}_{q,s,\lambda}(\alpha_1)f(z)} = -\gamma - (1-\gamma)q(z) + (\alpha_1+1). \tag{2.4}$$

Differentiating (2.4) logarithmically with respect to z, we have

$$\begin{aligned}\frac{z\left(I^{m,\ell}_{q,s,\lambda}(\alpha_1+1)f(z)\right)^{'}}{I^{m,\ell}_{q,s,\lambda}(\alpha_1+1)f(z)} &= \frac{z\left(I^{m,\ell}_{q,s,\lambda}(\alpha_1)f(z)\right)^{'}}{I^{m,\ell}_{q,s,\lambda}(\alpha_1)f(z)} + \frac{(1-\gamma)zq^{'}(z)}{(1-\gamma)q(z)+\gamma-(\alpha_1+1)} \\ &= -\gamma - (1-\gamma)q(z) + \frac{(1-\gamma)zq^{'}(z)}{(1-\gamma)q(z)+\gamma-(\alpha_1+1)}.\end{aligned}$$

Let

$$\Phi(u, v) = (1-\gamma)u - \frac{(1-\gamma)v}{(1-\gamma)u+\gamma-(\alpha_1+1)}$$

with $u = q(z) = u_1 + iu_2$ and $v = zq^{'}(z) = v_1 + iv_2$. Then
(i) $\Phi(u, v)$ *is continuous in* $D = \left(\mathbb{C} \backslash \frac{1+\alpha_1-\gamma}{1-\gamma}\right) \times \mathbb{C}$;
(ii) $(1, 0) \in D$ *and* $\Re e \left\{\Phi(1, 0)\right\} = 1 - \gamma > 0$;

(iii) for all $(iu_2, v_1) \in D$ *and such that* $v_1 \leq -\dfrac{(1+u_2^2)}{2}$, we have

$$\Re e\{\Phi(iu_2,v_1)\} = \Re e\left\{-\frac{(1-\gamma)v_1}{(1-\gamma)iu_2+\gamma-(\alpha_1+1)}\right\} = \frac{(1-\gamma)\left[\alpha_1+1-\gamma\right]v_1}{(1-\gamma)^2u_2^2+(\gamma-\alpha_1-1)^2}$$
$$\leq -\frac{(1-\gamma)\left(\alpha_1+1-\gamma\right)\left(1+u_2^2\right)}{2\left[(1-\gamma)^2u_2^2+(\gamma-\alpha_1-1)^2\right]} < 0,$$

which shows that $\Phi(u,v)$ satisfies the hypotheses of Lemma 1. Consequently, we easily obtain the inclusion relationship (2.2).

By using arguments similar to those detailed above, together with (1.18) and $\Phi(u,v)$ *is continuous in* $D = \left(\mathbb{C}\backslash\frac{1+\frac{\ell}{\lambda}-\gamma}{1-\gamma}\right)\times\mathbb{C}$, we can also prove the right part of Theorem 1, that is, that

$$MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma) \subset MS_{q,s,\alpha_1}^{m+1,\ell,\lambda}(\gamma). \tag{2.5}$$

Combining the inclusion relationships (2.2) and (2.5), we complete the proof of Theorem 1. ■

Theorem 4 *Let* $0 \leq \gamma < 1$. *Then*

$$MC_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\gamma) \subset MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma) \subset MS_{q,s,\alpha_1}^{m+1,\ell,\lambda}(\gamma). \tag{2.6}$$

Proof. Let $f(z) \in MC_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\gamma)$. Then, by Definition 2, we have

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z) \in MC(\gamma) \quad (0 \leq \gamma < 1).$$

Furthermore, in view of the relationship (1.6), we find that

$$-z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)' \in MS^*(\gamma),$$

that is, that

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)\left(-zf'(z)\right) \in MS^*(\gamma).$$

Thus, by using Definition 1 and Theorem 1, we have

$$-zf'(z) \in MS_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\gamma) \subset MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma),$$

which implies that

$$MC_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\gamma) \subset MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma).$$

The right part of Theorem 2 can be proved by using similar arguments. The proof of Theorem 2 is thus completed. ■

Theorem 5 *Let* $0 \leq \gamma, \eta < 1$. *Then*

$$MK_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\eta,\gamma) \subset MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma) \subset MK_{q,s,\alpha_1}^{m+1,\ell,\lambda}(\eta,\gamma). \tag{2.7}$$

Proof. Let us begin by proving that

$$MK_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\eta,\gamma) \subset MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma) \quad (0 \leq \gamma, \eta < 1). \tag{2.8}$$

Let $f(z) \in MK_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\eta,\gamma)$.

Then there exists a function $\psi(z) \in MS^*(\gamma)$ such that

$$\Re e\left(\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)'}{\psi(z)}\right) < -\eta \quad (z \in U).$$

We put $I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)=\psi(z)$, so that we have

$$g(z)\in MS_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\gamma) \text{ and } \Re e\left(\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)}\right)<-\eta \ (z\in U).$$

We next put

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}=-\eta-(1-\eta)q(z), \tag{2.9}$$

where $q(z)$ is given by (1.21). Thus, by using identity (1.17), we obtain

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)}=\frac{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)\left(zf^{'}(z)\right)}{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)}=\frac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)\right]^{'}+(\alpha_1+1)\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)\right]}{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)\right]^{'}+(\alpha_1+1)\,I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}$$

$$=\frac{\dfrac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)\right]^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}+(\alpha_1+1)\,\dfrac{I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}}{\dfrac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)\right]^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}+(\alpha_1+1)}.$$

Since $g(z)\in MS_{q,s,\alpha_1+1}^{m,\ell,\lambda}(\gamma)$, by using Theorem 1, we can put

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)\right)^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}=-\gamma-(1-\gamma)G(z),$$

where

$$G(z)=g_1(x,y)+ig_2(x,y) \text{ and } \Re e\,(G(z))=g_1(x,y)>0 \quad (z\in U).$$

Then

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)}=\frac{\frac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)\right]^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}-(\alpha_1+1)\left[\eta+(1-\eta)q(z)\right]}{-\gamma-(1-\gamma)G(z)+(\alpha_1+1)}. \tag{2.10}$$

We thus find from (2.9) that

$$z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)^{'}=-I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)\left[\eta+(1-\eta)q(z)\right]. \tag{2.11}$$

Differentiating both sides of (2.11) with respect to z, we obtain

$$z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)\right]^{'}=-z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)\right]^{'}\left[\eta+(1-\eta)q(z)\right]-(1-\eta)zq^{'}(z)I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)$$

so that

$$\frac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(zf^{'}(z)\right)\right]^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}=-(1-\eta)zq^{'}(z)-\left[\eta+(1-\eta)q(z)\right]\frac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)\right]^{'}}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}$$

$$=-(1-\eta)zq^{'}(z)+\left[\eta+(1-\eta)q(z)\right]\left[\gamma+(1-\gamma)G(z)\right]. \tag{2.12}$$

By substituting (2.12) into (2.10), we have

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)} = \frac{-(1-\eta)zq'(z)+[\eta+(1-\eta)q(z)][\gamma+(1-\gamma)G(z)]-(\alpha_1+1)[\eta+(1-\eta)q(z)]}{-\gamma-(1-\gamma)G(z)+(\alpha_1+1)}$$
$$= \frac{-(1-\eta)zq'(z)}{-\gamma-(1-\gamma)G(z)+(\alpha_1+1)} - [\eta+(1-\eta)q(z)],$$

then

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1+1)g(z)} + \eta = -\left\{(1-\eta)q(z) - \frac{(1-\eta)zq'(z)}{(1-\gamma)G(z)+\gamma-(\alpha_1+1)}\right\}.$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, we define the function $\Phi(u, v)$ by

$$\Phi(u,v) = (1-\eta)u - \frac{(1-\eta)v}{(1-\gamma)G(z)+\gamma-(\alpha_1+1)}, \tag{2.13}$$

where $(u, v) \in D = (\mathbb{C}\backslash D^*) \times \mathbb{C}$ and

$$D^* = \left\{z : z \in \mathbb{C} \text{ and } \Re e\,(G(z)) = g_1(x,y) > 1 + \frac{\alpha_1}{1-\gamma}\right\}.$$

Then it follows from (2.13) that

(i) $\Phi(u, v)$ *is continuous in* D;

(ii) $(1, 0) \in D$ *and* $\Re e\,\{\Phi(1, 0)\} = 1 - \eta > 0$;

(iii) for all $(iu_2, v_1) \in D$ *and such that* $v_1 \le -\frac{(1+u_2^2)}{2}$, we have

$$\Re e\,\{\Phi(iu_2,v_1)\} = \Re e\left\{-\frac{(1-\eta)v_1}{(1-\gamma)G(z)+\gamma-(\alpha_1+1)}\right\} = \frac{(1-\eta)v_1\,[\alpha_1+1-\gamma-(1-\gamma)g_1(x,y)]}{[(1-\gamma)g_1(x,y)+\gamma-\alpha_1-1]^2+[(1-\gamma)g_2(x,y)]^2}$$
$$\le -\frac{(1-\eta)(1+u_2^2)\,[\alpha_1+1-\gamma-(1-\gamma)g_1(x,y)]}{2\,[(1-\gamma)g_1(x,y)+\gamma-\alpha_1-1]^2+2\,[(1-\gamma)g_2(x,y)]^2} < 0,$$

which shows that $\Phi(u, v)$ satisfies the hypotheses of Lemma 1. Thus, in light of (2.9), we easily deduce the inclusion relationship (2.8).

Using the arguments similar to those detailed above, we can prove the second part of the inclusion. We therefore choose to omit the details involved. ■

Theorem 6 *Let* $0 \le \gamma, \eta < 1$. *Then*

$$MK_{q,s,\alpha_1+1}^{*,m,\ell,\lambda}(\eta,\gamma) \subset MK_{q,s,\alpha_1}^{*,m,\ell,\lambda}(\eta,\gamma) \subset MK_{q,s,\alpha_1}^{*,m+1,\ell,\lambda}(\eta,\gamma). \tag{2.14}$$

Proof. Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (1.6), we can also prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (1.9). ■

3 A set of integral-preserving properties

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $J_c(f)(z)$ defined by

$$J_c(f)(z) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt \quad \left(c > 0;\ f \in \sum\right), \tag{3.1}$$

which satisfies the following relationship:

$$z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)\right)' = cI_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z) - (c+1)\,I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z). \tag{3.2}$$

Theorem 7 *Let* $c>0$ *and* $0\le\gamma<1$. *If* $f(z)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$, *then* $J_c(f)(z)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$.

Proof. Suppose that $f(z)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$ and let

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)}=-\frac{1+(1-2\gamma)w(z)}{1-w(z)}, \tag{3.3}$$

where $w(0)=0$. Then, by using (3.2) and (3.3), we have

$$\frac{I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)}=\frac{c-(c+2-2\gamma)w(z)}{c(1-w(z))}. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z, we obtain

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}=\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)}-\frac{(c+2-2\gamma)\,zw'(z)}{c-(c+2-2\gamma)w(z)}+\frac{zw'(z)}{1-w(z)}$$

$$=-\frac{1+(1-2\gamma)w(z)}{1-w(z)}+\frac{zw'(z)}{1-w(z)}-\frac{(c+2-2\gamma)\,zw'(z)}{c-(c+2-2\gamma)w(z)} \tag{3.5}$$

so that

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}+\gamma=\frac{(\gamma-1)\,(1+w(z))}{1-w(z)}+\frac{zw'(z)}{1-w(z)}-\frac{(c+2-2\gamma)\,zw'(z)}{c-(c+2-2\gamma)w(z)}. \tag{3.6}$$

Now, assuming that $\underset{|z|\le|z_0|}{max}|w(z)|=|w(z_0)|=1\quad(z_0\in U)$ and applying Jack's lemma, we have

$$zw'(z_0)=\zeta w(z_0)\quad(\zeta\ge 1). \tag{3.7}$$

If we set $w(z_0)=e^{i\theta}\ (\theta\in\mathbb{R})$ in (3.6) and observe that

$$\Re e\left\{\frac{(\gamma-1)\,(1+w(z))}{1-w(z)}\right\}=0,$$

then we obtain

$$\Re e\left\{\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)}+\gamma\right\}=\Re e\left\{\frac{z_0w'(z_0)}{1-w(z_0)}-\frac{(c+2-2\gamma)\,z_0w'(z_0)}{c-(c+2-2\gamma)w(z_0)}\right\}$$

$$=\Re e\left\{-\frac{2\,(1-\gamma)\,\zeta e^{i\theta}}{(1-e^{i\theta})\,[c-(c+2-2\gamma)e^{i\theta}]}\right\}=\frac{2\zeta\,(1-\gamma)\,(c+1-\gamma)}{c^2-2c(c+2-2\gamma)\cos\theta+(c+2-2\gamma)^2}\ge 0,$$

which obviously contradicts the hypothesis $f(z)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$. Consequently, we can deduce that $|w(z)|<1\ (z\in U)$, which, in view of (3.3), proves the integral-preserving property asserted by Theorem 5. ■

Theorem 8 *Let* $c>0$ *and* $0\le\gamma<1$. *If* $f(z)\in MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$, *then* $J_c(f)(z)\in MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$.

Proof. By applying Theorem 5, it follows that

$$f(z)\in MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)\Leftrightarrow -zf'(z)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$$
$$\Rightarrow J_c\left(-zf'(z)\right)\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)\Leftrightarrow\quad -z\,(J_cf(z))'\in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)\Rightarrow J_c\,(f)\,(z)\in MC_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma),$$

which proves Theorem 6. ■

Theorem 9 *Let* $c>0$ *and* $0\le\gamma,\eta<1$. *If* $f(z)\in MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma)$, *then* $J_c(f)(z)\in MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma)$.

Proof. Suppose that $f(z) \in MK_{q,s,\alpha_1}^{m,\ell,\lambda}(\eta,\gamma)$. Then, by Definition 3, there exists a function $g(z) \in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$ such that

$$\Re e\left(\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)}\right) < -\eta \quad (z \in U).$$

Thus, upon setting

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(f)(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z)} + \eta = -(1-\eta)q(z), \tag{3.8}$$

where $q(z)$ is given by (1.21), we find from (3.2) that

$$\begin{aligned}\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)} &= -\frac{I_{q,s,\lambda}^{m,\ell}(\alpha_1)\left(-zf'(z)\right)}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)} = -\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(-zf'(z))\right)' + (c+1)\,I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(-zf'(z))}{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g(z)\right)' + (c+1)\,I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c g(z)} \\ &= -\frac{\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(-zf'(z))\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c g(z)} + (c+1)\,\frac{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(-zf'(z))}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c g(z)}}{\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c g(z)} + (c+1)}.\end{aligned}$$

Since $g(z) \in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$, we know from Theorem 5 that $J_c g(z) \in MS_{q,s,\alpha_1}^{m,\ell,\lambda}(\gamma)$. So we can set

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z)} + \gamma = -(1-\gamma)G(z), \tag{3.9}$$

where

$$G(z) = g_1(x,y) + ig_2(x,y) \text{ and } \Re e\,(G(z)) = g_1(x,y) > 0 \quad (z \in U).$$

Then we have

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)} = \frac{\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(-zf'(z))\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c g(z)} + (c+1)\left[\eta + (1-\eta)q(z)\right]}{\gamma + (1-\gamma)G(z) - (c+1)}. \tag{3.10}$$

We also find from (3.8) that

$$I_{q,s,\lambda}^{m,\ell}(\alpha_1)z\left(J_c(f)(z)\right)' = -\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z)\right)\left[\eta + (1-\eta)q(z)\right]. \tag{3.11}$$

Differentiating both sides of (3.11) with respect to z, we obtain

$$\begin{aligned} z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)z\left(J_c(f)(z)\right)'\right]' &= -(1-\eta)zq'(z)I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z) \\ &\quad - z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z)\right)'\left[\eta + (1-\eta)q(z)\right], \end{aligned} \tag{3.12}$$

that is,

$$\frac{z\left[I_{q,s,\lambda}^{m,\ell}(\alpha_1)z\left(J_c(f)(z)\right)'\right]'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)J_c(g)(z)} = -(1-\eta)zq'(z) + \left[\eta + (1-\eta)q(z)\right]\left[\gamma + (1-\gamma)G(z)\right]. \tag{3.13}$$

Substituting (3.13) into (3.10), we find that

$$\frac{z\left(I_{q,s,\lambda}^{m,\ell}(\alpha_1)f(z)\right)'}{I_{q,s,\lambda}^{m,\ell}(\alpha_1)g(z)} + \eta = -(1-\eta)q(z) + \frac{(1-\eta)zq'(z)}{(1-\gamma)G(z) + \gamma - (c+1)}. \tag{3.14}$$

Then, by setting $u = q(z) = u_1 + iu_2$ and $v = zq^{'}(z) = v_1 + iv_2$, we can define the function $\Phi(u, v)$ by

$$\Phi(u,v) = (1-\eta)u - \frac{(1-\eta)v}{(1-\gamma)G(z)+\gamma-(c+1)}. \tag{3.15}$$

The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved. ■

Theorem 10 *Let* $c > 0$ *and* $0 \leq \gamma, \eta < 1$. *If* $f(z) \in MK^{*,m,\ell,\lambda}_{q,s,\alpha_1}(\eta,\gamma)$, *then* $J_c(f)(z) \in MK^{*,m,\ell,\lambda}_{q,s,\alpha_1}(\eta,\gamma)$.

Proof. Just as we derived Theorem 6 from Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 from Theorem 7. ■

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References

[1] M. K. Aouf. A certain subclass of meromorphically starlike functions with positive coefficients, Rend. Mat, 9(1989): 225-235.

[2] M. K. Aouf. On a certain class of meromorphic univalent functions with positive coefficients, Rend. Mat., 7(11)(1991): 209-219.

[3] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni. Certain integral operators applied to meromorphic p-valent functions, J. Nat. Geom., 24(2003): 111-120.

[4] J. Clunie. On meromorphic Schlicht functions, J. London Math. Soc., 34(1959): 215-216.

[5] R. M. El-Ashwah. A note on certain meromorphic p-valent functions, Appl. Math. Letters, 22(2009): 1756-1759.

[6] R. M. El-Ashwah. Inclusion relationships for certain subclasses of meromorphic functions defined by using the extended multiplier transformations, Internat. Scholarly Research Network Geometry, (2011): 812541, 1-13.

[7] R. M. El-Ashwah. Properties of certain class of p-valent meromorphic functions associated with new integral operator, Acta Univ. Apulensis, 29(2012): 255-264.

[8] I. S. Jack. Functions starlike and convex of order α, J. London Math. Soc. 3(1971): 469-474.

[9] O. P. Juneja and T. R. Reddy. Meromorphic starlike univalent functions with positive coefficients, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 39(1985): 65-76.

[10] A. Y. Lashin. On certain subclass of meromorphic functions associated with certain integral operators, Comput. Math Appl., 59(2010): 524-531.

[11] J. E. Miller. Convex meromorphic mapping and related functions, Proc. Amer. Math. Soc., 25 (1970): 220 228.

[12] S. S. Miller and P. T. Mocanu. Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(1978): 289-305.

[13] Ch. Pommerenke. On meromorphic starlike functions, Pacific J. Math., 13(1963): 221 235.

[14] H. M. Srivastava and P. W. Karlsson. Multiple gaussian hypergeometric series, Halsted Press, Ellis Horwood Limited,Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.