ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.19(2015) No.2,pp.113-120

Approximate Solutions of Fractional Linear and Nonlinear Differential Equations Using Laplace Homotopy Analysis Method

V. G. Gupta[1] *, Pramod Kumar[2]
[1] Department of Mathematics, University of Rajasthan, Jaipur-302004, India
[2] Department of Mathematics, Jaipur National University, Jaipur-302024, India
(Received 2 April 2012 , accepted 10 March 2013)

Abstract: In this paper, a Laplace homotopy analysis method which is based on homotopy analysis method and Laplace transform is applied to obtain the approximate solutions of fractional linear and non-linear differential equations. The proposed algorithm presents a procedure of constructing the set of base functions and gives a high-order deformation equation in simple form. Numerical examples are included to illustrate preciseness and effectiveness of the proposed method.

Keywords: fractional differential equations; Laplace transform; homotopy analysis method; Laplace homotopy analysis method

1 Introduction

Fractional order ordinary and partial differential equations as generalizations of classical integer order ordinary and partial differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, engineering and others. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [11]. Consequently, considerable attention has been given to the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest [4, 10, 12, 13, 17, 18, 20]. Most non-linear fractional differential equations do not have exact analytic solutions, so approximate and numerical technique must be used. In recent years, much study has been focused on the numerical solution of fractional differential equations. Some numerical methods have been developed, such as Laplace transform method [12, 18], differential transform method [1, 19, 26], Variation iteration method [16], Adomian decomposition method [21, 25], homotopy perturbation method [14, 15], homotopy perturbation transform method [6]. However, the region of convergence of the corresponding result is rather small as shown in this paper. Recently, Liao [7] proposed a powerful analytical method, namely the homotopy analysis method, for solving linear and non-linear differential and integral equations. Different from perturbation techniques, the HAM does not depend upon any small or large parameters. A systematic and clear exposition on HAM is given in [8]. This method has been successfully applied to solve many types of non-linear problems such as non-linear Riccati differential equation with fractional order [3], non-linear Vaknnenko equation [23], the Glauert-jet problem [2], fractional KdV-Bergers Kuromoto equation [22] and so on.

The objective of the present paper is to apply the modified homotopy analysis method [27] to provide symbolic approximate solutions for linear and non-linear fractional initial value problems. The Laplace homotopy analysis method is a combination of HAM and Laplace transform. This method is characterized by choosing the identity auxiliary linear operator. The organization of this paper is as follows: A brief review of the fractional calculus is given in next section. The Laplace homotopy analysis method is presented in section 3; five numerical examples are given to show the applicability of the considered method in section 4.

*Corresponding author. *E-mail address*: pramodgupta472@gmail.com

IJNS.2015.04.15/859

2 Basic definitions

For the concept of fractional derivatives, we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variable and their integral order which is the case in most physical processes. Some basic definitions and properties of fractional calculus theory which we have used in this paper are given in this section.

Definition 1 *A real function f(x), x ¿ 0 is said to be in the space C_μ, $\mu \in R$, if there exist a real number p (¿ μ) such that f(x) = $x^p\, f_1(x)$, where $f_1(x) \in C\,[0,\infty)$, and it is said to be in the space* $\mathbb{C}_\mu^m iff f^m \in \mathbb{C}_\mu,\ m \in N \cup \{0\}$.

Definition 2 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as*

$$\mathbb{J}^\alpha f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1}\,℧(≈)≈, \alpha > 0,\ ↶ > 0 \quad \ldots \tag{1}$$

$$\mathbb{J}^0 f(x) = f(x) \quad \ldots \tag{2}$$

Properties of the operator $\mathbb{J}^\alpha$ *can be found in [18], we mention only the following:*
(i) $\mathbb{J}^\alpha\, \mathbb{J}^\beta\, f(x) = \mathbb{J}^{\alpha+\beta}\, f(x)$
(ii) $\mathbb{J}^\alpha\, \mathbb{J}^\beta\, f(x) = \mathbb{J}^\beta\, \mathbb{J}^\alpha\, f(x)$
(iii) $\mathbb{J}^\alpha\, x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\, x^{\alpha+\gamma}$
For $℧ \in \mathbb{C}_\mu,\ \mu \geq -1,\ \alpha,\beta \geq 0∂⋉\gamma > -1.$

Definition 3 *The fractional derivative of f(x) in the Caputo sense is defined as [4]*

$$\mathbb{D}_*^\alpha f(x) = \mathbb{J}^{m-\alpha}\, \mathbb{D}_*^m f(x) = \frac{1}{\Gamma(⋗-\alpha)}\int_0^x (x-t)^{m-\alpha-1}℧^m(≈)≈ \quad \ldots \tag{3}$$

For $m-1 < \alpha \leq ⋗, ⋗ \in \mathbb{N}, ↶ > ⊬, ℧ \in \mathbb{C}_{-1}^m.$

Also, we need here three basic properties
(i) $\mathbb{D}_*^\alpha \mathbb{J}^\alpha f(x) = ℧(↶)$
(ii) $\mathbb{J}^\alpha \mathbb{D}_*^\alpha\, f(x) = ℧(↶) - \sum_{⅂=0}^{m-1} ℧^{(⅂)}(0^+)\, \frac{x^⅂}{⅂!}, \qquad ↶ > 0$
(iii) $\mathbb{D}_*^\alpha ↶^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\, x^{\gamma-\alpha}\ ;\ ↶ > ⊬, \gamma > 0.$
For $m-1 < \alpha \leq ⋗, ⋗ \in \mathbb{N}, \mu \geq -1∂⋉℧ \in \mathbb{C}_\mu^m.$

Lemma 1 *If* $m-1 < \alpha \leq ⋗, ⋗ \in \mathbb{N}$, *then the Laplace transform of the fractional derivative* $\mathbb{D}_*^\alpha ℧(≈)$ *is*

$$\mathbb{L}(\mathbb{D}_*^\alpha ℧(≈)) = \sim^\alpha \bar{℧}(\sim) - \sum_{⅂=0}^{m-1} ℧^{(⅂)}(0^+)\sim^{\alpha-⅂-1},\ ≈ > 0 \quad \ldots, \tag{4}$$

where $\bar{℧}(\sim)$ *is the Laplace transform of f (t).*

3 Laplace homotopy analysis method

The homotopy analysis method which provides an analytical approximate solution has been applied to various non-linear problems by many workers. In this section, we apply the modified homotopy analysis method [27]. This modified method is based on the Laplace transform of the fractional derivatives. To illustrate the basic idea of this method, let us consider the following fractional differential equation

$$\mathbb{D}_*^\alpha ≊(≈) = ℧(≈, ≊(≈), ≊'(≈)), \qquad ≈ \geq ⊬,\ 1 < \alpha \leq 2 \quad \ldots \tag{5}$$

Subject to the initial conditions

$$≊(⊬) = {}_1∂⋉≊'(⊬) = {}_2 \quad \ldots \tag{6}$$

Applying the Laplace transform on both sides of (5), we get

$$\mathbb{L}(\mathbb{D}_*^\alpha\, ≊(≈)) = \mathbb{L}(℧(≈, ≊(≈), ≊'(≈)) \quad \ldots \tag{7}$$

Using (4), we have

$$\sim^{\alpha} \bar{\approxeq}(\sim) - \sim^{\alpha-1} \approxeq(\nvdash) - \sim^{\alpha-2} \approxeq'(\nvdash) = \mathbb{L}(\mho(\approx, \approxeq(\approx), \approxeq'(\approx)) \quad \ldots \tag{8}$$

$$\bar{\approxeq}(\sim) = \frac{1}{\sim} + \frac{2}{\sim^2} + \frac{1}{\sim^{\alpha}} \mathbb{L}(\mho(\approx, \approxeq(\approx), \approxeq'(\approx)) \quad \ldots, \tag{9}$$

where $\mathbb{L}(\approxeq(\approx)) = \bar{\approxeq}(\sim)$.

The so called deformation of Laplace equation (9) has the form

$$(1 - \shortparallel)[\bar{\phi}(\sim, \shortparallel) - \bar{\approxeq}_0(\sim)] = \shortparallel \eqsim \left[\bar{\phi}(\sim, \shortparallel) - \frac{1}{\sim} - \frac{2}{\sim^2} - \frac{1}{\sim^{\alpha}} \mathbb{L}(\mho(\approx, \phi(\approx, \shortparallel), \underset{\approx}{-} \phi(\approx, \shortparallel)) \right]$$

$\ldots$,(10) where q $\in$ [0, 1] is an embedding parameter, when q = 0 and q = 1, we have $\bar{\phi}(\sim, \nvdash) = \bar{\approxeq}_0(\sim) \eth \ltimes \bar{\phi}(\sim, \nVdash) = \bar{\approxeq}(\sim)$.

Thus, as q increases from 0 to 1, $\bar{\phi}(\sim, \shortparallel)$ varies from $\bar{\approxeq}_0(\sim) \approx \rtimes \bar{\approxeq}(\sim)$.Expanding $\bar{\phi}(\sim, \shortparallel)$ in Taylor series with respect to q, one has

$$\bar{\phi}(\sim, \shortparallel) = \bar{\approxeq}_0(\sim) + \sum_{m=1}^{\infty} \bar{\approxeq}_m(\sim) \shortparallel^m \quad \ldots, \tag{10}$$

where

$$\bar{\approxeq}_m(\sim) = \frac{1}{\gtrdot!} \frac{\partial^m \bar{\phi}(\sim, \shortparallel)}{\partial \shortparallel^m} \bigg|_{\shortparallel=0} \quad \ldots \tag{11}$$

If the auxiliary parameter h and initial guesses $\bar{\approxeq}_0(\sim)$are so properly chosen, then the series (10) is converses at q = 1 and one has

$$\bar{\approxeq}(\sim) = \bar{\approxeq}_0(\sim) + \sum_{m=1}^{\infty} \bar{\approxeq}_m(\sim) \quad \ldots \tag{12}$$

Define the vectors

$$\vec{\bar{\approxeq}}_m(\sim) = \{\bar{\approxeq}_0(\sim), \bar{\approxeq}_1(\sim), ..., \bar{\approxeq}_m(\sim)\} \quad \ldots \tag{13}$$

Differentiating equation (10) m times with respect to q, and then setting q = 0, h = -1 and finally dividing them by m!, we have the so-called mth -order deformation equation

$$\bar{\approxeq}_m(\sim) = \chi_m \bar{\approxeq}_{m-1} - \mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim)) \quad \ldots, \tag{14}$$

where

$$\mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim)) = \bar{\approxeq}_{m-1}(\sim) - \frac{1}{\sim^{\alpha}} \left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial \shortparallel^{m-1}} \{\mathbb{L}(\mho(\approx, \phi(\approx, \shortparallel), \right.$$

$$\left. \underset{\approx}{-} \phi(\approx, \shortparallel))\}_{\shortparallel=0} \right] - \left(\frac{1}{\sim} + \frac{2}{\sim^2} \right) (1 - \chi_m) \quad \ldots \tag{15}$$

and $\chi_m = \begin{cases} 0 \ , \ m \leq 1 \\ 1 \ , \ m > \nVdash \end{cases} \ldots (17)$

Applying the inverse Laplace transforms of (14), we have a power series solution $\approxeq(\approx) = \sum_{\sqsupset=0}^{\infty} \approxeq_{\sqsupset}(t)$ of (5).

4 Numerical examples

In order to assess the advantage and the accuracy of the Laplace homotopy analysis method presented in this paper for fractional linear and non-linear differential equations, we applied it to the following problems:

Example 1 $\mathbb{D}^{\alpha} \approxeq = - \approxeq \quad ; \nvdash < \alpha \ \leq \ 2 \ldots$ *(18)*

Subject to the initial conditions

$$\approxeq(\nvdash) = \nVdash, \approxeq'(\nvdash) = 0 \quad \ldots \tag{16}$$

Taking the Laplace transform of both sides of equation (18) and using (16), we have

$$\mathbb{S}^{\alpha} \bar{\approxeq}(\sim) - \sim^{\alpha-1} = - \bar{\approxeq}(\sim)$$

$$*\,\approxeq(\sim) + \frac{1}{\sim^{\alpha}}\approxeq(\sim) - \frac{1}{\sim} \,=\, 0$$

In view of the equation (14) and (15), we have

$$\bar{\approxeq}_m(\sim) \,=\, \chi_m\,\bar{\approxeq}_{m-1}(\sim) - \mathbb{R}_m[\vec{\bar{\approxeq}}_{m-1}(\sim)],$$

where

$$\mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim) = \bar{\approxeq}_{m-1}(\sim) + \frac{1}{\sim^{\alpha}}\bar{\approxeq}_{m-1}(\sim) - \frac{1}{\sim}\,(1-\chi_m)$$

So

$$\bar{\approxeq}_m(\sim) \,=\, \chi_m\,\bar{\approxeq}_{m-1}(\sim) - \left[\bar{\approxeq}_{m-1}(\sim) + \frac{1}{\sim^{\alpha}}\,\bar{\approxeq}_{m-1}(\sim) - \frac{1}{\sim}(1-\chi_m)\right] \quad \ldots \tag{17}$$

$$\bar{\approxeq}_0(\sim) = \frac{1}{\sim}, \quad \bar{\approxeq}_1(\sim) = -\,\frac{1}{\sim^{\alpha}+1}, \quad \bar{\approxeq}_2(\sim) = \frac{1}{\sim^{2\alpha+1}}, \quad \bar{\approxeq}_{\nvDash}(\sim) = -\,\frac{1}{\sim^{\nvDash\alpha+1}}$$

$$\bar{\approxeq}(\sim) = \bar{\approxeq}_0(\sim) + \bar{\approxeq}_1(\sim) + \bar{\approxeq}_2(\sim) + \bar{\approxeq}_{\nvDash}(\sim) + \ldots$$

$$\bar{\approxeq}(\sim) = \frac{1}{\sim} - \frac{1}{\sim^{\alpha+1}} + \frac{1}{\sim^{2\alpha+1}} - \frac{1}{\sim^{3\alpha+1}} \quad \ldots \tag{18}$$

Now taking the inverse Laplace transform of both sides of (18), we have

$$\approxeq(\approx) = \nvDash - \frac{\approx^{\alpha}}{\Gamma(\alpha+1)} + \frac{\approx^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\approx^{\nvDash\alpha}}{\Gamma(3\alpha+1)} + \ldots$$

$$\approxeq(\approx) = \sum_{\ell=0}^{\infty}\frac{(-\approx^{\alpha})^{\ell}}{\Gamma(\alpha\ell+1)} \quad \ldots \tag{19}$$

This is the exact series solution of (18).

Example 2 *Let us consider the following non-linear fractional initial value problem*

$$\mathbb{D}^{\alpha}\approxeq = \approxeq^2 + 1 \qquad 0 \,<\, \alpha \,\leq\, 1, \quad 0 \,<\, \approx < \nvDash \quad \ldots \tag{20}$$

Subject to the initial condition

u (0) = 0 . . . (25)

Taking the Laplace transform of equation (20) and using (25), we have

$$\bar{\approxeq}(\sim) = \frac{1}{\sim^{\alpha}}\mathbb{L}[\approxeq^2] + \frac{1}{\sim^{\alpha+1}} \quad \ldots \tag{21}$$

In view of equation (14) and (15), we have

$$\bar{\approxeq}_m(\sim) = \chi_m\,\bar{\approxeq}_{m-1}(\sim) - \mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim)),$$

where

$$\mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim)) = \bar{\approxeq}_{m-1}(\sim) - \frac{1}{\sim^{\alpha}}\mathbb{L}\left[\sum_{\daleth=0}^{m-1}\approxeq_{\daleth}\approxeq_{m-1-\daleth}\right] - \frac{1}{\sim^{\alpha+1}}(1-\chi_m) \quad \ldots \tag{22}$$

$$\bar{\approxeq}_0(\sim) = \frac{1}{\sim^{\alpha+1}}, \quad \bar{\approxeq}_1(\sim) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}\,\frac{1}{\sim^{\nvDash\alpha+1}}, \quad \bar{\approxeq}_2(\sim) = \frac{2\,\Gamma(2\alpha+1)\,\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\,\Gamma(3\alpha+1)}\,\frac{1}{\sim^{\ntrianglelefteq\alpha+1}}$$

$$\bar{\approxeq}(\sim) = \bar{\approxeq}_0(\sim) + \bar{\approxeq}_1(\sim) + \bar{\approxeq}_2(\sim) + \ldots \quad \ldots \tag{23}$$

Taking the inverse Laplace transform of (23), we have HAM series solution of the initial value problem (20)

$$\approxeq(\approx) = \frac{\approx^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\,\frac{1}{\Gamma(\alpha+1)^2}\approx^{\nvDash\alpha} + \frac{\Gamma(2\alpha+1)\,\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\,\Gamma(5\alpha+1)}\approx^{5\alpha} \quad \ldots \tag{24}$$

For α = 1, the exact solution is u (t) = tan (t).

Example 3 *Consider the following time fractional advection non-homogeneous equation [6]*

$$\mathbb{D}^{\alpha}_{\approx}\approxeq(\curvearrowleft,\approx)+\approxeq\approxeq_x=\nvDash\approx+x+\approx^{\nVdash}+\curvearrowleft\approx^2 \qquad ; \quad 0\ <\ \alpha\ \le\ 1 \quad \ldots \tag{25}$$

Subject to the initial condition

$$\approxeq(\curvearrowleft,\nvdash)=0 \quad \ldots \tag{26}$$

Taking the Laplace transform of both sides of equation (25) and using (26), we have

$$\bar{\approxeq}(\curvearrowleft,\sim)=\frac{1}{\sim^{\alpha}}\left(\frac{2}{\sim^2}+\frac{x}{\sim}+\frac{6}{\sim^{\ntrianglerighteq}}+\frac{\nvDash\curvearrowleft}{\sim^{\nVdash}}\right)-\frac{1}{\sim^{\alpha}}\mathbb{L}(\approxeq\approxeq_x) \quad \ldots \tag{27}$$

In view of the equation (14) and (15),we have

$$\bar{\approxeq}_m(\sim)=\chi_m\,\bar{\approxeq}_{m-1}(\sim)-\mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim)),$$

where

$$\mathbb{R}_m(\vec{\bar{\approxeq}}_{m-1}(\sim))=\bar{\approxeq}_{m-1}(\curvearrowleft,\sim)+\frac{1}{\sim^{\alpha}}\mathbb{L}\left[\sum_{\daleth=0}^{m-1}\approxeq_{\daleth}\approxeq'_{m-1-\daleth}\right]$$
$$-\frac{1}{\sim^{\alpha}}\left[\frac{2}{\sim^2}+\frac{x}{\sim}+\frac{6}{\sim^{\ntrianglerighteq}}+\frac{\nvDash\curvearrowleft}{\sim^{\nVdash}}\right](1-\chi_m)$$

So $\bar{\approxeq}_0(\curvearrowleft,\sim)=\frac{2}{\sim^{\alpha+2}}+\frac{x}{\sim^{\alpha+1}}+\frac{6}{\sim^{\alpha+4}}+\frac{2x}{\sim^{\alpha+3}}$

$$\bar{\approxeq}_1(\curvearrowleft,\sim)=-\left[\frac{2\,\Gamma(\nvDash\alpha+\nvDash)}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\frac{1}{\sim^{\nVdash\alpha+2}}+\frac{x\,\Gamma(\nvDash\alpha+\nVdash)}{\Gamma(1+\alpha)^2}\frac{1}{\sim^{3\alpha+1}}+\frac{6\,\Gamma(2\alpha+4)}{\Gamma(\alpha+1)\Gamma(\alpha+4)}\frac{1}{\sim^{\nVdash\alpha+4}}\right.$$
$$\left.+\frac{4\Gamma(2\alpha+4)}{\Gamma(\alpha+2)\Gamma(\alpha+3)}\frac{1}{\sim^{\nVdash\alpha+4}}+\frac{\ntrianglerighteq\curvearrowleft\Gamma(2\alpha+3)}{\Gamma(1+\alpha)\Gamma(3+\alpha)}\frac{1}{\sim^{\nVdash\alpha+3}}+\frac{12\,\Gamma(2\alpha+6)}{\Gamma(\alpha+3)\Gamma(\alpha+4)}\frac{1}{\sim^{\nVdash\alpha+6}}+\frac{\ntrianglerighteq\curvearrowleft\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^2}\frac{1}{\sim^{\nVdash\alpha+5}}\right]$$

So

$$\bar{\approxeq}(\curvearrowleft,\sim)=\bar{\approxeq}_0(\curvearrowleft,\sim)+\bar{\approxeq}_1(\curvearrowleft,\sim)+... \tag{28}$$

Now taking the inverse Laplace transform of both sides of equation (28)*, we have*

$$\approxeq(\curvearrowleft,\approx)=\left(\frac{2}{\Gamma(\alpha+\nvDash)}\approx^{\alpha+1}+\frac{x}{\Gamma(\alpha+1)}\approx^{\alpha}+\frac{\ntriangleleft}{\Gamma(\alpha+4)}\approx^{\alpha+3}+\frac{\nvDash\curvearrowleft}{\Gamma(\alpha+3)}\approx^{\alpha+2}\right)$$
$$-\left[\frac{\curvearrowleft\Gamma(2\alpha+1)}{\Gamma(1+\alpha)^2\Gamma(3\alpha+1)}\approx^{\nVdash\alpha}+\frac{2\Gamma(2\alpha+4)}{\Gamma(\alpha+1)\Gamma(\alpha+2)(3\alpha+2)}\approx^{\nVdash\alpha+1}\right.$$
$$+\frac{6\,\Gamma(2\alpha+4)}{\Gamma(\alpha+1)\Gamma(\alpha+4)\Gamma(3\alpha+4)}\approx^{\nVdash\alpha+3}+\frac{4\Gamma(2\alpha+4)}{\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(3\alpha+4)}\approx^{\nVdash\alpha+3}$$
$$+\frac{\ntrianglerighteq\curvearrowleft\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}\approx^{\nVdash\alpha+2}+\frac{12\,\Gamma(2\alpha+6)}{\Gamma(\alpha+3)\Gamma(\alpha+4)\Gamma(3\alpha+6)}\approx^{\nVdash\alpha+5}$$
$$\left.+\frac{\ntrianglerighteq\curvearrowleft\Gamma(2\alpha+5)}{\Gamma(\alpha+3)\Gamma(3\alpha+5)}\approx^{\nVdash\alpha+4}\right]+... \quad \ldots \tag{29}$$

If we take α = 1, then the exact solution is

$$\approxeq(\curvearrowleft,\approx)=\approx^2+\curvearrowleft\approx.$$

This was given in [6].

Example 4 *Consider the following time space fractional non-linear Fokker-Planck equation [24]*

$$\mathbb{D}^{\alpha}_{\approx}\approxeq(\curvearrowleft,\approx)=-\,\mathbb{D}^{\beta}_{x}\left(\frac{\ntrianglerighteq\approxeq^2}{x}-\frac{x^{\ntrianglerighteq}}{3}\right)+\mathbb{D}^{2\beta}_{x}(\approxeq^2) \qquad ; \quad 0\ <\ \alpha,\beta\ \le\ 1. \quad \ldots \tag{30}$$

Subject to the initial condition

$$\approxeq(\curvearrowleft,\nvdash)=\curvearrowleft^2 \qquad \ldots \tag{31}$$

Taking the Laplace transform of both sides of equation (30) and using (31), we have

$$\sim^{\alpha}\tilde{\approxeq}(\curvearrowleft,\sim)-\sim^{\alpha-1}x^2=\mathbb{L}\left[-\mathbb{D}_x^{\beta}\left(\frac{\ntrianglerighteq\approxeq^2}{x}-\frac{\curvearrowleft\approxeq}{\nVdash}\right)+\mathbb{D}_x^{2\beta}(\approxeq^2)\right]$$

$$\tilde{\approxeq}(\curvearrowleft,\sim)=\frac{1}{\sim}x^2+\frac{1}{\sim^{\alpha}}\mathbb{L}\left[-\mathbb{D}_*^{\beta}\left(\frac{\ntrianglerighteq\approxeq^2}{x}-\frac{x^{\ntrianglerighteq}}{3}\right)+\mathbb{D}_x^{2\beta}(\approxeq^2)\right]\quad\ldots \tag{32}$$

In view of equation (14) and (15), we have

$$\tilde{\approxeq}_m(\curvearrowleft,\sim)=\chi_m\tilde{\approxeq}_{m-1}(\curvearrowleft,\sim)-\mathbb{R}_m(\vec{\tilde{\approxeq}}_{m-1}(\curvearrowleft,\sim)),$$

where

$$\mathbb{R}_m(\vec{\tilde{\approxeq}}_{m-1}(\curvearrowleft,\sim))=\tilde{\approxeq}_{m-1}(\curvearrowleft,\sim)-\frac{1}{\sim^{\alpha}}\mathbb{L}\left[-\mathbb{D}_x^{\beta}\left\{\frac{4}{x}\sum_{\daleth=0}^{m-1}\approxeq_{\daleth}\approxeq_{m-1-\daleth}-\frac{x}{\nVdash}\approxeq_{m-1}\right\}\right.$$

$$\left.+\mathbb{D}_x^{2\beta}\sum_{\daleth=0}^{m-1}\approxeq_{\daleth}\approxeq_{m-1-\daleth}\right]-\frac{x^2}{\sim}(1-\chi_m)$$

So $\tilde{\approxeq}_m(\curvearrowleft,\sim)=\chi_m\tilde{\approxeq}_{m-1}(\curvearrowleft,\sim)-\left[\tilde{\approxeq}_{m-1}(\curvearrowleft,\sim)-\frac{1}{\sim^{\alpha}}\mathbb{L}\left(-\mathbb{D}_x^{\beta}\left\{\frac{4}{x}\sum_{\zeta=0}^{m-1}\approxeq_{\daleth}\approxeq_{m-1-\daleth}-\frac{x}{\nVdash}\approxeq_{m-1}\right\}\right.\right.$

$$\left.\left.+\mathbb{D}_x^{2\beta}\sum_{\daleth=0}^{m-1}\approxeq_{\daleth}\approxeq_{m-1-\daleth}\right)-\frac{x^2}{\sim}(1-\chi_m)\right]\quad\ldots \tag{33}$$

$$\tilde{\approxeq}_0(\curvearrowleft,\sim)=\frac{x^2}{\sim},\quad \tilde{\approxeq}_1(\curvearrowleft,\sim)=\frac{1}{\sim^{\alpha+1}}\left[\frac{\nvDash\ntrianglerighteq}{\Gamma(5-2\beta)}x^{\ntrianglerighteq-2\beta}-\frac{22}{\Gamma(4-\beta)}x^{\nVdash-\beta}\right]$$

$$\tilde{\approxeq}_2(\curvearrowleft,\sim)=\frac{1}{\sim^{2\alpha+1}}\left[-\frac{184\,\Gamma(6-2\beta)x^{\ntrianglelefteq-3\beta}}{\Gamma(6-3\beta)\Gamma(5-2\beta)}+\frac{48\Gamma(7-2\beta)}{\Gamma(7-4\beta)\Gamma(5-2\beta)}x^{\ntriangleleft-4\beta}\right.$$

$$\left.+\frac{506\Gamma(5-\beta)x^{\ntrianglerighteq-2\beta}}{3\Gamma(5-2\beta)\,\Gamma(4-\beta)}-\frac{44\,\Gamma(6-\beta)}{\Gamma(6-3\beta)\Gamma(4-\beta)}x^{\ntrianglelefteq-3\beta}\right]$$

$$\tilde{\approxeq}(\curvearrowleft,\sim)=\tilde{\approxeq}_0(\curvearrowleft,\sim)+\tilde{\approxeq}_1(\curvearrowleft,\sim)+\tilde{\approxeq}_2(\curvearrowleft,\sim)+\ldots\quad\ldots \tag{34}$$

Using above, and taking inverse Laplace transform of both sides of equation (34), we have

$$\approxeq(\curvearrowleft,\approx)=\curvearrowleft^2+\left[\frac{\nvDash\ntrianglerighteq\curvearrowleft^{\ntrianglerighteq-2\beta}}{\Gamma(5-2\beta)}-\frac{22x^{\nVdash-\beta}}{\Gamma(4-\beta)}\right]\frac{\approx^{\alpha}}{\Gamma(1+\alpha)}+\left[-\frac{184\,\Gamma(6-2\beta)}{\Gamma(6-3\beta)\Gamma(5-2\beta)}x^{\ntrianglelefteq-2\beta}\right.$$

$$\left.+\frac{48\,\Gamma(7-2\beta)}{\Gamma(7-4\beta)\Gamma(5-2\beta)}x^{\ntriangleleft-4\beta}+\frac{506\,\Gamma(5-\beta)x^{\ntrianglerighteq-2\beta}}{3\,\Gamma(5-2\beta)\Gamma(4-\beta)}-\frac{44\Gamma(6-\beta)x^{\ntrianglelefteq-3\beta}}{\Gamma(6-3\beta)\Gamma(4-\beta)}\right]\frac{\approx^{2\alpha}}{\Gamma(1+2\alpha)}+\ldots$$

If we take $\alpha=\beta=1$*, then we have*

$$\approxeq(\curvearrowleft,\approx)=\curvearrowleft^2+x^2\approx+\frac{x^2\approx^2}{2\,!}+\ldots$$

$$\approxeq(\curvearrowleft,\approx)=x^{2\approx}$$

This was given in [24].

Example 5 *Our last Example covers the inhomogeneous linear equation*

$$\mathbb{D}^{\alpha}\curvearrowright(\approx) = \frac{2}{\Gamma(3-\alpha)}\approx^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}\approx^{1-\alpha} - \curvearrowright(\approx) + \approx^2 - \approx \quad ; \; 0 < \; \alpha \leq \nVdash, \approx > 0 \quad \ldots \tag{35}$$

Subject to the initial conditions

$$y(0) = 0 \tag{36}$$

Taking the Laplace transform of the equation (35) and using (36), we have

$$\bar{\curvearrowright}(\backsim) = \frac{2}{\backsim^{\nVdash}} - \frac{1}{\backsim^{2}} - \frac{1}{\backsim^{\alpha}}\bar{\curvearrowright}(\backsim) + \frac{2}{\backsim^{\nVdash+\alpha}} - \frac{1}{\backsim^{2+\alpha}} \quad \ldots \tag{37}$$

In view of equation (14) and (15), we have

$$\bar{\curvearrowright}_m(\backsim) = \chi_m \, \bar{\curvearrowright}_{m-1}(\backsim) - \mathbb{R}_m(\vec{\bar{\approx}}_{m-1}(\backsim))$$

$$\mathbb{R}_m(\vec{\bar{\curvearrowright}}_{m-1}(\backsim)) = \bar{\curvearrowright}_{m-1}(\backsim) + \frac{1}{\backsim^{\alpha}}\bar{\curvearrowright}_{m-1}(\backsim) - \left(\frac{2}{\backsim^{\nVdash}} - \frac{1}{\backsim^{2}} + \frac{2}{\backsim^{\nVdash+\alpha}} - \frac{1}{\backsim^{2+\alpha}}\right)(1-\chi_m)$$

$$\bar{\curvearrowright}_m(\backsim) = \chi_m\bar{\curvearrowright}_{m-1}(\backsim) - \left[\bar{\curvearrowright}_{m-1}(\backsim) + \frac{1}{\backsim^{\alpha}}\bar{\curvearrowright}_{m-1}(\backsim) - \left(\frac{2}{\backsim^{\nVdash}} - \frac{1}{\backsim^{2}} + \frac{2}{\backsim^{\nVdash+\alpha}} - \frac{1}{\backsim^{2+\alpha}}\right)(1-\chi_m)\right]$$

$$\bar{\curvearrowright}_0(\backsim) = \frac{2}{\backsim^{\nVdash}} - \frac{1}{\backsim^{2}} + \frac{2}{\backsim^{\nVdash+\alpha}} - \frac{1}{\backsim^{2+\alpha}}$$

$$\bar{\curvearrowright}_1(\backsim) = -\left[\frac{2}{\backsim^{\nVdash+\alpha}} - \frac{1}{\backsim^{2+\alpha}} + \frac{2}{\backsim^{\nVdash+2\alpha}} - \frac{1}{\backsim^{2+2\alpha}}\right]$$

$$\bar{\curvearrowright}_2(\backsim) = -\left[\frac{2}{\backsim^{\nVdash+2\alpha}} - \frac{1}{\backsim^{2+2\alpha}} + \frac{2}{\backsim^{\nVdash+\nVdash\alpha}} - \frac{1}{\backsim^{2+3\alpha}}\right]$$

$$\bar{\curvearrowright}(\backsim) = \bar{\curvearrowright}_0(\backsim) + \bar{\curvearrowright}_1(\backsim) + \bar{\curvearrowright}_2(\backsim) + \ldots \quad .. \tag{38}$$

Taking the inverse Laplace transform of both sides of (38), we have the exact solution

$$\curvearrowright(\approx) = \approx^2 - \approx$$

5 Conclusion

In present paper, a Laplace homotopy analysis method which is based on homotopy analysis method and Laplace transform is used to solve linear and non-linear differential equations of fractional order. The non-linear terms can be easily handled by m-th order deformation equation of HAM. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical method while still maintaining the high accuracy of the result, the size reduction amounts to an improvement of the performance of the approach. However, the proposal approach can be further implemented to solve other linear and non-linear problems in the fractional calculus field.

References

[1] A. Al-rabtah, Ertürk, V. S. Momani S. Solutions of a fractional oscillator by using differential transform method. *Computers and Mathematics with Applications,* 59(3): (2010), 1356-1362.

[2] Y. Bouremel. Explicit Series Solution for the Glauert-jet problem by means of homotopy analysis method. *Int. J. Non-linear Sci. Numer. Simulat.*, 12(5) :(2007), 714-724.

[3] J. Cang, Y. Tan, H. Xu, S. Liao. Series solution of non-linear Riccati differential equations with fractional order, Chaos, Soliton. Fract. 40(1) (2009), 1-9.

[4] M. Caputo. Linear models of dissipation whose Q is almost frequency independent. *Part III, J. Roy. Astr. Soc.*, 13 :(1967), 529-539.

[5] F. R. Goren, F. Mainardi. Fractional Calculus : integral and differential equations of fractional order in : A. Carpinteri, F. Mainardi (Eds). *Fractals and Fractional calculus in Continuum Mechanics, Springer, New York*, (1997), 223-276.

[6] Y. Khan, B. Q. Wu. Homotopy perturbation transform method for non-linear equations using He's polynomials. *Computers and Mathematics with Applications,* 61: (2011), 1963-1967.

[7] S. J. Liao. The proposed homotopy analysis techniques for the solution of non-linear problems. *Ph.D. Thesis, Shanghai Jiao Tong University*, (1992).

[8] S. J. Liao, P. Beyond. Introduction to homotopy analysis method. *CRC Press/Champan & Hall, Boca Raton*, (2003).

[9] Y. Luchko, F. R. Goren. The initial value problems for some fractional differential equations with the Caputo derivative. *Preprint Series AU8-9b, Fachbreicn Mathematik and Informatic, Freic, Universitat Berlin*, (1991).

[10] F. Mainardi. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos Solitons Fractals,* 7 :(1996), 1461-1477.

[11] F. Mainardi. Fractional calculus: Some basic problems in continuum and statistical mechanics. *Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York*, (1997), 291-348.

[12] K. S. Miller, B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations. *Willey, New York*, (1993).

[13] S. Momani. Non-perturbative analytical solutions of the space and time-fractional Burgers equations. *Chaos, Solitons Fractals*, 28(4): (2006), 930-937.

[14] S. Momani, Z. Odibat. Homotopy perturbation method for non-linear differential equations of fractional order. *Physics Letters A*, 365: (2007), no.5, 345-350.

[15] S. Momani, Z. Odibat. Numerical approach to differential equations of fractional order. *J. Comput. Apple. Main.*, 207(1): (2007), 96-110.

[16] Z. Odibat, S. Momani. Application of variation iteration method to non-linear differential equations of fractional order. *Int. J. Nonlin. Sci. Numer. Simulat.*, 1 (7):(2006), 15-27.

[17] K. B. Oldham, J. Spanier. The Fractional Calculus. *Academic Press, New York*, (1974).

[18] I. Podlubny. Fractional Differential Equations. *Academic Press, San Diago*, (1999).

[19] G. E. Pukhov. Computational structure for solving differential equations by Taylor transformation. *Cybernetics and System Analysis*, 14(3): (1978),383-390.

[20] W. Schneider, W. Wyss. Fractional diffusion equation and wave equations. *J. Math. Phys.*, 30 :(1989), 134-144.

[21] N. T. Shawagfeh. Analytical Approximate solutions for non-linear fractional differential equations. *Appl. Math. Comput.*, 131 (2):(2002), 517-529.

[22] C. Song, H. Zhang. Application of homotopy analysis method to fractional KdV-Burgers-Kurumoto equation. *Phys. Lett. A*, 367 (1-2) :(2007), 88-94.

[23] Y. Y. Wu, S. J. Liao. Solving the one loop soliton solution of the Vakhnenko equation by means of the homotopy analysis method. *Chaos. Soliton, Fract.*, 23(5) :(2004), 1733-1740.

[24] A. Yildirim. Applications of the homotopy perturbation method for the Fokker-Planck equation. *Communications in Numerical Methods in Engineering*, 26 :(2010), 1144-1154.

[25] E. Yusufoglu. Numerical solution of Duffing equation by the Laplace decomposition algorithm. *Appl. Math. Comput.*, 177: (2006), 572-680.

[26] J. K. Zhou. Differential Transformation and Its Applications for Electrical Circuits. *Huazhong University Press, Wuhan, China in Chinese,*(1986).

[27] M. Zurigat. Solving Fractional Oscillators Using Laplace homotopy analysis method. *Annals. of the Uni. of Craiova. Mathematics and Computer Science Series*, 38(4) :(2011), 1-11.