The Approximate Solution for BBM Equation under Slowly Varying Medium

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Abstract: In this paper, we consider the soliton dynamics on the potential internal for the BBM equation under a slowly varying medium. We construct an approximate solution for this equation and prove that the error term due to the approximate solution can be controlled.

Keywords: BBM equation; approximate solution; slowly varying medium.

1 Introduction

In this work, we consider the following BBM equation under slowly varying medium,

\begin{equation}
(1 - \frac{1}{2} \partial^2_x) u_t + (u_{xx} - u + a \epsilon x^2) x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\end{equation}

Here \( u = u(t, x) \) is a real-valued function and \( a = a(\epsilon x) \) satisfies the following conditions. There exist constants \( K, \gamma > 0 \) such that

\begin{equation}
\begin{cases}
1 < a(r) < 2, a'(r) > 0, \forall r \in \mathbb{R}, \\
0 < a(r) - 1 < Ke^{\gamma r}, \forall r \leq 0, \\
0 < 2 - a(r) < Ke^{-\gamma r}, \forall r > 0.
\end{cases}
\end{equation}

In particular, \( \lim_{r \to -\infty} a(r) = 1 \) and \( \lim_{r \to +\infty} a(r) = 2. \)

We construct the approximate solution of the equation on the internal of \([T_x, T_x] \) and then prove that the error term due to the approximate solution can be controlled under \( O(\epsilon^2 e^{-\gamma|t|}). \)

Many relevant works have been done. Kaup and Newell [1] considered the study of perturbations of integrable e-
quations, in particular, they considered the perturbed gKdV equation. Grimshaw [2,3] introduced slowly varying solitary
waves for the Korteweg-de-Vries equation and nonlinear Schrödinger equation. K.Ko and H.H.Kuehl [4] had a research
on the Korteweg-de Vries equation with slowly varying coefficients for a soliton initial condition. Recently, C. Muñoz
made many contributions to this work. He [5-7] researched the soliton dynamics under a slowly varying medium and
inelastic character of solitons for generalized KdV equations. At the same time, Muñoz [8,9] studied the soliton dynamics
and sharp inelastic character under slowly varying medium for nonlinear Schrödinger equations.

2 Preliminary

2.1 Soliton solution of BBM equation

Recall the so-called Benjamin-Bona-Mahony equation,

\begin{equation}
(1 - \frac{1}{2} \partial^2_x) u_t + (u_{xx} - u + u^2) x = 0, \quad t, x \in \mathbb{R}.
\end{equation}
This equation has the soliton solutions as follows:

$$u(t, x) = Q_c(x - ct), \quad Q_c = (1 + c)Q\left(\sqrt{\frac{1 + c}{1 + \frac{c^2}{4}}}; x\right).$$  \hspace{1cm} (2.2)

where $Q(x) = \frac{3}{2} \cosh^{-2}\left(\frac{x}{2}\right)$, solves $Q'' + Q^2 = Q$.  \hspace{1cm} (2.3)

2.2 Definition of (IP) property

We say that $A_c(\epsilon t, x)$ satisfies the (IP) property if and only if:

(i) Any spatial derivative of $A_c(\epsilon t, x)$ is a localized $Y$-function.

(ii) There exists $K, \gamma > 0$ such that $||A_c(\epsilon t, x)||_{L^\infty(R)} \leq Ke^{-\gamma|t|}$ for all $t \in R$.

$Y$-function means the set of functions $f \in C^\infty(R, R)$ such that

$$\forall j \in N, \exists C_j, r_j > 0, \forall x \in R, \left|f^{(j)}(x)\right| \leq C_j(1 + |x|)^r e^{-|x|}.$$  \hspace{1cm} (2.4)

2.3 The characters of $Q_c$ and the properties of the new operator $L$

**Lemma 1** For $Q_c = (1 + c)Q\left(\sqrt{\frac{1 + c}{1 + \frac{c^2}{4}}}; x\right)$.

(i)

$$(1 + \frac{1}{2}c)Q''_c + Q^2_c = (1 + c)Q_c, \quad \Lambda Q_c = \left(\frac{d}{dc}Q_c\right)\big|_{c = c} = \frac{1}{1 + c}(Q_c + \frac{1}{4}(1 + c)^2 yQ'_c).$$  \hspace{1cm} (2.5)

$$\int Q^2_c = (1 + c)^2(1 + \frac{1}{2}c) \int Q^2, \quad \int Q^2 = 6, \quad \int (Q'_c)^2 = (1 + c)^2(1 + \frac{1}{2}c)^{-2} \int (Q')^2. \hspace{1cm} (2.6)

(ii) set

$$\phi(x) = -\frac{Q'_c(x)}{Q_c(x)}, \quad \phi_c(x) = -\frac{Q'_c(x)}{Q_c(x)} = \sqrt{\frac{1 + c}{1 + \frac{c^2}{4}}} \phi\left(\sqrt{\frac{1 + c}{1 + \frac{c^2}{4}}}; x\right). \hspace{1cm} (2.7)

We have

$$\lim_{x \to -\infty} \phi_c = -\sqrt{\frac{1 + c}{1 + \frac{c^2}{4}}}, \quad \lim_{x \to +\infty} \phi_c = \sqrt{\frac{1 + c}{1 + \frac{c^2}{4}}}. \hspace{1cm} (2.8)

**Lemma 2** Set

$$Lf = -(1 + \frac{1}{2}c)f'' + (1 + c)f - 2Q_c f. \hspace{1cm} (2.9)

(i) The kernel of $L$ is spawned by $Q'_c$.

(ii) Inverse. For all $h = h(y) \in L^2(R)$ such that $\int_R h Q'_c = 0$, there exists a unique $\hat{h} \in H^2(R)$ such that $\int_R \hat{h} Q'_c = 0$ and $L\hat{h} = h$. Moreover, if $h$ is even (resp. odd), then $\hat{h}$ is even (resp. odd).

**Proof.** The proofs of Lemma 1 and 2 are similar to the Claim A.2 in the paper [10]. So it is omitted. \hspace{1cm} ■

3 Construction of a soliton-like solution

3.1 Decomposition of the approximate solution

Set

$$T_c = e^{\frac{1}{2}c - \frac{\pi}{1 + \frac{c^2}{4}}}. \hspace{1cm} (3.1)

We look for the $\delta(t, x)$, the approximate solution for (1.1) on the interval of time $[-T_c, T_c]$,

$$y = x - \rho(t), \quad R = \frac{Q_c}{a(\varepsilon\rho(t))}. \hspace{1cm} (3.2)$$

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Recall the proof where we can get the following results.

\[ u = R + w = R + \varepsilon A_c(\varepsilon t, y), \]

where \( A_c(\varepsilon t, x) \) satisfies (IP) property. We want to measure the size of error produced by inserting \( \bar{u}(t, x) \) as defined in (3.4) in the equation of (1.1). For this, let

\[ S[\bar{u}] = (1 - \frac{1}{2} \partial^2_x)\bar{u}_t + (\bar{u}_{xx} - \bar{u} + a_c \bar{u}^2)_x. \]

We can get the following results.

**Proposition 3** For \( \forall t \in [-T^*_c, T^*_c] \), the nonlinear decomposition of the error term \( S[\bar{u}] \) holds:

\[ S[\bar{u}] = \varepsilon[F - (LA_c)_y] + \varepsilon^2[\frac{a''}{2a^2}(y^2Q_c^2)_y + 2\frac{a'}{a}(yA_c)_x + (1 - \frac{1}{2} \partial^2_x)(\Lambda A_c, c' + (A_c)_t)] + O(\varepsilon^2 e^{-\gamma |t|}). \]

where

\[ F = (1 - \frac{1}{2} \partial^2_x)Q_c - (1 - \frac{1}{2} \partial^2_x)\frac{a'}{a}Q_c + \frac{a'}{a^2}(y^2Q_c^2)_y. \]

**Proof.** This proposition is proved explicitly in the next four Lemmas.

**Lemma 4** Set

\[ S[\bar{u}] = I + II + III. \]

where, \( I = S[R] = (1 - \frac{1}{2} \partial^2_x)R_t + (R_{xx} - R + a_c R^2)_x, \quad II = (1 - \frac{1}{2} \partial^2_x)w_t + (w_{xx} - w + 2a_c w R)_x, \quad III = \{a_c w^2\}_x. \]

**Proof.** Recall \( \bar{u} = R + w \) and this lemma is just proved by the binomial theorem.

**Lemma 5**

\[ I = \varepsilon \left[\frac{a'}{a}(1 - \frac{1}{2} \partial^2_x)\Lambda Q_c - \frac{a''}{a^2}(1 - \frac{1}{2} \partial^2_x)Q_c + \frac{a'}{a^2}(y^2Q_c^2)_x + \frac{a''}{2a^2}(y^2Q_c^2)_x + O_{H^2}(\varepsilon^3). \right] \]

**Proof.** By \( y = x - \rho(t), R = \frac{Q_c}{a(\rho(t))} \) and \( \partial_t(\rho(t) = c(t)) \). We have

\[ I = (1 - \frac{1}{2} \partial^2_x)(\Lambda Q_c)c\varepsilon - (Q_c)_x + \frac{a''}{a^2}(y^2Q_c^2)_x + \frac{a''}{a^2}(y^2Q_c^2)_x + O_{H^2}(\varepsilon^3). \]

Via a Taylor expansion

\[ (a(\varepsilon x)Q_c^2)_x = a(\varepsilon \rho(t))(Q_c)_x + a\varepsilon'(\varepsilon \rho(t))(y^2Q_c^2)_x + \frac{1}{2} \varepsilon^2 a''(\varepsilon \rho(t))(y^2Q_c^2)_x + \frac{1}{6} \varepsilon^3 a'''(\varepsilon(\rho(t) + \theta y))(y^2Q_c^2)_x. \]

In the term of \( a'''(\varepsilon(\rho(t) + \theta y))(y^2Q_c^2)_x \), thus \( |a'''| \leq k, (y^2Q_c^2)_x \in Y \).

So

\[ (a(\varepsilon x)Q_c^2)_x = a(\varepsilon \rho(t))(Q_c)_x + a\varepsilon'(\varepsilon \rho(t))(y^2Q_c^2)_x + \frac{1}{2} \varepsilon^2 a''(\varepsilon \rho(t))(y^2Q_c^2)_x + O_{H^2}(\varepsilon^3). \]

**Proof.** By \( y = x - \rho(t), R = \frac{Q_c}{a(\rho(t))} \) and \( \partial_t(\rho(t) = c(t)) \). We have

\[ I = (1 - \frac{1}{2} \partial^2_x)(\Lambda Q_c)c\varepsilon - (Q_c)_x + \frac{a''}{a^2}(y^2Q_c^2)_x + \frac{a''}{a^2}(y^2Q_c^2)_x + O_{H^2}(\varepsilon^3). \]
Lemma 6

\[ II = \varepsilon^2 \left( 1 - \frac{1}{2} \partial_x^2 \right) (\Lambda \alpha c' \varepsilon + (A_c)_t) + 2 \frac{a'}{a} (y A_c Q)_x - \varepsilon (LA_c)_y + O_{H^2(R)}(\varepsilon^3 e^{-\gamma \varepsilon |t|}). \]

Proof. We compute

\[ II = (1 - \frac{1}{2} \partial_x^2)w_t + (w_{xx} - w + 2a_c w R)_x = \varepsilon \left( 1 - \frac{1}{2} \partial_x^2 \right) (A_c(\varepsilon t, y))_t + \varepsilon [(A_c)_{yy} - A_c + 2 \frac{a'(\varepsilon x)}{a(\varepsilon \rho)} A_c Q_x]_x. \]

Use the same method, Taylor expansion just like Lemma 5

\[ III = \varepsilon^2 (1 - \frac{1}{2} \partial_x^2)(\Lambda \alpha c' \varepsilon + (A_c)_t) + 2 \varepsilon^2 \frac{a'}{a} (y A_c Q)_x + \varepsilon \varepsilon (1 + c) A_c + \left( 1 + \frac{1}{2} \partial_x \right) (A_c)_{yy} + 2 Q_c A_c y + O_{H^2(R)}(\varepsilon^3 e^{-\gamma \varepsilon |t|}). \]

Lemma 7

\[ III = \{a_c w^2\}_x = \varepsilon^2 \varepsilon (a(x) A^2_c)_x = \varepsilon^2 a'(\varepsilon x) A^2_c + \varepsilon^2 a_c (A^2_c)' = O(\varepsilon^2 e^{-\gamma \varepsilon |t|}). \]

Proof. Note that \( (A^2_c)' \in Y \) because (IP) property holds for \( A_c \). So, we can get

\[ III = \{a_c w^2\}_x = \varepsilon^2 (a(x) A^2_c)_x = \varepsilon^2 a'(\varepsilon x) A^2_c + \varepsilon^2 a_c (A^2_c)' = O(\varepsilon^2 e^{-\gamma \varepsilon |t|}). \]

Now we collect the estimate from Lemma 4, Lemma 5, and Lemma 6. We finally get

\[ S[\bar{u}] = \varepsilon [F - (LA_c)_y] + \varepsilon^2 \left( \frac{a''}{2a^2} (y^2 Q^2_y)_y + 2 \frac{a'}{a} (y A_c Q)_x + (1 - \frac{1}{2} \partial_x^2)(\Lambda \alpha c' + (A_c)_t) + O(\varepsilon^2 e^{-\gamma \varepsilon |t|}). \]

Due to Lemma 4, Lemma 5, and Lemma 6, the Proposition 3 is proved. ■

Note that if we want to improve the approximation \( \bar{u} \), the unknown function \( A_c \) must be chosen such that

\[ F - (LA_c)_y = 0, \text{for all } y \in R \quad (\Omega). \]

Then the error term will be reduced to the second order quantity

\[ S[\bar{u}] = \varepsilon^2 \left( \frac{a''}{2a^2} (y^2 Q^2_y)_y + 2 \frac{a'}{a} (y A_c Q)_x + (1 - \frac{1}{2} \partial_x^2)(\Lambda \alpha c' + (A_c)_t) \right) + O(\varepsilon^2 e^{-\gamma \varepsilon |t|}). \]

We prove such a solvability result in the next part.

3.2 Resolution of \( \Omega \)

Lemma 8 (Existence theory for \( \Omega \))

Suppose \( F \in Y \) even and satisfying the orthogonality condition

\[ \int_R F Q_c = 0. \]

Let \( \beta = \frac{1}{\sqrt{1 + \sqrt{1 + \epsilon}}} \int_R F \), the problem of \( \Omega \) has a bounded solution \( A_c \) of the form \( A_c = \beta \phi_c + \delta + A_1(y) \), with \( A_1(y) \in Y \).

Proof. Let us write \( A_c = \beta \phi_c + \delta + A_1(y) \), where \( \beta, \delta \in R \) and \( A_1(y) \in Y \) are to be determined. We have \( L A_1(y) = H(y) - \beta L \phi_c - \gamma \), where \( H(y) = \int_{-\infty}^{y} F(s) ds \) and \( \gamma = |A_1| - \int_{-\infty}^{y} F(s) ds \).

Without loss of generality, we can suppose the constant term \( \gamma = -\sqrt{1 + \epsilon} \beta \). The problem of \( \Omega \) is solvable if and only
So put these three parts together and from Lemma 1 and Lemma 2 to get orthogonality condition, we impose

$$\int_R |H(y) - \beta(L\phi_c + 1)|Q_c' = \int_R HQ_c' = - \int_R FQ_c = 0.$$ 

Namely recall that \((LQ_c' = 0)\) thus there exists a solution \(A_1(y)\) satisfying \(\int_R A_1 Q_c = 0\). Since

$$\lim_{y \to -\infty} (H(y) - \beta(L\phi_c + \sqrt{1 + c/1 + 2c})) = 0, \quad \lim_{y \to +\infty} (H(y) - \beta(L\phi_c + \sqrt{1 + c/1 + 2c})) = \int_R F - 2\sqrt{1 + c/1 + 2c}.$$ 

So we get \(A_1(y) \in Y\) provided \(\beta = \frac{1}{2} \sqrt{1 + \frac{c}{1 + 2c}} \int_R F\). This finishes the proof. ■

According to the Lemma 8, it suffices to verify the orthogonality conditions.

**Lemma 9** There exists a solution \(A_c\) of the problem \((\Omega)\) satisfying \((IP)\) and such that

$$A_c = \beta(\phi_c - \sqrt{1 + c/1 + 2c}) + A_1(y), \quad \lim_{y \to +\infty} A_c = 0.$$ 

$$\beta = \frac{1}{2} \sqrt{1 + \frac{c}{1 + c}} \int_R F \quad \frac{1}{2} \sqrt{1 + \frac{c}{1 + c}} (\frac{c'}{a(1 + c)} - \frac{a'}{a(1 + c)^2}) \int_R Q_c.$$

**Proof.** We prove this Lemma in next three Lemmas. ■

**Lemma 10** (The imposed condition)

To get orthogonality condition \(\int_R FQ_c = 0\), the parameter of \(c, a\) satisfy the following conditions

$$\frac{c'}{\left(\frac{11}{10} + \frac{3}{5}c - \frac{1}{150}(1 + c)^2/2 + 2c\right)} = \frac{a'}{a(1 + c)} - \frac{a'}{a(1 + c)^2} \int_R Q_c' = 0.$$ 

**Proof.** Note that

$$F = (1 - \frac{1}{2} \frac{\partial^2}{a^2}) \frac{c'}{a} \Lambda Q_c - (1 - \frac{1}{2} \frac{\partial^2}{a^2}) \frac{a'}{a} \Lambda Q_c + \frac{1}{2} \frac{\partial^2}{a^2} (yQ_c')_y.$$ 

We just compute these three terms \(\int_R (1 - \frac{1}{2} \frac{\partial^2}{a^2}) \Lambda Q_c, \int_R (1 - \frac{1}{2} \frac{\partial^2}{a^2}) \Lambda Q_c\), and \(\int_R (yQ_c')_y Q_c\).

$$\int_R (1 - \frac{1}{2} \frac{\partial^2}{a^2}) \Lambda Q_c = \int_R (1 + \frac{1}{2} \frac{\partial^2}{a^2}) \Lambda Q_c = \int_R \left(\frac{1}{1 + c} \int_R \left(\frac{1}{2 + c} Q_c + \frac{1}{2 + c} Q_c^2\right)\right) \frac{1}{1 + c} \int_R Q_c' = 0.$$ 

Note that \(\int Q_c yQ_c' = \frac{1}{2} \int yQ_c^2\) and \(\int Q_c yQ_c^2 = \frac{1}{2} \int yQ_c\) and \(\int Q_c^2 = 0\). So

$$\int_R (1 - \frac{1}{2} \frac{\partial^2}{a^2}) \Lambda Q_c = \int_R \frac{1}{1 + c} \int_R \left(\frac{1}{2 + c} \frac{1}{1 + c} \int_R \left(\frac{1}{2 + c} Q_c + \frac{1}{2 + c} Q_c^2\right)\right) \frac{1}{1 + c} \int_R Q_c' = 0.$$ 

So put these three parts together and from Lemma 1 and Lemma 2 to get orthogonality condition, we impose

$$\frac{c'}{\left(\frac{11}{10} + \frac{3}{5}c - \frac{1}{150}(1 + c)^2/2 + 2c\right)} = \frac{a'}{a(1 + c)} - \frac{a'}{a(1 + c)^2} \int_R Q_c' = 0.$$ 

■

**Lemma 11**

$$\beta = \frac{1}{2} \sqrt{1 + \frac{c}{1 + c}} \int_R F \quad \frac{1}{2} \sqrt{1 + \frac{c}{1 + c}} \frac{c'}{a(1 + c)} (1 - \frac{1}{4} \frac{a'}{a(1 + c)^2} \int_R Q_c.$$
Proof.

\[ F = (1 - \frac{1}{2} \partial_x^2) \frac{a'}{a} \Lambda c + (1 - \frac{1}{2} \partial_x^2) \frac{a'}{a} c + \frac{a'}{a^2} (y Q_c^y). \]

We just compute the \( \int (1 - \frac{1}{2} \partial_x^2) \Lambda c \), \( \int (1 - \frac{1}{2} \partial_x^2) Q_c \), and \( \int (y Q_c^y) \). Because \( a, c \) is independent of \( x \).

\[
\int (1 - \frac{1}{2} \partial_x^2) \Lambda c = \frac{1}{1 + c} \int (1 - \frac{1}{2} \partial_x^2) (Q_c + \frac{1}{4} (1 + c)^\frac{3}{2} y Q_c') = \frac{1}{1 + c} \int Q_c - \frac{1}{4} \frac{1 + c)^\frac{3}{2}} { (1 + \frac{1}{2} c)^2} \int Q_c.
\]

Due to \( \int (1 - \frac{1}{2} \partial_x^2) Q_c = \int Q_c - \frac{1}{2} \int Q_c'' = \int Q_c \) and \( \int (y Q_c^y) = 0 \).

\[
\beta = \frac{1}{2} \sqrt{1 + \frac{2}{c}} \left( \frac{c'}{a(1 + c)} (1 - \frac{1}{4} \frac{1 + c)^\frac{3}{2}} { (1 + \frac{1}{2} c)^2} \right) \int Q_c.
\]

Lemma 12

\[
\delta = -\beta \sqrt{\frac{1 + c}{1 + \frac{2}{c}}}
\]

Proof. Finally, to get \( \lim_{y \to +\infty} A_e = 0 \) by Lemma 1, we choose \( \delta = -\beta \sqrt{\frac{1 + c}{1 + \frac{2}{c}}} \).

According to Lemma 9, Lemma 10, and Lemma 11, We have \( A_e = \beta (\phi_c - \sqrt{\frac{1 + c}{1 + \frac{2}{c}}} + A_1(y), A_1 \in Y \), this finishes the proof of Lemma 8. This proves the problem of \( \Omega \).

3.3 Correction to the solution of problem of \( \Omega \)

Consider the cutoff function \( \eta \in C^\infty(R) \) satisfying the following properties,

\[
\begin{align*}
0 \leq \eta(s) &\leq 1, 0 \leq \eta'(s) \leq 1, \forall s \in R, \\
\eta(s) &\equiv 0, \forall s \leq -1, \\
\eta(s) &\equiv 1, \forall s \geq 1.
\end{align*}
\]

(3.8)

Define

\[
\eta_e(y) = \eta(\varepsilon y + 2).
\]

(3.9)

And for \( A_e = A_e(\varepsilon t, y) \) solution of \( \Omega \), denote

\[
A_\# = \eta_e A_e(\varepsilon t, y).
\]

(3.10)

Now redefine

\[
\tilde{u} = R + w = R + \varepsilon A_\#.
\]

(3.11)

The following Proposition, which deals with the error associated to the cutoff function and the new approximate solution \( \tilde{u} \), is the main result.

Proposition 13 There exist constants \( \varepsilon_0, K > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), the following holds.

(i) (a) New behavior. For all \( t \in [-T_\varepsilon, T_\varepsilon] \),

\[
\begin{align*}
A_\#(\varepsilon t, y) = 0, \forall y \leq -\frac{3}{\varepsilon}, \\
A_\#(\varepsilon t, y) = A_e(\varepsilon t, y), \forall y \geq \frac{1}{\varepsilon}.
\end{align*}
\]

(3.12)

(b) Integrable solution. For all \( t \in [-T_\varepsilon, T_\varepsilon] \), \( A_\# \in H^1(R) \) with

\[
\|\varepsilon A_\#\|_{H^1(R)} \leq K \varepsilon^{\frac{1}{2}} e^{-\gamma(\varepsilon)},
\]

(3.13)

(ii) The error associated to the new function \( \tilde{u} \) satisfies

\[
\|S[\tilde{u}]\|_{H^2(R)} \leq K \varepsilon^{\frac{1}{2}} e^{-\gamma(\varepsilon)}.
\]

(3.14)
and the following integral estimate holds
\[ \int_R \|S[\tilde{u}]\|_{H^2(R)} \, dt \leq K \epsilon^{\frac{3}{2}}, \] (3.15)

**Proof.** The proof of first part of this proposition is similar to proof of proposition 13 in [5], so we omit this part. We will prove the second part of this proposition in the next Lemma. \( \blacksquare \)

**Lemma 14**

\[ S[\tilde{u}] = I + II' + III' \]

where
\[ II' = -\epsilon\eta_c(LA_c)y + O_{H^2(R)}(\epsilon^{\frac{3}{2}}e^{-\gamma|t|}) \quad \text{and} \quad \|III'\|_{H^2(R)} \leq K \epsilon^2 e^{-\gamma|t|}. \]

**Proof.**

\[ III' = \{a(\epsilon x)w^2\}_x = \epsilon^2(a(\epsilon x)\eta^2_cA^2_c)_x = \epsilon^3a'(\epsilon x)\eta^2_cA^2_c + 2\epsilon^3a(\epsilon x)\eta_c\eta''_cA^2_c + 2\epsilon^3a(\epsilon x)\eta^2_cA_c'A_c'. \]

With \( A_c, A'_c \in Y, \|\eta''_c\|_{L^2(R)} \leq K \epsilon^{-1/2}, \) uniformly \( t \in [-T_c, T_c]. \) Moreover, we have the estimate
\[ \|III'\|_{H^2(R)} \leq K \epsilon^2 e^{-\gamma|t|}. \]

Note that
\[ (1 - \frac{1}{2}\frac{\partial^2}{\partial t^2})(\epsilon A\#)_t = \epsilon(1 - \frac{1}{2}\frac{\partial^2}{\partial t^2})(-\epsilon \eta_cA_c - c(A_c)y\eta_c + \epsilon(A_c)\eta_c - \epsilon\eta_cLA_c') = \epsilon^2(1 - \frac{1}{2}\frac{\partial^2}{\partial t^2})(-\epsilon \eta_cA_c + (A_c)\eta_c - \epsilon\eta_cLA_c') - \epsilon(1 - \frac{1}{2}\frac{\partial^2}{\partial t^2})(\epsilon(A_c)y\eta_c). \]

Thus
\[ (\epsilon A\#)_{xx} - \epsilon A\# + 2a_x\epsilon A\# R)_{x} = (\epsilon\eta_cA_c x - \epsilon\eta_cA_c + 2a_xR\eta_cA_c)_x = \epsilon(\eta_c(A_c)_{yy} + 2\eta''_c(A_c)_{y} + \epsilon^2\eta''_c(A_c) - \epsilon\eta_cA_c + 2a_xR\eta_cA_c)_x = \epsilon\eta_c((A_c)_{yy} - A_c + 2\frac{a(\epsilon x)}{a(\epsilon \rho)}Q_cA_c)_x + \epsilon^2(2\eta''_c(A_c)_{y} + \epsilon\eta''_cA_c)_y = \epsilon\eta_c((A_c)_{yy} - A_c + 2\frac{a(\epsilon x)}{a(\epsilon \rho)}Q_cA_c)_x + \epsilon^2(2\eta''_c(A_c)_{y} + \epsilon\eta''_cA_c)_y + \epsilon^2(3\eta''_c(A_c)_{y} + 3\eta''_cA_c - A_c + 2\frac{a(\epsilon x)}{a(\epsilon \rho)}Q_cA_c)_x = \epsilon\eta_c((A_c)_{yy} - A_c + 2Q_cA_c)_x + \epsilon^2(2\eta''_c(A_c)_{y} + \epsilon\eta''_cA_c)_y + \epsilon^2(3\eta''_c(A_c)_{y} + 3\eta''_cA_c - A_c + 2\frac{a(\epsilon x)}{a(\epsilon \rho)}Q_cA_c)_x + O(\epsilon^3\eta_c(y^2Q_cA_c)_y). \]

From the (IP) property to estimate as follows
\[ \left\| 2\epsilon^2\eta_c\frac{a'(\epsilon x)}{a(\epsilon \rho)}(yQ_cA_c)_y \right\|_{H^2(R)} \leq K \epsilon^2 e^{-\gamma|t|} \left\| O(\epsilon^3\eta_c(y^2Q_cA_c)_y) \right\|_{H^2(R)} \leq K \epsilon^3. \]
\[ \left\| \epsilon^4\eta''_cA_c \right\|_{H^2(R)} \leq \epsilon^2 e^{-\gamma|t|}, \left\| \epsilon^2\eta''_cA_c \right\|_{H^2(R)} \leq K \epsilon^3 e^{-\gamma|t|}, \left\| \epsilon^2(3\epsilon^2\eta''_c(A_c)_y + 3\eta''_c(A_c)_{yy} + 2\eta''_cQ_cA_c) \right\|_{H^2(R)} \leq K \epsilon^2 e^{-\gamma|t|}. \]

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Therefore
\[
((\varepsilon A_x)_x - \varepsilon A_y + 2a\varepsilon A_x R)_x = \varepsilon \eta_c [(A_c)_{yy} - A_c + 2Q_c A_c]_x + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma|t|} + \varepsilon^3).
\]
So, we get
\[
II' = -\varepsilon \eta_c (LA_c)_y + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma|t|}).
\]

Note that
\[
S[\tilde{u}] = \varepsilon [F - \eta_c (LA_c)]_y + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma|t|}) = \varepsilon (1 - \eta_c) F + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma|t|}).
\]
For every \(t \in [-T, T]\), \(1 - \eta_c \subseteq (-\infty, -\frac{1}{2})\), \(\|F\|_{H^2(R)} \leq Ke^{-\gamma|y| - \gamma|t|}\).
So we gain
\[
\|\varepsilon (1 - \eta_c) F\|_{H^2(R)} \leq Ke^{-\frac{\gamma}{4} - e^{-\gamma|t|}} < K\varepsilon^{10}.
\]
\[
\|S[\tilde{u}]\|_{H^2(R)} \leq K\varepsilon^{\frac{3}{2}} e^{-\gamma|t|}.
\]

(3.15) is just from integration of the formula of (3.14).
\[
\int_R \|S[\tilde{u}]\|_{H^2(R)} dt \leq K\varepsilon^{\frac{3}{2}}.
\]
This finishes the second proof of Proposition 3.

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References