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# Homotopy Analysis Method for Computing Eigenvalues of Sturm-Liouville Problems

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**Abstract:** In this paper, we apply homotopy analysis method (HAM) for computing the eigenvalues of Sturm-Liouville problems. The parameter $h$, in this method, helps us to adjust and control the convergence region. The results show that this method has validity and high accuracy with less iteration number in compare to Variation Iteration Method (VIM) and Adomian decomposition method (ADM). Moreover it is illustrated that this method is independent of eigenvalues indexes.

**Keywords:** Sturm-Liouville equation; eigenvalue; homotopy analysis method

## 1 Introduction

Most phenomena in real world are described through Sturm-Liouville and Schrodinger equations and these type of equations have attracted lots of attention among scientists. A classical Sturm-Liouville equation, named Schrodinger equation, is a real second-order linear differential equation of the form

$$y''(x) = (q(x) - \lambda)y(x), \tag{1}$$

with separated boundary conditions

$$a_0 y(a) + b_0 y'(a) = 0, \quad a_1 y(b) + b_1 y'(b) = 0, \tag{2}$$

where $q(x)$ and $y(x)$ are called the potential function and eigenfunction respectively on the finite interval $[a, b]$[1,2], and $\lambda$ is the eigenvalue of the equation.
In relation (2), $a_0$ and $b_0$ are not both zero, and similarly for $a_1$, $b_1$.

This linear second order differential equation describes a lot of important physical phenomena which exhibit a pronounced oscillatory character; behavior of pendulum-like systems, vibrations, resonances and wave propagation are all phenomena of this type in classical mechanics, while the same is true for the typical behavior of quantum particles.

Large class of nonlinear equations such as Sturm-Liouville equations do not have a precise analytic solution, so numerical methods have largely been used to handle these equations [3,8].

In this paper we extend the HAM for determining the eigenvalues of Sturm-Liouville equation.

HAM firstly was developed by S. J. Liao in 1992. This method can be applied to different type of problems in many engineering and physical applications[9,14]. HAM contains a certain auxiliary parameter $h$, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. This method properly overcomes restrictions of perturbation techniques because it does not need any small or large parameters to be contained in the problem. Moreover some methods such as ADM and the Homotopy Perturbation Method (HPM) are special cases of HAM[15,17].

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## 2 Homotopy Analysis Solution

For convenience of the readers, we will first present a brief description of the standard HAM. To achieve our goal, let us assume the nonlinear differential equations be in the form of

$$N[u(t), \lambda] = 0, \tag{3}$$

where $N$ are nonlinear operators, $t$ is an independent variable, $u(t)$ and $\lambda$ are eigenfunction and eigenvalues of our equation respectively. By means of generalizing the traditional homotopy method, Liao construct the zeroth-order deformation equation as follows

$$(1-q)L[\phi(t,q) - u_0(t)] = qhH(t)N[\phi(t,q), \Lambda(q)], \tag{4}$$

where $q \in [0,1]$ is an embedding parameter, $h$ is a auxiliary parameter and $H(t)$ is nonzero auxiliary function, $L$ and $N$ are linear and nonlinear operators, $u_0(t)$ and $\lambda_0$ are initial guesses of $u(t)$ and $\lambda$, $\phi(t;q)$ and $\Lambda(q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects such as $h$ and $L$ in HAM; this freedom plays an important role in establishing the keyston of validity and flexibility of HAM as shown in this paper. Obviously, when $q = 0$ and $q = 1$, both

$$\phi(t,0) = u_0(t) \quad \text{and} \quad \phi(t,1) = u(t), \tag{5}$$

$$\Lambda(0) = \lambda_0 \quad \text{and} \quad \Lambda(1) = \lambda, \tag{6}$$

hold. Thus as $q$ increases from 0 to 1, the solutions of $\phi(t;q)$ and $\Lambda(q)$ change from the initial guesses $u_0(t)$ and $\lambda_0$ to the solutions $u(t)$ and $\lambda$ . Expanding $\phi(t;q)$ in Taylor series with respect to $q$, one has

$$\phi(t,q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m, \tag{7}$$

$$\Lambda(q) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m q^m, \tag{8}$$

where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t,q)}{\partial q^m}\Big|_{q=0}, \tag{9}$$

$$\lambda_m = \frac{1}{m!} \frac{\partial^m \Lambda(q)}{\partial q^m}\Big|_{q=0}. \tag{10}$$

If the auxiliary linear operator, the initial guesses, the auxiliary parameters $h$, and the auxiliary function is so properly chosen, then the series (7) and (8) converges at $q = 1$, one has

$$\phi(t,1) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t), \tag{11}$$

$$\Lambda(1) = \lambda_0 + \sum_{m=1}^{+\infty} \lambda_m, \tag{12}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao. Define the vectors

$$\vec{u}_n(t) = \{u_0(t), u_1(t), ..., u_n(t)\}, \tag{13}$$

$$\vec{\lambda}_n(t) = \{\lambda_0, \lambda_1, ..., \lambda_n\}. \tag{14}$$

Differentiating (4), $m$ times with respect to the embedding parameter $q$ and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called $mth$-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hR_m(\vec{u}_{m-1}, \vec{\lambda}_{m-1}), \tag{15}$$

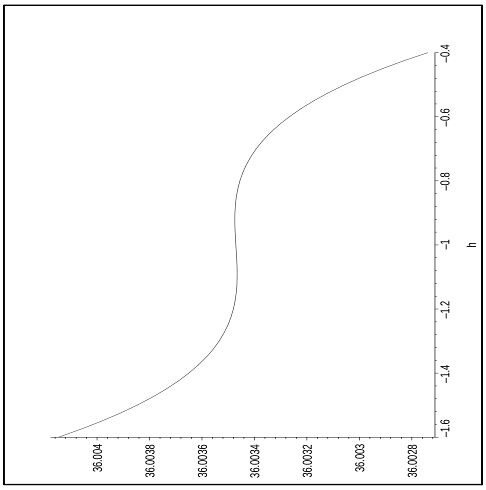


Figure 1: Approximation of $\lambda_5$ when $-1.2 \leq h \leq -0.8$

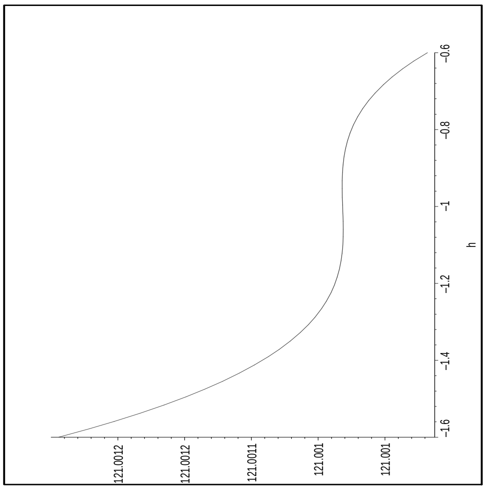


Figure 2: Approximation of $\lambda_{10}$ when $-1.2 \leq h \leq -0.8$

with the conditions

$$u_m(a) = u_m(b) = 0, \tag{16}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t,q), \Lambda(q)]}{\partial q^{m-1}} \Big|_{q=0}, \tag{17}$$

and

$$\chi_{\mathrm{m}} = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{18}$$

It should be emphasized that $u_m(t)$ is governed by the linear equations (4) and (15) with the linear boundary conditions that come from the original problem. These equations can be easily solved by symbolic computation softwares such as Maple and Mathematica.

# 3 Applications

In this section, some examples are given to confirm the validity and high accuracy for the proposed method.

**Example 1** *Let us consider the equation (1) with $q(x) = 0$ and Dirichlet boundary conditions*

$$y(0) = y(\pi) = 0. \tag{19}$$

*The solution of $u(t)$ can be expressed by a set of base functions*

$$\{\sin(kmt), \;\; m = 0, 1, ...\}. \tag{20}$$

*To obey the base function (20) and boundary condition (19), it is straightforward to choose the initial guess as*

$$u_0(t) = \sin(kt). \tag{21}$$

*By following the process of previous section, we choose the auxiliary linear operator $L$ and $N$, as follows*

$$L[\phi(t,q)] = \frac{\partial^2 \phi(t,q)}{\partial t^2} + k^2 \phi(t,q), \tag{22}$$

*which has the property*

$$L[c_1 sin(kt) + c_2 cos(kt)] = 0, \tag{23}$$

*and*

$$N[\phi(t,q)] = -\frac{\partial^2 \phi(t,q)}{\partial t^2} - \Lambda(q)\phi(t,q). \tag{24}$$

*Then we have*

$$R_m(\vec{u}_{m-1}, \vec{\lambda}_{m-1}) = -u''_{m-1}(t) - \sum_{j=0}^{m-1} \lambda_m u_{m-1-j}(t). \tag{25}$$

*Note that both $u_m(t)$ and $\lambda_{m-1}$ are unknown, but we have only one differential equation for $u_m(t)$. So, the problem is not closed and an additional algebraic equation is needed to determine $\lambda_{m-1}$. Assume that $H(t)$ is properly chosen so that the right-hand side term of the high-order deformation equation (25) can be expressed by*

$$hH(t)R_m(\vec{u}_{m-1}, \vec{\lambda}_{m-1}) = \sum_{n=0}^{\mu_m} b_{m,n}\lambda_{m-1}\sin(knt), \tag{26}$$

*where $b_{m,n}\lambda_{m-1}$ is a coefficient and the positive integer $\mu_m$ depends upon $H(t)$ and $m$. According to the property of $L$, when $b_{m,0}\lambda_{m-1} \neq 0$, the solution of the $mth$-order deformation equation (26) contains the term*

$$t\sin(kt), \tag{27}$$

*which disobeys the rule of solution expression denoted by (20). To avoid this, we have to enforce*

$$b_{m,0}\lambda_{m-1} = 0, \tag{28}$$

*which provides us with an additional algebraic equation for $\lambda_{m-1}$. In this way, the problem is closed. Therefore, it is easy to gain the solution of Equation (15), say,*

$$u_m(t) - \chi_m u_{m-1}(t) = \sum_{n=1}^{\mu_m} \frac{b_{m,n}}{k^2(1-n^2)}\sin(nkt) + C_1\sin(kt) + C_2\cos(kt) \tag{29}$$

*where $C_1$ and $C_2$ are coefficients. Under the rule of solution expression denoted by (19), $C_2$ must be zero. In this way, we gain $\lambda_{m-1}$ and $u_m(t)$ successively. At the $Nth$-order of approximation, we have*

$$u(t) = u_0(t) + \sum_{m=1}^{N} u_m(t), \tag{30}$$

$$\lambda = \lambda_0 + \sum_{m=1}^{N} \lambda_m. \tag{31}$$

*For the sake of simplicity, we choose here*

$$H(t) = 1. \tag{32}$$

*Then, using (26),we have*

$$hR_1(\vec{u}_0, \vec{\lambda}_0) = h(-\sin(kx)k^2 + \lambda_0\sin(kx)) \tag{33}$$

*which gives according to above illustration*

$$b_{1,0} = \lambda_0 - k^2 \quad b_{1,1} = 0. \tag{34}$$

*Then*

$$\lambda_0 - k^2 = 0 \Rightarrow \lambda_0 = k^2, \tag{35}$$

*and*

$$\lambda_k = 0, \quad k \geq 1 \quad \textit{then} \quad \lambda = k^2, \tag{36}$$

*where are the exact values of eigenvalues problems.*

.

**Example 2** *Let us consider example 1 with Neuman-Dirichlet boundary conditions:*

$$y^{'}(0) = 0, \quad y(1) = 0. \tag{37}$$

*By choosing the following base function for $u(t)$*

$$\{cos((k\pi + \frac{\pi}{2})mt), \;\; m = 0, 1, ...\}, \tag{38}$$

*it is straightforward to choose the initial guess as*

$$u_0(t) = \cos((k\pi + \frac{\pi}{2})t), \tag{39}$$

*and we choose the linear operator $L$ as follows*

$$L[\phi(t, q)] = \frac{\partial^2 \phi(t, q)}{\partial t^2} + (k\pi + \frac{\pi}{2})^2 \phi(t, q), \tag{40}$$

*with has the following property*

$$L[c_1 sin(k\pi + \frac{\pi}{2}) + c_2 cos(k\pi + \frac{\pi}{2})] = 0. \tag{41}$$

*Using the same relation (24) and (25), we write*

$$hH(t)R_m(\vec{u}_{m-1}, \vec{\lambda}_{m-1}) = \sum_{n=0}^{\mu_m} b_{m,n}\lambda_{m-1} cos((k\pi + \frac{\pi}{2})mt). \tag{42}$$

*if we take $b_{m,0}\lambda_{m-1} \neq 0$, The solution of the $mth$-order deformation equation (42) contains the term*

$$t \cos((k\pi + \frac{\pi}{2})t). \tag{43}$$

*To avoid this term we enforce*

$$b_{m,0}\lambda_{m-1} = 0. \tag{44}$$

*Hence by using similar computations in example 1, we reach the following recurrence relation*

$$u_m(t) \quad - \quad \chi_m u_{m-1}(t) = \frac{1}{(k\pi + \frac{\pi}{2})^2} + \sum_{n=2}^{\mu_m} \frac{b_{m,n}}{(k\pi + \frac{\pi}{2})(1 - n^2)} \cos((k\pi + \frac{\pi}{2})mt) + \cos(k\pi + \frac{\pi}{2}). \tag{45}$$

*By letting $H(t) = 1$ and applying (42) we obtain*

$$hR_1(\vec{u}_0, \vec{\lambda}_0) = h[-\cos((k\pi + \frac{\pi}{2})x)(k\pi + \frac{\pi}{2})^2 + \lambda_0 \cos((k\pi + \frac{\pi}{2})x)], \tag{46}$$

*according to (26), which gives*

$$b_{1,0} = \lambda_0 - \frac{\pi^2(2k+1)^2}{4} \quad b_{1,1} = 0, \tag{47}$$

*and*

$$\lambda_k = 0, \quad k \geq 1 \;\; then \quad \lambda = \frac{\pi^2(2k+1)^2}{4}. \tag{48}$$

*Again we obtain the exact solution.*

In the next example, we show the effectiveness of our method by choosing an oscillatory function.

Table 1: computation of eigenvalues by HAM and comparison them to other methods with iteration $N = 4$.

| $\lambda_k$ | Exact Solution | HAM for various of h | VIM | ADM |
|---|---|---|---|---|
| $k = 0$ | 0.918058176625213 | 1.145989294 h=-0.6 | 0.91799210213 | 0.9194733525 |
| k=1 | 4.031921988127956 | 4.031929949 h=-0.9 | not available | 2.555191113 |
| k=2 | 9.014301750061977 | 9.014020152 h=-0.8 | ” | not available |
| k=3 | 16.00793923538222 | 16.00784612 h=-0.85 | ” | ” |
| k=4 | 25.00505118570864 | 25.00500426 h=-0.85 | ” | ” |
| k=5 | 36.00349672552254 | 36.00346592 h=-1.1 | ” | ” |
| k=6 | 49.00256418923281 | 49.00254604 h=-0.85 | ” | ” |
| k=7 | 64.00196082281326 | 64.00195162 h=-0.9 | ” | ” |
| k=8 | 81.00154800647830 | 81.00154125 h=-1 | ” | ” |
| k=9 | 100.0012531428047 | 100.0012406 h=-0.8 | ” | ” |
| k=10 | 121.0010352022962 | 121.0010316 h=-0.9 | ” | ” |
| k=20 | 441.0002836076016 | 441.0002831 h=-1 | ” | ” |

**Example 3** *In this example we consider(1) with $q(x) = \cos(x)$ and Dirichlet boundary (19). By using the base functions*

$$\{sin[(k+m-1)t],\ \ m = 0, 1, ...\}. \tag{49}$$

*and choosing the initial guess*

$$\sin(kt), \tag{50}$$

*We consider a linear operator as follows*

$$L[\phi(t,q)] = \frac{\partial^2\phi(t,q)}{\partial t^2} + k^2\phi(t,q), \tag{51}$$

*with the possession of*

$$L[c_1 sin(kt) + c_2 cos(kt)] = 0, \tag{52}$$

*and*

$$N[\phi(t,q)] = -\frac{\partial^2\phi(t,q)}{\partial t^2} + [\cos(t) - \Lambda(q)]\phi(t,q). \tag{53}$$

*Then we obtain*

$$R_m(\vec{u}_{m-1}, \vec{\lambda}_{m-1}) = -u''_{m-1}(t) + \cos(t)u_{m-1}(t) + \sum_{j=0}^{m-1} \lambda_m u_{m-1-j}(t). \tag{54}$$

*Using the previous discussion, we have the final recurrence*

$$u_m(t) - \chi_m u_{m-1}(t) = \frac{b_{m,0}}{(k+1)^2 - k^2}\sin(kt) + \sum_{n=2}^{\mu_M} \frac{b_{m,n}}{(k+1)^2 - (k+n)^2}\sin[(k+n)(t)] + c_1\sin((k+1)t). \tag{55}$$

So, in this way, we obtain $\lambda_0, u_1, \lambda_1, u_2, \lambda_2, u_3$,..., one after the other. This is easy to do by means of the symbolic computation software, and determine the convergence region in following diagrams for some of them by the parameter $h$, as well as the comparison of our results between ADM and VIM.

Our convergence region are drawn for some cases in figures (1) and (2) and results are shown in Table (1). (the exact solution refereed to MATSLISE software package[30] )

# 4 Conclusion

In this paper, we approximate the eigenvalues for Sturm-Liouville problems with various initial conditions when $q(x) = 0$ to show the validity of HAM and for oscillating potential function to show the effectiveness of it. The convergence region for eigenvalues are determined by the parameter $h$, which provides us a great freedom to choose convenient value for it. The comparison of our results with the results of ADM and VIM, show that this method has higher accuracy and fast convergence as well as more availability of eigenvalues.

# References

[1] V. Ledoux. Study of Special Algorithms for solving Sturm-Liouville and Schrodinger Equations. *doctora thesis Gent University*, (2006-2007).

[2] E. Schrodinger. An undulatory theory of the mechanics of atoms and molecules. *Phys. Rev.*, 28 (1926): 1049-1070.

[3] D. Altintan, O. Ugur. Variational iteration method for Sturm-Liouville differential equations. *Computers and Mathematics with Applications*, 58 (2009): 322-328.

[4] Q. M. Al-Mdallal. An efficient method for solving fractional Sturm-Liouville problems. *Chaos, Solitons and Fractals*, 40 (2009): 183-189.

[5] M. I. Syam, H. I. Siyyam. An efficient technique for finding the eigenvalues of fourth-order Sturm-Liouville problems. *Chaos, Solitons and Fractals*, 39 (2009): 659-665.

[6] L. Aceto, P. Ghelardoni, C. Magherini. Boundary Value Methods as an extension of Numerovs method for Sturm-Liouville eigenvalue estimates. *Applied Numerical Mathematics*, 59 (2009): 1644-1656.

[7] B. Chanane. Computing the eigenvalues of a class of nonlocal Sturm-Liouville problems. *Mathematical and Computer Modelling* ,50 (2009): 225-232.

[8] B. Chanane. Computing the spectrum of non-self-adjoint Sturm-Liouville problems with parameter-dependent boundary conditions. *Journal of Computational and Applied Mathematics*, 206 (2007): 229-237.

[9] DR. G. Domairry, A. Mohsenzadeh, M. Famouri. The application of homotopy analysis method to solve nonlinear differential equation governing Jefferyamel flow. *Communications in Nonlinear Science and Numerical Simulation*, 14 (2009): 85-95.

[10] G. Domairry, M. Fazeli. Homotopy analysis method to determine the fin efficiency of convective straight fins with temperature-dependent thermal conductivity. *Communications in Nonlinear Science and Numerical Simulation*, 14 (2009): 489-499.

[11] L. Song, H. Zhang. Application of homotopy analysis method to fractional KdV-Burgers uramoto equation. *Physics Letters A* ,367 (2007): 88-94.

[12] H. Jafari, A. Golbabai, S. Seifi, K. Sayevand. Homotopy Analysis Method for Solving multi-term linear and nonlinear diffusion -wave equations of fractional order. *Computers and Mathematics with Applications*, 59(2009):1337-1344.

[13] S. Abbasbandy. Homotopy analysis method for heat radiation equations. *International Communications in Heat and Mass Transfer*, 34(2007):380-387.

[14] F. J. Farahani, J. D. Seader. Use of homotopy-continuation method in stability analysis of multiphase, reacting systems. *Computers and Chemical Engineering*, 24(2000):1997-2008.

[15] A. Golbabai, B. Keramati. Solution of non-linear Fredholm integral equations of the first kind using modified homotopy perturbation method. *Chaos, Solitons and Fractals*, 39(5)(2009):2316-2321.

[16] A. Golbabai, M. Javidi. Modified homotopy perturbation method for solving nonlinear Fredholm integral equations. *Chaos, Solitons and Fractals*,40(3)(2009):1408-1412.

[17] B. S. Attili. The Adomian decomposition method for computing eigenelements of Sturm-Liouville two point boundary value problems. *Applied Mathematics and Computation*, 168(2005):1306-1316.