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The Approximate Solution for BBM Equation under Slowly Varying Medium

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Abstract: In this paper, we consider the soliton dynamics on the potential internal for the BBM equation under a slowly varying medium. We construct an approximate solution for this equation and prove that the error term due to the approximate solution can be controlled.

Keywords: BBM equation; approximate solution; slowly varying medium.

1 Introduction

In this work, we consider the following BBM equation under slowly varying medium,

$$(1-\frac{1}{2}\partial_x^2)u_t+(u_{xx}-u+a_\varepsilon u^2)_x=0, \qquad (t,x)\in \mathrm{R}_t\times \mathrm{R}_x. \tag{1.1}$$

Here $u=u(t,x)$ is a real-valued function and $a_\varepsilon=a(\varepsilon x)$ satisfies the following conditions. There exist constants $K,\gamma>0$ such that

$$\begin{cases} 1<a(r)<2, a'(r)>0, \forall r\in R,\\ 0<a(r)-1<Ke^{\gamma r}, \forall r\leq 0,\\ 0<2-a(r)<Ke^{-\gamma r}, \forall r>0. \end{cases} \tag{1.2}$$

In particular, $\lim_{r\to-\infty}a(r)=1$ and $\lim_{r\to+\infty}a(r)=2$.

We construct the approximate solution of the equation on the internal of $[-T_\varepsilon, T_\varepsilon]$ and then prove that the error term due to the approximate solution can be controlled under $O(\varepsilon^{\frac{3}{2}}e^{-\gamma\varepsilon|t|})$.

Many relevant works have been done. Kaup and Newell [1] considered the study of perturbations of integrable equations, in particular, they considered the perturbed gKdV equation. Grimshaw [2,3] introduced slowly varying solitary waves for the Korteweg-de-Vries equation and nonlinear Schrödinger equation. K.Ko and H.H.Kuehl [4] had a research on the Korteweg-de Vries equation with slowly varying coefficients for a soliton initial condition. Recently, C. Muñoz made many contributions to this work. He [5-7] researched the soliton dynamics under a slowly varying medium and inelastic character of solitons for generalized KdV equations. At the same time, Muñoz [8,9] studied the soliton dynamics and sharp inelastic character under slowly varying medium for nonlinear Schrödinger equations.

2 Preliminaies

2.1 Soliton solution of BBM equation

Recall the so-called Benjamin-Bona-Mahony equation,

$$(1-\frac{1}{2}\partial_x^2)u_t+(u_{xx}-u+u^2)_x=0, \quad t,x\in R. \tag{2.1}$$

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This equation has the soliton solutions as follows:

$$u(t,x) = Q_c(x-ct), \qquad Q_c = (1+c)Q(\sqrt{\frac{1+c}{1+\frac{1}{2}c}}x). \tag{2.2}$$

where $Q(x) = \frac{3}{2}\cosh^{-2}(\frac{x}{2})$, solves $Q'' + Q^2 = Q$. (2.3)

2.2 Definition of (IP) property

We say that $A_c(\varepsilon t, x)$ satisfies the **(IP)** property if and only if :

(i) Any spatial derivative of $A_c(\varepsilon t, x)$ is a localized Y-function.

(ii) There exists $K, \gamma > 0$ such that $\|A_c(\varepsilon t, x)\|_{L^\infty(R)} \leq K e^{-\gamma\varepsilon|t|}$ for all $t \in R$.

Y-function means the set of functions $f \in C^\infty(R,R)$ such that

$$\forall j \in N,\ \ \exists C_j, r_j > 0,\ \ \forall x \in R,\quad \left|f^{(j)}(x)\right| \leq C_j(1+|x|)^{r_j}e^{-|x|}.$$

2.3 The characters of Q_c and the properties of the new operator L

Lemma 1 *For* $Q_c = (1+c)Q(\sqrt{\frac{1+c}{1+\frac{1}{2}c}}x)$.

(i)

$$(1+\frac{1}{2}c)Q_c'' + Q_c^2 = (1+c)Q_c, \quad \Lambda Q_c = (\frac{d}{dc'}Q_c)_{|c'=c} = \frac{1}{1+c}(Q_c + \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{3}{2}}}yQ_c'). \tag{2.4}$$

$$\int Q_c^2 = (1+c)^{\frac{3}{2}}(1+\frac{1}{2}c)^{\frac{1}{2}}\int Q^2, \quad \int Q^2 = 6, \quad \int (Q_c')^2 = (1+c)^{\frac{5}{2}}(1+\frac{1}{2}c)^{-\frac{1}{2}}\int (Q')^2. \tag{2.5}$$

$$r\int Q^r = \frac{2r+1}{3}\int Q^{r+1}, \ \ r\int Q_c^r = \frac{2r+1}{3(1+c)}\int Q_c^{r+1}, \quad \int \Lambda Q_c = [\frac{1}{1+c} - \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{3}{2}}}]\int Q_c. \tag{2.6}$$

(ii)set

$$\phi(x) = -\frac{Q'(x)}{Q(x)}, \ \ \phi_c(x) = -\frac{Q_c'(x)}{Q_c(x)} = \sqrt{\frac{1+c}{1+\frac{1}{2}c}}\phi(\sqrt{\frac{1+c}{1+\frac{1}{2}c}}x). \tag{2.7}$$

We have

$$\lim_{x\to-\infty}\phi_c = -\sqrt{\frac{1+c}{1+\frac{1}{2}c}}, \quad \lim_{x\to+\infty}\phi_c = \sqrt{\frac{1+c}{1+\frac{1}{2}c}}. \tag{2.8}$$

Lemma 2 *Set*

$$Lf = -(1+\frac{1}{2}c)f'' + (1+c)f - 2Q_c f. \tag{2.9}$$

(i)The kernel of L is spawned by Q_c'.

(ii)Inverse. For all $h = h(y) \in L^2(R)$ such that $\int_R hQ_c' = 0$, there exists a unique $\hat{h} \in H^2(R)$ such that $\int_R \hat{h}Q_c' = 0$ and $L\hat{h} = h$. Moreover, if h is even (resp.odd), then $\hat{h}$ is even(resp.odd).

Proof. The proofs of Lemma 1 and 2 are similar to the Claim A.2 in the paper [10]. So it is omitted. ■

3 Construction of a soliton-like solution

3.1 Decomposition of the approximate solution

Set

$$T_\varepsilon = \varepsilon^{-1-\frac{1}{100}}. \tag{3.1}$$

We look for the $\tilde{u}(t,x)$, the approximate solution for (1.1) on the interval of time $[-T_\varepsilon, T_\varepsilon]$,

$$y = x - \rho(t), \qquad R = \frac{Q_c}{a(\varepsilon\rho(t))}. \tag{3.2}$$

where

$$Q_c = (1+c)Q(\sqrt{\frac{1+c}{1+\frac{1}{2}c}}(x-\rho(t))), \quad \rho(t) = \int_{-T_\varepsilon}^{t} c(\varepsilon s)ds - T_\varepsilon. \tag{3.3}$$

The form of $\tilde{u}(t,x)$ will be the sum of the soliton plus a correction term:

$$\tilde{u} = R + w = R + \varepsilon A_c(\varepsilon t, y), \tag{3.4}$$

where $A_c(\varepsilon t, x)$ satisfies (IP) property. We want to measure the size of error produced by inserting $\tilde{u}(t,x)$ as defined in (3.4) in the equation of (1.1). For this, let

$$S[\tilde{u}] = (1 - \frac{1}{2}\partial_x^2)\tilde{u}_t + (\tilde{u}_{xx} - \tilde{u} + a_\varepsilon \tilde{u}^2)_x. \tag{3.5}$$

We can get the following results.

Proposition 3 *For $\forall t \in [-T_\varepsilon, T_\varepsilon]$, the nonlinear decomposition of the error term $S[\tilde{u}]$ holds:*

$$S[\tilde{u}] = \varepsilon[F - (LA_c)_y] + \varepsilon^2[\frac{a''}{2a^2}(y^2Q_c^2)_y + 2\frac{a'}{a}(yA_cQ)_x + (1-\frac{1}{2}\partial_x^2)(\Lambda A_c c' + (A_c)_t)] + O(\varepsilon^2 e^{-\gamma\varepsilon|t|}). \tag{3.6}$$

where

$$F = (1-\tfrac{1}{2}\partial_x^2)\tfrac{c'}{a}\Lambda Q_c - (1-\tfrac{1}{2}\partial_x^2)\tfrac{a'}{a^2}cQ_c + \tfrac{a'}{a^2}(yQ_c^2)_y. \tag{3.7}$$

Proof. This proposition is proved explicitly in the next four Lemmas. ■

Lemma 4 *Set*

$$S[\tilde{u}] = I + II + III.$$

where, $I = S[R] = (1-\frac{1}{2}\partial_x^2)R_t + (R_{xx} - R + a_\varepsilon R^2)_x$, $\quad II = (1-\frac{1}{2}\partial_x^2)w_t + (w_{xx} - w + 2a_\varepsilon wR)_x$, $\quad III = \{a_\varepsilon w^2\}_x$.

Proof. Recall $\tilde{u} = R + w$ and this lemma is just proved by the binomial theorem. ■

Lemma 5

$$I = \varepsilon[\frac{c'}{a}(1-\frac{1}{2}\partial_x^2)\Lambda Q_c - \frac{a'c}{a^2}(1-\frac{1}{2}\partial_x^2)Q_c + \frac{a'}{a^2}(yQ_c^2)_x] + \varepsilon^2\frac{a''}{2a^2}(y^2Q_c^2)_x + O_{H^2(R)}(\varepsilon^3).$$

Proof. By $y = x - \rho(t)$, $R = \frac{Q_c}{a(\varepsilon\rho(t))}$ and $\partial_t\rho(t) = c(\varepsilon t)$. We have

$$\begin{aligned} I &= (1-\frac{1}{2}\partial_x^2)R_t + (R_{xx} - R + a_\varepsilon R^2)_x \\ &= (1-\frac{1}{2}\partial_x^2)\frac{(\Lambda Q_c c'\varepsilon - Q_c'c)a - Q_c a'\varepsilon c}{a^2} + \frac{1}{a}Q_c''' - \frac{Q_c'}{a} + \frac{1}{a^2}(a(\varepsilon x)Q_c^2). \end{aligned}$$

Via a Taylor expansion

$$(a(\varepsilon x)Q_c^2)_x = a(\varepsilon\rho(t))(Q_c^2)_x + \varepsilon a'(\varepsilon\rho(t))(yQ_c^2)_x + \frac{1}{2}\varepsilon^2 a''(\varepsilon\rho(t))(y^2Q_c^2)_x + \frac{1}{6}\varepsilon^3 a'''(\varepsilon(\rho(t)+\theta y))(y^3Q_c^2)_x.$$

In the term of $a'''(\varepsilon(\rho(t)+\theta y))(y^3Q_c^2)_x$, thus $|a'''| \le k$, $(y^3Q_c^2)_x \in Y$.

So

$$(a(\varepsilon x)Q_c^2)_x = a(\varepsilon\rho(t))(Q_c^2)_x + \varepsilon a'(\varepsilon\rho(t))(yQ_c^2)_x + \frac{1}{2}\varepsilon^2 a''(\varepsilon\rho(t))(y^2Q_c^2)_x + O_{H^2(R)}(\varepsilon^3).$$

$$\begin{aligned} I &= \frac{(\Lambda Q_c c'\varepsilon - Q_c'c)a - Q_c a'\varepsilon c}{a^2} - \frac{1}{2}\frac{(\Lambda Q_c'' c'\varepsilon - Q_c'''c)a - Q_c'' a'\varepsilon c}{a^2} \\ &+ \frac{1}{a}Q_c''' - \frac{Q_c'}{a} + \frac{1}{a^2}[a(Q_c^2)_x + \varepsilon a'(yQ_c^2)_x + \frac{1}{2}\varepsilon^2 a''(y^2Q_c^2)_x + O_{H^2(R)}(\varepsilon^3). \end{aligned}$$

$$\begin{aligned} I &= \frac{1}{a}(Q_c^2 - (1+c)Q_c + (1+\frac{1}{2}c)Q_c'')' + \varepsilon[\frac{c'}{a}(1-\frac{1}{2}\partial_x^2)\Lambda Q_c - \frac{a'c}{a^2}(1-\frac{1}{2}\partial_x^2)Q_c + \frac{a'}{a^2}(yQ_c^2)_x] \\ &+ \varepsilon^2\frac{a''}{2a^2}(y^2Q_c^2)_x + O_{H^2(R)}(\varepsilon^3). \end{aligned}$$

$$I = \varepsilon[\frac{c'}{a}(1-\frac{1}{2}\partial_x^2)\Lambda Q_c - \frac{a'c}{a^2}(1-\frac{1}{2}\partial_x^2)Q_c + \frac{a'}{a^2}(yQ_c^2)_x] + \varepsilon^2\frac{a''}{2a^2}(y^2Q_c^2)_x + O_{H^2(R)}(\varepsilon^3).$$

■

Lemma 6

$$II=\varepsilon^2[(1-\frac{1}{2}\partial_x^2)(\Lambda A_c c'\varepsilon+(A_c)_t)+2\frac{a'}{a}(yA_cQ)_x]-\varepsilon(LA_c)_y+O_{H^2(R)}(\varepsilon^3e^{-\gamma\varepsilon|t|}).$$

Proof. We compute

$$II=(1-\frac{1}{2}\partial_x^2)w_t+(w_{xx}-w+2a_\varepsilon wR)_x=\varepsilon(1-\frac{1}{2}\partial_x^2)(A_c(\varepsilon t,y))_t+\varepsilon[(A_c)_{yy}-A_c+2\frac{a(\varepsilon x)}{a(\varepsilon\rho)}A_cQ_c]_x.$$

Use the same method, Taylor expansion just like Lemma 5

$$\begin{aligned}II &= \varepsilon(1-\frac{1}{2}\partial_x^2)[\Lambda A_c c'\varepsilon+(A_c)_t\varepsilon-(A_c)_yc]+\varepsilon[(A_c)_{yy}-A_c+2A_cQ_c+2\varepsilon\frac{a'}{a}yA_cQ_c]_x+O_{H^2(R)}(\varepsilon^3e^{-\gamma\varepsilon|t|})\\ &= \varepsilon^2(1-\frac{1}{2}\partial_x^2)(\Lambda A_c c'\varepsilon+(A_c)_t)+2\varepsilon^2\frac{a'}{a}(yA_cQ)_x+\varepsilon[-(1+c)A_c+(1+\frac{1}{2}c)(A_c)_{yy}+2Q_cA_c]_y\\ &+ O_{H^2(R)}(\varepsilon^3e^{-\gamma\varepsilon|t|})\\ &= \varepsilon^2[(1-\frac{1}{2}\partial_x^2)(\Lambda A_c c'\varepsilon+(A_c)_t)+2\frac{a'}{a}(yA_cQ)_x]-\varepsilon(LA_c)_y+O_{H^2(R)}(\varepsilon^3e^{-\gamma\varepsilon|t|}).\end{aligned}$$

■

Lemma 7

$$III=\{a_\varepsilon w^2\}_x=\varepsilon^2(a(\varepsilon x)A_c^2)_x=\varepsilon^3a'(\varepsilon x)A_c^2+\varepsilon^2a_\varepsilon(A_c^2)'=O(\varepsilon^2e^{-\gamma\varepsilon|t|}).$$

Proof. Note that $(A_c^2)'\in Y$ because (IP) property holds for A_c. So, we can get

$$III=\{a_\varepsilon w^2\}_x=\varepsilon^2(a(\varepsilon x)A_c^2)_x=\varepsilon^3a'(\varepsilon x)A_c^2+\varepsilon^2a_\varepsilon(A_c^2)'=O(\varepsilon^2e^{-\gamma\varepsilon|t|}).$$

Now we collect the estimate from Lemma 4, Lemma 5, and Lemma 6. We finally get

$$S[\tilde{u}]=\varepsilon[F-(LA_c)_y]+\varepsilon^2[\frac{a''}{2a^2}(y^2Q_c^2)_y+2\frac{a'}{a}(yA_cQ)_x+(1-\frac{1}{2}\partial_x^2)(\Lambda A_cc'+(A_c)_t)]+O(\varepsilon^2e^{-\gamma\varepsilon|t|}).$$

Due to Lemma 4, Lemma 5, and Lemma 6, the Proposition 3 is proved. ■
Note that if we want to improve the approximation $\tilde{u}$,the unknown function A_c must be chosen such that
$F-(LA_c)_y=0$, for all $y\in R$ (Ω).
Then the error term will be reduced to the second order quantity

$$S[\tilde{u}]=\varepsilon^2[\frac{a''}{2a^2}(y^2Q_c^2)_y+2\frac{a'}{a}(yA_cQ)_x+(1-\frac{1}{2}\partial_x^2)(\Lambda A_cc'+(A_c)_t)]+O(\varepsilon^2e^{-\gamma\varepsilon|t|}).$$

We prove such a solvability result in the next part.

3.2 Resolution of Ω

Lemma 8 *(Existence theory for Ω)*
Suppose $F\in Y$ even and satisfying the orthogonality condition

$$\int_R FQ_c=0.$$

Let $\beta=\frac{1}{2}\sqrt{\frac{1+\frac{1}{2}c}{1+c}}\int_R F$, the problem of Ω has a bounded solution A_c of the form $A_c=\beta\phi_c+\delta+A_1(y)$, with $A_1(y)\in Y$.

Proof. Let us write $A_c=\beta\phi_c+\delta+A_1(y)$, where $\beta,\delta\in R$ and $A_1(y)\in Y$ are to be determined. We have $LA_1(y)=H(y)-\beta L\phi_c-\gamma$, where $H(y)=\int_{-\infty}^y F(s)ds$ and $\gamma=LA_1(0)-\int_{-\infty}^0 F(s)ds$.
Without loss of generality , we can suppose the constant term $\gamma=-\sqrt{\frac{1+c}{1+\frac{1}{2}c}}\beta$. The problem of Ω is solvable if and only

if

$$\int_R [H(y) - \beta(L\phi_c + 1)]Q_c' = \int_R HQ_c' = -\int_R FQ_c = 0.$$

Namely recall that $(LQ_c' = 0)$ thus there exists a solution $A_1(y)$ satisfying $\int_R A_1 Q_c' = 0$.
Since

$$\lim_{y\to-\infty}(H(y) - \beta(L\phi_c + \sqrt{\frac{1+c}{1+\frac{1}{2}c}})) = 0, \qquad \lim_{y\to+\infty}(H(y) - \beta(L\phi_c + \sqrt{\frac{1+c}{1+\frac{1}{2}c}})) = \int_R F - 2\sqrt{\frac{1+c}{1+\frac{1}{2}c}}\beta.$$

So we get $A_1(y) \in Y$ provided $\beta = \frac{1}{2}\sqrt{\frac{1+\frac{1}{2}c}{1+c}}\int_R F$. This finishes the proof. ■
According to the Lemma 8, it suffices to verify the orthogonality conditions.

Lemma 9 *There exists a solution A_c of the problem (Ω) satisfying (IP) and such that*

$$A_c = \beta(\phi_c - \sqrt{\frac{1+c}{1+\frac{1}{2}c}}) + A_1(y), \qquad \lim_{y\to+\infty} A_c = 0.$$

$$\beta = \frac{1}{2}\sqrt{\frac{1+\frac{c}{2}}{1+c}}\int_R F = \frac{1}{2}\sqrt{\frac{1+\frac{c}{2}}{1+c}}(\frac{c'}{a(1+c)}(1 - \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{c}{2})^{\frac{3}{2}}}) - \frac{a'}{a^2}c)\int Q_c.$$

Proof. We prove this Lemma in next three Lemmas. ■

Lemma 10 *(The imposed condition)*
To get orthogonality condition $\int_R FQ_c = 0$, the parameter of c, a satisfy the following conditions

$$c'[(\frac{11}{10} + \frac{3}{5}c - \frac{1}{40}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{c}{2})^{\frac{3}{2}}}(\frac{9}{2} + 2c)] - \frac{a'}{a}[c(1+c)(\frac{11}{10} + \frac{3}{5}c) - \frac{2}{5}(1+c)^2(1+\frac{c}{2})] = 0.$$

Proof. Note that

$$F = (1 - \frac{1}{2}\partial_x^2)\frac{c'}{a}\Lambda Q_c - (1 - \frac{1}{2}\partial_x^2)\frac{a'}{a^2}cQ_c + \frac{a'}{a^2}(yQ_c^2)_y.$$

We just compute these three terms $\int_R (1 - \frac{1}{2}\partial_x^2)\Lambda QQ_c$, $\int_R (1 - \frac{1}{2}\partial_x^2)Q_cQ_c$ and $\int_R (yQ_c^2)_y Q_c$.

$$\int_R (1 - \frac{1}{2}\partial_x^2)\Lambda QQ_c \quad = \quad \int_R (1 - \frac{1}{2}\partial_x^2)Q_c\Lambda Q_c = \frac{1}{1+c}\int(\frac{1}{2+c}Q_c + \frac{1}{2+c}Q_c^2)(Q_c + \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{3}{2}}}yQ_c').$$

Note that $\int Q_c yQ_c' = \frac{1}{2}\int yd(Q_c^2) = -\frac{1}{2}\int Q_c^2$ and $\int Q_c^2 yQ_c' = \frac{1}{3}\int yd(Q_c^3) = -\frac{1}{3}\int Q_c^3$.
So

$$\int_R (1 - \frac{1}{2}\partial_x^2)\Lambda QQ_c = \frac{1}{1+c}[\int(\frac{1}{2+c} - \frac{1}{8}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{5}{2}}})Q_c^2 + (\frac{1}{2+c} - \frac{1}{24}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{5}{2}}})Q_c^3.$$

$$\int_R (1 - \frac{1}{2}\partial_x^2)Q_cQ_c = \int[Q_c(Q_c - Q_c'')] = \int \frac{1}{2+c}Q_c^2 + \frac{1}{2+c}Q_c^3).$$

$$\int_R (yQ_c^2)_y Q_c = \int_R Q_c d(yQ_c^2) = -\int_R yQ_c^2 d(Q_c) = \frac{1}{3}\int_R Q_c^3.$$

So put these three parts together and from Lemma 1 and Lemma 2 to get orthogonality condition, we impose

$$c'[(\frac{11}{10} + \frac{3}{5}c - \frac{1}{40}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{c}{2})^{\frac{3}{2}}}(\frac{9}{2} + 2c)] - \frac{a'}{a}[c(1+c)(\frac{11}{10} + \frac{3}{5}c) - \frac{2}{5}(1+c)^2(1+\frac{c}{2})] = 0.$$

■

Lemma 11

$$\beta = \frac{1}{2}\sqrt{\frac{1+\frac{c}{2}}{1+c}}\int_R F = \frac{1}{2}\sqrt{\frac{1+\frac{c}{2}}{1+c}}(\frac{c'}{a(1+c)}(1 - \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{c}{2})^{\frac{3}{2}}}) - \frac{a'}{a^2}c)\int Q_c.$$

Proof.

$$F = (1-\frac{1}{2}\partial_x^2)\frac{c'}{a}\Lambda Q_c - (1-\frac{1}{2}\partial_x^2)\frac{a'}{a^2}cQ_c + \frac{a'}{a^2}(yQ_c^2)_y.$$

We just compute the $\int (1-\frac{1}{2}\partial_x^2)\Lambda Q_c$, $\int (1-\frac{1}{2}\partial_x^2)Q_cand\int (yQ_c^2)_y$. Because a, c is independent of x.

$$\int (1-\frac{1}{2}\partial_x^2)\Lambda Q_c = \frac{1}{1+c}\int (1-\frac{1}{2}\partial_x^2)(Q_c + \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{3}{2}}}yQ_c') = \frac{1}{1+c}[\int Q_c - \frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{1}{2}c)^{\frac{3}{2}}}\int Q_c].$$

Due to $\int (1-\frac{1}{2}\partial_x^2)Q_c = \int Q_c - \frac{1}{2}\int Q_c'' = \int Q_c$ and $\int (yQ_c^2)_y = 0$.

$$\beta = \frac{1}{2}\sqrt{\frac{1+\frac{c}{2}}{1+c}}(\frac{c'}{a(1+c)}(1-\frac{1}{4}\frac{(1+c)^{\frac{1}{2}}}{(1+\frac{c}{2})^{\frac{3}{2}}}) - \frac{a'}{a^2}c)\int Q_c.$$

■

Lemma 12

$$\delta = -\beta\sqrt{\frac{1+c}{1+\frac{1}{2}c}}.$$

Proof. Finally, to get $\lim_{y\to+\infty} A_c = 0$ by Lemma 1, we choose $\delta = -\beta\sqrt{\frac{1+c}{1+\frac{1}{2}c}}$.

■

According to Lemma 9, Lemma 10, and Lemma 11, We have $A_c = \beta(\phi_c - \sqrt{\frac{1+c}{1+\frac{1}{2}c}}) + A_1(y)$, $A_1 \in Y$, this finishes the proof of Lemma 8.This proves the problem of Ω.

3.3 Correction to the solution of problem of (Ω)

Consider the cutoff function $\eta \in C^\infty(R)$ satisfying the following properties,

$$\begin{cases} 0 \le \eta(s) \le 1, 0 \le \eta'(s) \le 1, \forall s \in R, \\ \eta(s) \equiv 0, \forall s \le -1, \\ \eta(s) \equiv 1, \forall s \ge 1. \end{cases} \tag{3.8}$$

Define

$$\eta_\varepsilon(y) = \eta(\varepsilon y + 2). \tag{3.9}$$

And for $A_c = A_c(\varepsilon t, y)$ solution of Ω , denote

$$A_\# = \eta_\varepsilon A_c(\varepsilon t, y). \tag{3.10}$$

Now redefine

$$\tilde{u} = R + w = R + \varepsilon A_\#. \tag{3.11}$$

The following Proposition, which deals with the error associated to the cutoff function and the new approximate solution $\tilde{u}$, is the main result.

Proposition 13 *There exist constants $\varepsilon_0, K > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the following holds.*
(i) (a)New behavior. For all $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\begin{cases} A_\#(\varepsilon t, y) = 0, \forall y \le -\frac{3}{\varepsilon}, \\ A_\#(\varepsilon t, y) = A_c(\varepsilon t, y), \forall y \ge \frac{1}{\varepsilon}. \end{cases} \tag{3.12}$$

(b)Integrable solution. For all $t \in [-T_\varepsilon, T_\varepsilon]$,$A_\# \in H^1(R)$ with

$$\|\varepsilon A_\#\|_{H^1(R)} \le K\varepsilon^{\frac{1}{2}}e^{-\gamma\varepsilon|t|}. \tag{3.13}$$

(ii)The error associated to the new function $\tilde{u}$ satisfies

$$\|S[\tilde{u}]\|_{H^2(R)} \le K\varepsilon^{\frac{3}{2}}e^{-\gamma\varepsilon|t|}. \tag{3.14}$$

and the following integral estimate holds

$$\int_R \|S[\tilde{u}]\|_{H^2(R)}\, dt \leq K\varepsilon^{\frac{1}{2}}. \tag{3.15}$$

Proof. The proof of first part of this proposition is similar to proof of proposition 13 in [5], so we omit this part. We will prove the second part of this proposition in the next Lemma. ∎

Lemma 14

$$S[\tilde{u}] = I + II' + III' \quad .$$

where

$$II' = -\varepsilon\eta_c(LA_c)_y + O_{H^2(R)}(\varepsilon^{\frac{3}{2}}e^{-\gamma\varepsilon|t|}) \quad , \quad \|III'\|_{H^2(R)} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}.$$

Proof.

$$III' = \{a(\varepsilon x)w^2\}_x = \varepsilon^2(a(\varepsilon x)\eta_c^2 A_c^2)_x = \varepsilon^3 a'(\varepsilon x)\eta_c^2 A_c^2 + 2\varepsilon^3 a(\varepsilon x)\eta_c\eta_c' A_c^2 + 2\varepsilon^2 a(\varepsilon x)\eta_c^2 A_c A_c'.$$

With $A_c, A_c' \in Y$,$\|\eta_c'\|_{L^2(R)} \leq K\varepsilon^{-1/2}$, uniformly $t \in [-T_\varepsilon, T_\varepsilon]$.Moreover ,we have the estimate

$$\|III'\|_{H^2(R)} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}.$$

Note that

$$\begin{aligned}(1-\frac{1}{2}\partial_x^2)(\varepsilon A_\#)_t &= \varepsilon(1-\frac{1}{2}\partial_x^2)(-\varepsilon c\eta_\varepsilon' A_c - c(A_c)_y\eta_\varepsilon + \varepsilon(A_c)_t\eta_\varepsilon - \varepsilon\eta_\varepsilon\Lambda A_c c')\\ &= \varepsilon^2(1-\frac{1}{2}\partial_x^2)(-c\eta_\varepsilon' A_c + (A_c)_t\eta_\varepsilon - \eta_\varepsilon\Lambda A_c c') - \varepsilon(1-\frac{1}{2}\partial_x^2)(c(A_c)_y\eta_\varepsilon).\end{aligned}$$

Thus

$$\begin{aligned}&((\varepsilon A_\#)_{xx} - \varepsilon A_\# + 2a_\varepsilon\varepsilon A_\# R)_x\\ =\ & ((\varepsilon\eta_\varepsilon A_c)_{xx} - \varepsilon\eta_\varepsilon A_c + 2\varepsilon a_\varepsilon R\eta_\varepsilon A_c)_x\\ =\ & \varepsilon[(\eta_\varepsilon(A_c)_{yy} + 2\varepsilon\eta_\varepsilon'(A_c)_y + \varepsilon^2\eta_\varepsilon'' A_c) - \varepsilon\eta_\varepsilon A_c + 2\varepsilon a_\varepsilon R\eta_\varepsilon A_c]_x\\ =\ & \varepsilon[\eta_\varepsilon((A_c)_{yy} - A_c + 2\frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c A_c)]_x + \varepsilon^2(2\eta_\varepsilon'(A_c)_y + \varepsilon\eta_\varepsilon'' A_c)_y\\ =\ & \varepsilon\eta_\varepsilon[(A_c)_{yy} - A_c + 2\frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c A_c]_x + \varepsilon^2\eta_\varepsilon'((A_c)_{yy} - A_c + 2\frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c A_c)\\ +\ & \varepsilon^2(2\varepsilon\eta_\varepsilon''(A_c)_y + 2\eta_\varepsilon'(A_c)_{yy} + \varepsilon\eta_\varepsilon''(A_c)_y + \varepsilon^2\eta_\varepsilon''' A_c)\\ =\ & \varepsilon\eta_\varepsilon[(A_c)_{yy} - A_c + 2\frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c A_c]_x + \varepsilon^2(3\varepsilon\eta_\varepsilon''(A_c)_y + 3\eta_\varepsilon'(A_c)_{yy} + \varepsilon^2\eta_\varepsilon''' A_c - A_c + 2\frac{a(\varepsilon x)}{a(\varepsilon\rho)}Q_c A_c)\\ =\ & \varepsilon\eta_\varepsilon[(A_c)_{yy} - A_c + 2Q_c A_c]_x + 2\varepsilon^2\eta_\varepsilon\frac{a'}{a}(yQ_c A_c)_y\\ +\ & \varepsilon^2(3\varepsilon\eta_\varepsilon''(A_c)_y + 3\eta_\varepsilon'(A_c)_{yy} + \varepsilon^2\eta_\varepsilon''' A_c - \eta_\varepsilon' A_c + 2\eta_\varepsilon' Q_c A_c) + O(\varepsilon^3\eta_\varepsilon(y^2 Q_c A_c)_y).\end{aligned}$$

From the (IP) property to estimate as follows

$$\left\|2\varepsilon^2\eta_\varepsilon\frac{a'}{a}(yQ_c A_c)_y\right\|_{H^2(R)} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|} \left\|O(\varepsilon^3\eta_\varepsilon(y^2 Q_c A_c)_y)\right\|_{H^2(R)} \leq K\varepsilon^3.$$

$$\left\|\varepsilon^4\eta_\varepsilon''' A_c\right\|_{H^2(R)} \leq \varepsilon^{\frac{7}{2}}e^{-\gamma\varepsilon|t|}, \left\|\varepsilon^2\eta_\varepsilon' A_c\right\|_{H^2(R)} \leq K\varepsilon^{\frac{3}{2}}e^{-\gamma\varepsilon|t|}.$$

$$\left\|\varepsilon^2(3\varepsilon\eta_\varepsilon''(A_c)_y + 3\eta_\varepsilon'(A_c)_{yy} + 2\eta_\varepsilon' Q_c A_c)\right\|_{H^2(R)} \leq K\varepsilon^2 e^{-\gamma\varepsilon|t|}.$$

Therefore

$$((\varepsilon A_{\#})_{xx} - \varepsilon A_{\#} + 2a_\varepsilon \varepsilon A_{\#} R)_x = \varepsilon\eta_\varepsilon[(A_c)_{yy} - A_c + 2Q_c A_c]_x + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|} + \varepsilon^3).$$

So, we get

$$II' = -\varepsilon\eta_c(LA_c)_y + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}).$$

■

Note that

$$S[\tilde{u}] = \varepsilon[F - \eta_\varepsilon(LA_c)_y] + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}) = \varepsilon(1-\eta_\varepsilon)F + O_{H^2(R)}(\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}).$$

For every $t \in [-T_\varepsilon, T_\varepsilon]$, $1 - \eta_\varepsilon \subseteq (-\infty, -\frac{1}{\varepsilon})$, $\|F\|_{H^2(R)} \leq Ke^{-\gamma|y|-\gamma\varepsilon|t|}$.
So we gain

$$\|\varepsilon(1-\eta_\varepsilon)F\|_{H^2(R)} \leq Ke^{-\frac{1}{\varepsilon}-\gamma\varepsilon|t|} << K\varepsilon^{10}.$$

$$\|S[\tilde{u}]\|_{H^2(R)} \leq K\varepsilon^{\frac{3}{2}} e^{-\gamma\varepsilon|t|}.$$

(3.15) is just from integration of the formula of (3.14).

$$\int_R \|S[\tilde{u}]\|_{H^2(R)}\, dt \leq K\varepsilon^{\frac{1}{2}}.$$

This finishes the second proof of Proposition 3.

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