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Exact Solitary Wave Solutions of Nonlinear Evolution Equations with a Positive Real Number Power Term

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Abstract: Positive solutions of a second order ODE with an arbitrary positive real number power nonlinearity are obtained by transformation of dependent variable and the $\left(\frac{G'}{G}\right)$-expansion method. In terms of the positive solutions of the ODE introduced here, solitary wave solutions of a number of nonlinear evolution equations with an arbitrary positive real number power term of dependent variable, which includes generalized KdV equation, generalized BBM equation, generalized Zakharov-Kuznetsov equation, generalized KP equation, generalized Klein-Gordon equation, generalized Boussinesq equation, generalized NLS equation and generalized Zakharov equations and so on, are obtained successfully. In the special cases, solitary wave solutions of some well-known important model equations, such as the KdV (mKdV as well) equation, BBM (mBBM as well) equation, ZK (mZK as well) equation, KP (mKP as well) equation, Klein-Gordon equation, Boussinesq equation, NLS equation and Zakharov equations and so on, are also rediscovered.

Keywords: Second order ODE with a positive real number power nonlinearity; positive solutions; nonlinear PDEs with a positive real number power term; important model equations; solitary wave solutions

1 Introduction

In the present paper we shall consider a class of nonlinear evolution equations with a positive real number power term of dependent variable, some examples of this class of nonlinear equations are listed below:

The generalized KdV equation (GKdV)[1, 2]

$$u_t + au^r u_x + bu_{xxx} = 0. \tag{1}$$

The generalized BBM equation (GBBM) [3]

$$u_t + u_x + au^r u_x + bu_{xxt} = 0. \tag{2}$$

The generalized Zakharov-Kuznetsov equation (GZK) [4]

$$u_t + au^r u_x + b(u_{xx} + u_{yy})_x = 0. \tag{3}$$

The generalized KP equation (GKP) [5, 6]

$$(u_t + au^r u_x + bu_{xxx})_x + \sigma^2 u_{yy} = 0. \tag{4}$$

The generalized (m+1)-dimensional Klein-Gordon equation (GKG) [7]

$$u_{tt} - c^2\Delta_m u + au + bu^{r+1} = 0, \Delta_m = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}. \tag{5}$$

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The generalized Boussinesq equation (GB) [8]

$$u_{xx} - u_{tt} + a\left(u^{r+1}\right)_{xx} + bu_{xxxx} = 0. \tag{6}$$

The generalized nonlinear Schrödinger equation (GNLS) [9]

$$iu_t + \frac{1}{2}u_{xx} + a\left|u\right|^r u = 0. \tag{7}$$

The generalized Zakharov equations (GZE) [10]

$$iE_t + E_{xx} = \alpha EH + b\left|E\right|^r E, \qquad (8)_1$$

$$H_{tt} - H_{xx} = \beta\left(\left|E\right|^r\right)_{xx} \qquad (8)_2$$

and so on, where r be an arbitrary positive real number, a and b are constants. In particular cases, if $r = 1$, Eqs. (1)-(6) become the important model equations in mathematical physics, to be more precise, Eqs. (1)-(6) become the celebrated KdV equation [11, 12], BBM equation [13], ZK equation [14], KP equation [15], Klein-Gordon equation [16], and Boussinesq equation [15], respectively. If $r = 2$, Eqs. (1)-(5) become the mKdV, mBBM, mKP, and Klein-Gordon equation with cubic nonlinear term, respectively, and Eqs. (7)-(8) become the standard NLS equation and the Zakharov equations, respectively. Each one of these model equations mentioned above has been investigated extensively by many authors during the past four decades or so. Since equations (1)-(8) are all the generalized form of well-known important model equations, therefore the investigation of this class of equations is of great significance both mathematically and physically.

The solitary wave solutions have been one of the most striking aspects of nonlinear evolution equations, which has attracted the notice of mathematicians and physicists during the past four decades or so. On account of the importance of solitary waves, our main goal in this paper is to look for the solitary wave solutions of Eqs. (1)-(8), meanwhile to discuss how the behaviour of solitary waves is affected by the signs of coefficients a and b appearing in Eqs. (1)-(8).

The rest of the paper is organized as follows: In section 2, a second order ODE with a positive real number power nonlinearity is introduced, and its positive solutions are found out by transformation of dependent variable and the $\left(\frac{G'}{G}\right)$-expansion method introduced recently. In subsequent sections, from section 3 to section 10, we shall apply the results obtained in section 2 to the GKdV, GBBM, GZK, GKP, GKG, GB, GNLS, and GZ equations, respectively, to obtain the solitary wave solutions of these equations. In section 11, the method used in the paper is briefly summarized.

2 A nonlinear ODE and its positive solutions

Let us consider the second order ODE in the form

$$F''(\xi) = AF(\xi) + BF^{r+1}(\xi), \tag{9}$$

where A and B are constants, $A > 0$, r be an arbitrary positive real number.

We will use the $\left(\frac{G'}{G}\right)$-expansion method introduced in [17] to solve ODE (9). First of all, making transformation of dependent variable in ODE (9),

$$F(\xi) = v(\xi)^{\frac{1}{r}}, \tag{10}$$

then the ODE (9) is transformed into the nonlinear ODE for new dependent variable $v = v(\xi)$

$$vv'' + \left(\frac{1}{r} - 1\right)v'^2 = Arv^2 + Brv^3. \tag{11}$$

Considering the homogeneous balance between vv'' (or v'^2) and v^3 appearing in ODE (11) ($2m + 2 = 3m \rightarrow m = 2$) We suppose that $v = v(\xi)$ is of the form

$$v(\xi) = a_2\left(\frac{G'}{G}\right)^2 + a_0, \tag{12}$$

where $G = G(\xi)$ satisfies the second order linear ODE

$$G'' + \beta G = 0, \tag{13}$$

a_2, a_0 and β are constants to be determined later. From (12) and (13), it is easily to derive that

$$v^2 = a_2^2\left(\frac{G'}{G}\right)^4 + 2a_2a_0\left(\frac{G'}{G}\right)^2 + a_0^2, \tag{14}$$

$$v^3 = a_2^3\left(\frac{G'}{G}\right)^6 + 3a_2^2a_0\left(\frac{G'}{G}\right)^4 + 3a_2a_0^2\left(\frac{G'}{G}\right)^2 + a_0^3, \tag{15}$$

$$v' = -2a_2\left(\frac{G'}{G}\right)^3 - 2a_2\beta\left(\frac{G'}{G}\right),$$

$$v'^2 = 4a_2^2\left(\frac{G'}{G}\right)^6 + 8a_2^2\beta\left(\frac{G'}{G}\right)^4 + 4a_2^2\beta^2\left(\frac{G'}{G}\right)^2, \tag{16}$$

$$v'' = 6a_2\left(\frac{G'}{G}\right)^4 + 8a_2\beta\left(\frac{G'}{G}\right)^2 + 2a_2\beta^2,$$

$$vv'' = 6a_2^2\left(\frac{G'}{G}\right)^6 + \left(6a_2a_0 + 8a_2^2\beta\right)\left(\frac{G'}{G}\right)^4 + \left(2a_2^2\beta^2 + 8a_2a_0\beta\right)\left(\frac{G'}{G}\right)^2 + 2a_2a_0\beta^2. \tag{17}$$

Substituting (14)-(17) into ODE (11), collecting the same terms of $\left(\frac{G'}{G}\right)^i$ $(i = 6, 4, 2, 0)$ together, and equating the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i = 6, 4, 2, 0)$ to zero, yields a set of algebraic equations for a_2, a_0 and β as follows

$$\left(\frac{G'}{G}\right)^6: \quad 6a_2^2 + 4\left(\frac{1}{r} - 1\right)a_2^2 = Bra_2^3, \tag{18}$$

$$\left(\frac{G'}{G}\right)^4: \quad 6a_2a_0 + \frac{8}{r}a_2^2\beta = Ara_2^2 + 3Bra_2^2a_0, \tag{19}$$

$$\left(\frac{G'}{G}\right)^2: \quad 2a_2^2\beta^2 + 8a_2a_0\beta + 4\left(\frac{1}{r} - 1\right)a_2^2\beta^2 = 2Ara_2a_0 + 3Bra_2a_0^2, \tag{20}$$

$$\left(\frac{G'}{G}\right)^0: \quad 2a_2a_0\beta^2 = Ara_0^2 + Bra_0^3. \tag{21}$$

Solving equations (18)-(21), we have

$$a_2 = \frac{2(r+2)}{Br^2}, a_0 = -\frac{A(r+2)}{2B}, \tag{22}$$

$$\beta = -\frac{Ar^2}{4}. \tag{23}$$

Substituting (23) into ODE (13), we can obtain the general solution of ODE (13) as follows

$$G(\xi) = C_1\cosh\frac{\sqrt{A}}{2}r\xi + C_2\sinh\frac{\sqrt{A}}{2}r\xi, \tag{24}$$

where C_1 and C_2 are arbitrary constants. From (24) it is derived that

$$\left(\frac{G'}{G}\right) = \frac{\sqrt{A}r}{2}\left(\frac{C_1\sinh\frac{\sqrt{A}}{2}r\xi + C_2\cosh\frac{\sqrt{A}}{2}r\xi}{C_1\cosh\frac{\sqrt{A}}{2}r\xi + C_2\sinh\frac{\sqrt{A}}{2}r\xi}\right). \tag{25}$$

Substituting (22) and (25) into (12) we have the solution of ODE (11) as follows

$$v(\xi) = \frac{A(r+2)}{2B}\left(\frac{C_1\sinh\frac{\sqrt{A}}{2}r\xi + C_2\cosh\frac{\sqrt{A}}{2}r\xi}{C_1\cosh\frac{\sqrt{A}}{2}r\xi + C_2\sinh\frac{\sqrt{A}}{2}r\xi}\right)^2 - \frac{A(r+2)}{2B}. \tag{26}$$

In order that the transformation (10) makes definite meaning for arbitrary positive real number r, we need positive solution $v = v(\xi)$ of ODE (11) only. We can select the parameters A, B, C_1 and C_2 appearing in (26) such that $v = v(\xi) > 0$. In fact, when $A > 0, B < 0$ and $C_1^2 - C_2^2 > 0$, (26) becomes

$$v(\xi) = -\frac{A(r+2)}{2B}\operatorname{sech}^2\frac{\sqrt{A}}{2}r\left(\xi+\xi_0\right), \xi_0 = \frac{2}{\sqrt{A}r}\tanh^{-1}\frac{C_2}{C_1}, \tag{27}$$

which is a positive bounded solution of ODE (11).

When $A > 0, B > 0$ and $C_2^2 - C_1^2 > 0$, (26) becomes

$$v(\xi) = \frac{A(r+2)}{2B}\operatorname{csch}^2\frac{\sqrt{A}}{2}r\left(\xi+\xi_0\right), \xi_0 = \frac{2}{\sqrt{A}r}\tanh^{-1}\frac{C_1}{C_2}, \tag{28}$$

which is a positive unbounded solution of ODE (11).

Substituting (27) and (28) into (10), respectively, we have the positive solutions of ODE (9) as follows

$$F(\xi) = \left[-\frac{A(r+2)}{2B}\operatorname{sech}^2\frac{\sqrt{A}}{2}r\left(\xi+\xi_0\right)\right]^{\frac{1}{r}}, \text{if } A > 0, B < 0, \tag{29}$$

and

$$F(\xi) = \left[\frac{A(r+2)}{2B}\operatorname{csch}^2\frac{\sqrt{A}}{2}r\left(\xi+\xi_0\right)\right]^{\frac{1}{r}}, \text{if } A > 0, B > 0, \tag{30}$$

respectively.

In the subsequent sections, Eq. (9) and its solutions (29) and (30) will be applied for finding the solitary wave solutions of Eqs. (1)-(8).

3 Solitary wave solution of GKdV

We begin with the GKdV equation (1) and seek its travelling wave solution in the form

$$u(x,t) = u(\xi), \xi = x - Vt, V = \text{Constant}, \tag{31}$$

where V represents velocity of the travelling wave. Substituting (31) into Eq. (1) yields an ODE for $u = u(\xi)$

$$-Vu' + au^r u' + bu''' = 0.$$

Integrating the ODE with respect to ξ once and taking the constant of integration to zero, we have a second order ODE for $u = u(\xi)$

$$u'' = \frac{V}{b}u - \frac{a}{b(r+1)}u^{r+1}. \tag{32}$$

ODE (32) belongs in the same type as ODE (9) introduced in section 2.

Using the solutions (29) and (30) of ODE (9) in section 2, the positive solutions of ODE (32), i.e., the solitary wave solutions of GKdV equation (1) can be given by

$$u(x,t) = \left[\frac{V(r+2)(r+1)}{2a}\operatorname{sech}^2\frac{r}{2}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right)\right]^{\frac{1}{r}}, \tag{33}$$

provided $a > 0, b > 0, V > 0$ or $a < 0, b < 0, V < 0$, and

$$u(x,t) = \left[\frac{-V(r+2)(r+1)}{2a}\operatorname{csch}^2\frac{r}{2}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right)\right]^{\frac{1}{r}}, \tag{34}$$

provided $a < 0, b > 0, V > 0$ or $a > 0, b < 0, V < 0$.

From (33) and (34) we can come to the conclusion that the behaviour of the solitary wave of GKdV (1) depends on the signs of nonlinear coefficient a and dispersive coefficient b .

In fact,

1. When both a and b are positive, GKdV (1) admits a bounded positive solitary wave with propagating velocity $V > 0$, travelling to the right (Figure 1(a)).

2. When both a and b are negative, GKdV (1) admits a bounded positive solitary wave with propagating velocity $V < 0$, travelling to the left (Figure 1(b)).

3. When $a > 0, b < 0$, GKdV (1) admits an unbounded positive solitary wave with propagating velocity $V < 0$, travelling to the left (Figure 1(c)).

4. When $a < 0, b > 0$, GKdV (1) admits an unbounded positive solitary wave with propagating velocity $V > 0$, travelling to the right (Figure 1(d)).

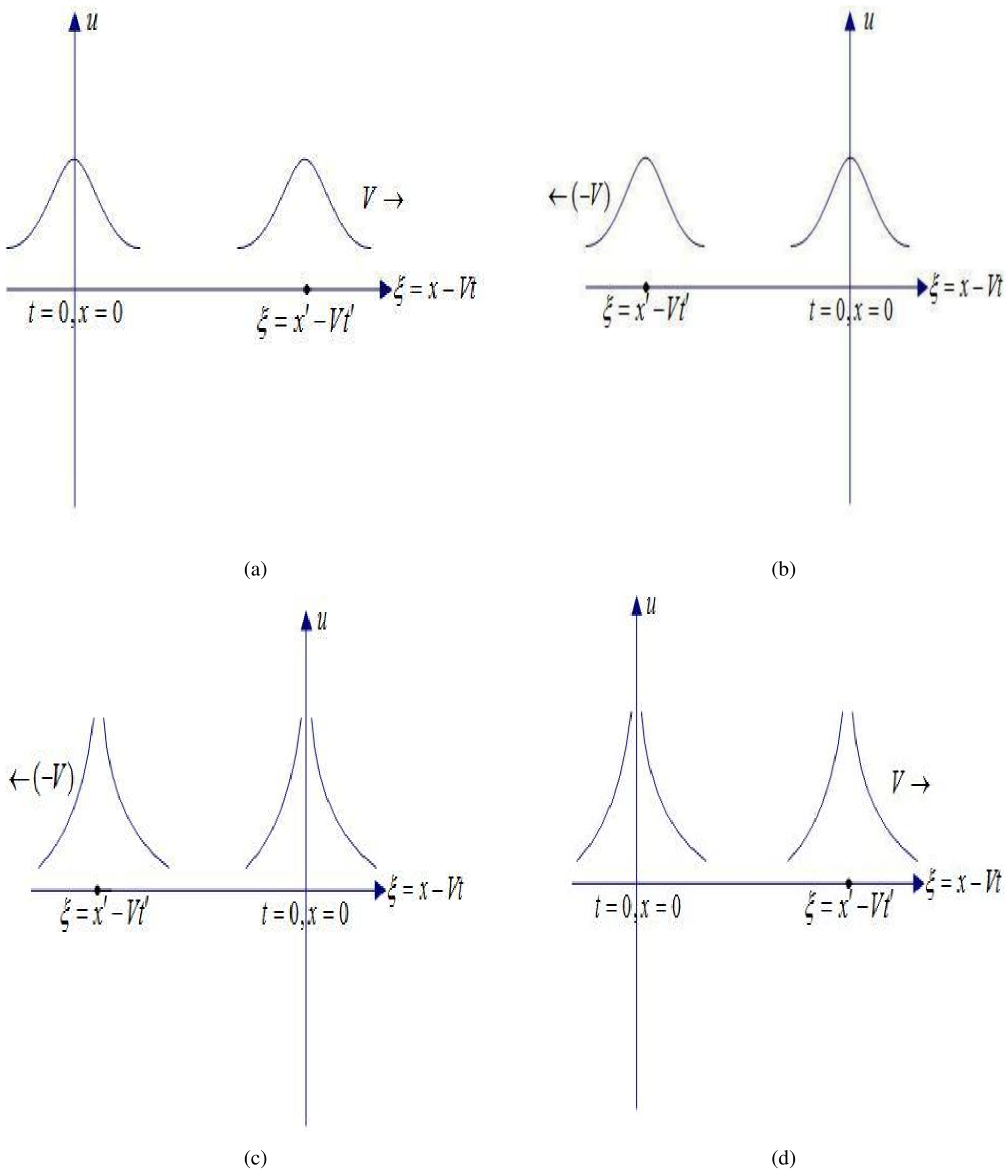

Figure 1:

In particular, if $r = 1$, (33) and (34) become

$$u(x,t) = \frac{3V}{a}\operatorname{sech}^2 \frac{1}{2}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right), (a > 0, b > 0, V > 0, \text{ or } a < 0, b < 0, V < 0) \tag{35}$$

and

$$u(x,t) = -\frac{3V}{a}\operatorname{csch}^2 \frac{1}{2}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right), (a < 0, b > 0, V > 0, \text{ or } a > 0, b < 0, V < 0) \tag{36}$$

respectively. Both (35) and (36) are solitary wave solutions of KdV equation

$$u_t + auu_x + bu_{xxx} = 0.$$

If $r = 2$, (33) and (34) become

$$u(x,t) = \sqrt{\frac{6V}{a}}\operatorname{sech}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right), (a > 0, b > 0, V > 0, \text{ or } a < 0, b < 0, V < 0) \tag{37}$$

and

$$u(x,t) = \sqrt{\frac{6V}{-a}}\left|\operatorname{csch}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right)\right|, (a < 0, b > 0, V > 0, \text{ or } a > 0, b < 0, V < 0) \tag{38}$$

respectively. Both (37) and (38) are solitary wave solutions of mKdV equation

$$u_t + au^2u_x + bu_{xxx} = 0.$$

If $r = \frac{1}{2}$, (33) and (34) become

$$u(x,t) = \frac{225}{64}\frac{V^2}{a^2}\operatorname{sech}^4 \frac{1}{4}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right), \tag{39}$$

and

$$u(x,t) = \frac{225}{64}\frac{V^2}{a^2}\operatorname{csch}^4 \frac{1}{4}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right), \tag{40}$$

respectively. Both (39) and (40) are solitary wave solutions of the equation [18]

$$u_t + a\sqrt{u}u_x + bu_{xxx} = 0.$$

If $r = \frac{2}{3}$, (33) and (34) become

$$u(x,t) = \frac{40\sqrt{5}}{27}\left(\frac{V}{a}\right)^{\frac{3}{2}}\operatorname{sech}^3 \frac{1}{3}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right), \tag{41}$$

and

$$u(x,t) = \frac{40\sqrt{5}}{27}\left(\frac{V}{a}\right)^{\frac{3}{2}}\left|\operatorname{csch} \frac{1}{3}\sqrt{\frac{V}{b}}\left(x - Vt + \xi_0\right)\right|^3, \tag{42}$$

respectively. Both (41) and (42) are solitary wave solutions of the equation

$$u_t + au^{\frac{2}{3}}u_x + bu_{xxx} = 0.$$

4 Solitary wave solution of GBBM

We seek travelling wave solution of GBBM equation (2) in the form

$$u(x,t) = u(\xi), \xi = x - Vt. \tag{43}$$

Substituting (43) into Eq. (2) yields an ODE for $u = u(\xi)$

$$(1 - V)u' + au^ru' - bVu''' = 0. \tag{44}$$

Integrating ODE (44) with respect to ξ once and taking the constant of integration to zero, we can obtain the ODE for $u = u(\xi)$

$$u'' = \frac{1-V}{bV}u + \frac{a}{bV(r+1)}u^{r+1}, \tag{45}$$

which belongs in the same type as Eq. (9) introduced in section 2 provided that $\frac{1-V}{bV} > 0$.

Appling the results (29) and (30) of ODE (9) in section 2, the solutions of ODE (45), i.e., the solitary wave solutions of GBBM equation (2) can be given by

$$u(x,t) = \left[\frac{(r+2)(r+1)(V-1)}{2a}\operatorname{sech}^2 \frac{r}{2}\sqrt{\frac{1-V}{bV}}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}}, \tag{46}$$

provided that $a > 0, b < 0, V > 1$, or $a < 0, b > 0, 0 < V < 1$, or $a < 0, b < 0, V < 0$, and

$$u(x,t) = \left[\frac{(r+2)(r+1)(1-V)}{2a}\operatorname{csch}^2 \frac{r}{2}\sqrt{\frac{1-V}{bV}}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}}, \tag{47}$$

provided that $a < 0, b < 0, V > 1$, or $a > 0, b > 0, 0 < V < 1$, or $a > 0, b < 0, V < 0$.

In particular case, if $r = 1$, (46) and (47) become

$$u(x,t) = \frac{3(V-1)}{a}\operatorname{sech}^2 \frac{1}{2}\sqrt{\frac{1-V}{bV}}\,(x - Vt + \xi_0)\,, \tag{48}$$

$$(a > 0, b < 0, V > 1, \text{ or } a < 0, b > 0, 0 < V < 1, \text{ or } a < 0, b < 0, V < 0)$$

and

$$u(x,t) = \frac{3(1-V)}{a}\operatorname{csch}^2 \frac{1}{2}\sqrt{\frac{V-1}{bV}}\,(x - Vt + \xi_0)\,, \tag{49}$$

$$(a < 0, b < 0, V > 1, \text{ or } a > 0, b > 0, 0 < V < 1, \text{ or } a > 0, b < 0, V < 0)$$

respectively. (48) and (49) are both the solitary wave solutions of the BBM equation

$$u_t + u_x + auu_x + bu_{xxt} = 0.$$

Similar to the GKdV case, from (46) and (47) we can also come to the conclusion that the behaviour of solitary wave of GBBM (2) is affected by the signs of nonlinear coefficient a and dispersive coefficient b :

1. When $a > 0, b < 0$, GBBM Eq. (2) admits not only a bounded positive solitary wave which travels to the right with velocity $V > 1$ (Figure 2(a)), but also an unbounded positive solitary wave which travels to the left (Figure 2(b)) .

2. When $a < 0, b > 0$, GBBM Eq. (2) admits only a bounded positive solitary wave which travels to the right with velocity $0 < V < 1$(Figure 2(c)).

3. When both a and b are positive,GBBM Eq. (2) admits only an unbounded positive solitary wave which travels to the right with velocity velocity $0 < V < 1$ (Figure 2(d)).

4. When both a and b are negative, GBBM Eq. (2) admits not only a bounded positive solitary wave travels to the left (Figure 2(e)), but also an unbounded positive solitary wave travels to the right with velocity $V > 1$ (Figure 2(f)).

5 Solitary wave solution of GZK equation

In order to seek travelling wave solution of GZK equation (3) we suppose that

$$u(x,y,t) = u(\xi), \xi = x + ky - Vt. \tag{50}$$

By using (50), Eq. (3) is transformed into an ODE for $u = u(\xi)$

$$-u' + au^r u' + b(1+k^2)u''' = 0.$$

Integrating the ODE with respect to ξ once and taking the constant of integration to zero, we can get

$$u'' = \frac{V}{b(1+k^2)}u - \frac{au^{r+1}}{b(r+1)(1+k^2)}. \tag{51}$$

Using the solutions (29) and (30) of ODE (9) in section 2, we can get the positive solutions of Eq. (51), i.e., the solitary wave solutions of GZK equation (3) as follows

$$u(x,y,t)=\left[\frac{V(r+2)(r+1)}{2a}\operatorname{sech}^2\frac{r}{2}\sqrt{\frac{V}{b(1+k^2)}}\,(x+ky-Vt+\xi_0)\right]^{\frac{1}{r}}, \tag{52}$$

provided $a>0,b>0,V>0$, or $a<0,b<0,V<0$, and

$$u(x,y,t)=\left[-\frac{V(r+2)(r+1)}{2a}\operatorname{csch}^2\frac{r}{2}\sqrt{\frac{V}{b(1+k^2)}}\,(x+ky-Vt+\xi_0)\right]^{\frac{1}{r}}, \tag{53}$$

provided $a<0,b>0,V>0$, or $a>0,b<0,V<0$, respectively.

From (52) and (53) we find that if $ab>0$, GZK admits a bounded solitary wave; if $ab<0$, GZK admits an unbounded solitary wave.

In particular, if $r=1$, (52) and (53) become

$$u(x,y,t)=\frac{3V}{a}\operatorname{sech}^2\frac{1}{2}\sqrt{\frac{V}{b(1+k^2)}}\,(x+ky-Vt+\xi_0)\,, \tag{54}$$

$$(a>0,b>0,V>0,\ \text{or}\ a<0,b<0,V<0)$$

and

$$u(x,y,t)=-\frac{3V}{a}\operatorname{csch}^2\frac{1}{2}\sqrt{\frac{V}{b(1+k^2)}}\,(x+ky-Vt+\xi_0)\,, \tag{55}$$

$$(a<0,b>0,V>0,\ \text{or}\ a>0,b<0,V<0)$$

respectively. Both (54) and (55) are the solitary wave solutions of the ZK equation

$$u_t+auu_x+b(u_{xx}+u_{yy})_x=0.$$

6 Solitary wave solution of GKP equation

We seek travelling wave solution of GKP equation (4) in the form

$$u(x,y,t)=u(\xi),\xi=x+ky-Vt, \tag{56}$$

where k and V are constants. Substituting (56) into Eq. (4) yields an ODE for $u=u(\xi)$

$$(-Vu'+au^ru'+bu''')'+\sigma^2k^2u''=0.$$

Integrating the ODE with respect to ξ twice and taking the constants of integration to zero, yields

$$u''=\frac{V-\sigma^2k^2}{b}u-\frac{au^{r+1}}{b(r+1)}. \tag{57}$$

Using the the solutions (29) and (30) of ODE (9) in section 2, we can get the positive solutions of ODE (57), i.e., the solitary wave solutions of GKP equation (4) as follows

$$u(x,y,t)=\left[\frac{(V-\sigma^2k^2)(r+2)(r+1)}{2a}\operatorname{sech}^2\frac{r}{2}\sqrt{\frac{V-\sigma^2k^2}{b}}\,(x+ky-Vt+\xi_0)\right]^{\frac{1}{r}}, \tag{58}$$

provided that $a>0,b>0,V>\sigma^2k^2$, or $a<0,b<0,V<\sigma^2k^2$, and

$$u(x,y,t)=\left[\frac{(\sigma^2k^2-V)(r+2)(r+1)}{2a}\operatorname{csch}^2\frac{r}{2}\sqrt{\frac{V-\sigma^2k^2}{b}}\,(x+ky-Vt+\xi_0)\right]^{\frac{1}{r}}, \tag{59}$$

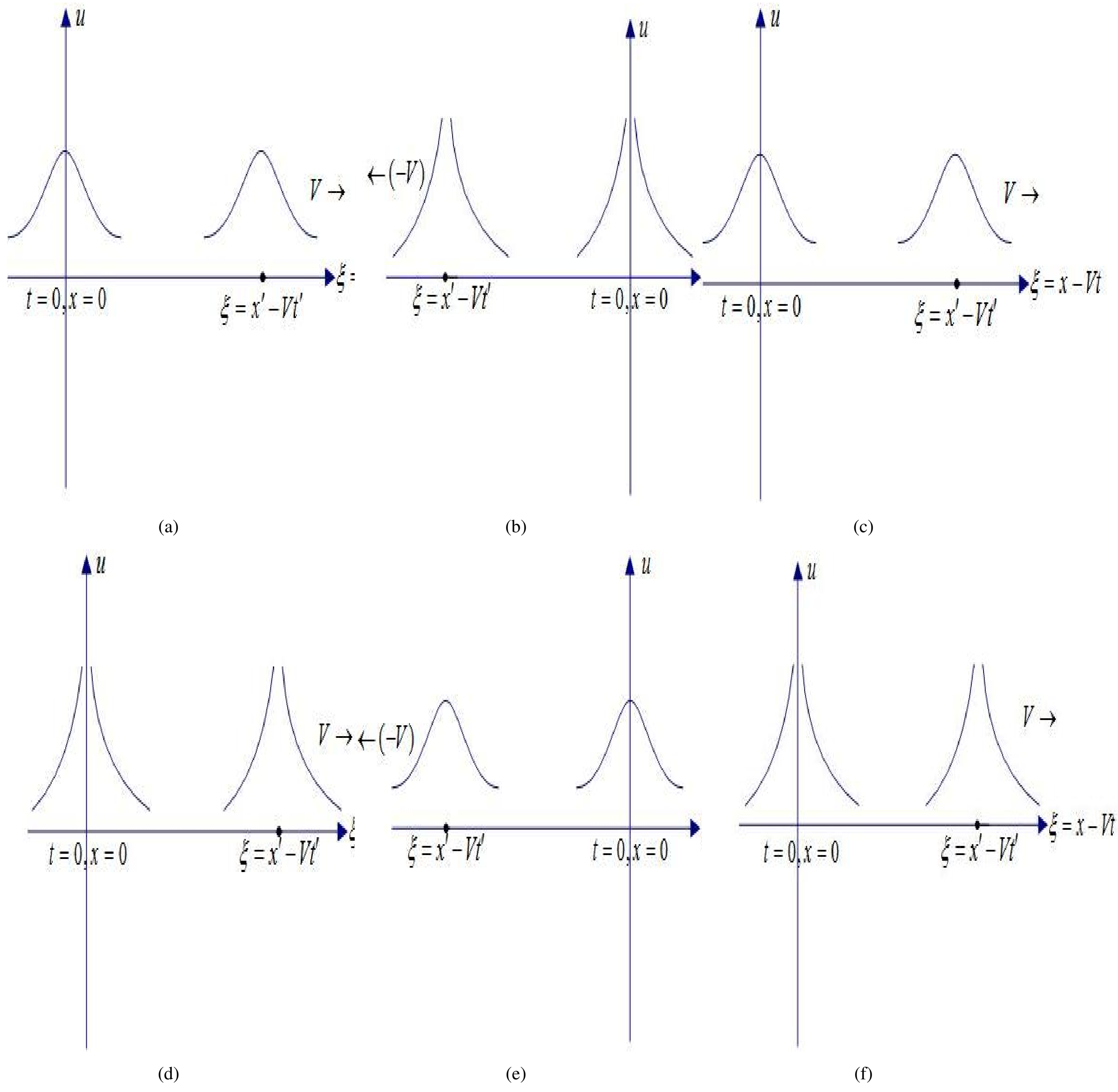

Figure 2:

provided that $a < 0, b > 0, V > \sigma^2 k^2$, or $a > 0, b < 0, V < \sigma^2 k^2$, respectively.

From (58) and (59) we also find that if $ab > 0$, GKP admits a bounded solitary wave solution; if $ab < 0$, GKP admits an unbounded solitary wave solution.

In particular, if $r = 1$, (58) and (59) become

$$u(x, y, t) = \frac{3(V - \sigma^2 k^2)}{a} \operatorname{sech}^2 \frac{1}{2}\sqrt{\frac{V - \sigma^2 k^2}{b}} (x + ky - Vt + \xi_0), \tag{60}$$

$$(a > 0, b > 0, V > \sigma^2 k^2, \text{ or } a < 0, b < 0, V < \sigma^2 k^2)$$

and

$$u(x, y, t) = \frac{3(\sigma^2 k^2 - V)}{a} \operatorname{csch}^2 \frac{1}{2}\sqrt{\frac{V - \sigma^2 k^2}{b}} (x + ky - Vt + \xi_0), \tag{61}$$

$$(a < 0, b > 0, V > \sigma^2 k^2, \text{ or } a > 0, b < 0, V < \sigma^2 k^2)$$

respectively. Both (60) and (61) are solitary wave solutions of the KP equation

$$(u_t + auu_x + bu_{xxx})_x + \sigma^2 u_{yy} = 0.$$

7 Solitary wave solution of GKG equation

In order to look for the travelling wave solution of generalized (m+1)-dimensional Klein-Gordon equation (5), we suppose that

$$u(x_1, x_2, \cdots, x_m, t) = u(\xi), \xi = \sum_{i=1}^{m} k_i x_i - Vt, \sum_{i=1}^{m} k_i^2 = 1. \tag{62}$$

By using (62), the GKG equation (5) is transformed into an ODE for $u = u(\xi)$

$$u'' = \frac{a}{c^2 - V^2} u + \frac{bu^{r+1}}{c^2 - V^2}, c^2 \neq V^2. \tag{63}$$

Using the solutions (29) and (30) of ODE (9) in section 2, the positive solutions of ODE (63), i.e., the solitary wave solutions of GKG equation (5) can be given by

$$u(x_1, x_2, \cdots, x_m, t) = \left[-\frac{a(r+2)}{2b} \operatorname{sech}^2 \frac{r}{2}\sqrt{\frac{a}{c^2 - V^2}} \left(\sum_{i=1}^{m} k_i x_i - Vt + \xi_0 \right) \right]^{\frac{1}{r}}, \tag{64}$$

if $a > 0, b < 0, V^2 < c^2$, or $a < 0, b > 0, V^2 > c^2$,

and

$$u(x_1, x_2, \cdots, x_m, t) = \left[\frac{a(r+2)}{2b} \operatorname{csch}^2 \frac{r}{2}\sqrt{\frac{a}{c^2 - V^2}} \left(\sum_{i=1}^{m} k_i x_i - Vt + \xi_0 \right) \right]^{\frac{1}{r}}, \tag{65}$$

if $a > 0, b > 0, V^2 < c^2$, or $a < 0, b < 0, V^2 > c^2$, respectively.

In (1+1)-dimensional case, i.e $m = 1$, the results (64) and (65) of generalized Klein-Gordon (5), are found to be agreement with that obtained by using exp-function method in [7].

In particular, if $r = 1$, (64) and (65) become

$$u(x_1, x_2, \cdots, x_m, t) = -\frac{3a}{2b} \operatorname{sech}^2 \frac{1}{2}\sqrt{\frac{a}{c^2 - V^2}} \left(\sum_{i=1}^{m} k_i x_i - Vt + \xi_0 \right), \tag{66}$$

$$(a > 0, b < 0, V^2 < c^2, \text{ or } a < 0, b > 0, V^2 > c^2)$$

and

$$u(x_1, x_2, \cdots, x_m, t) = \frac{3a}{2b} \operatorname{csch}^2 \frac{1}{2}\sqrt{\frac{a}{c^2 - V^2}} \left(\sum_{i=1}^{m} k_i x_i - Vt + \xi_0 \right), \tag{67}$$

$$(a > 0, b > 0, V^2 < c^2, \text{ or } a < 0, b < 0, V^2 > c^2)$$

respectively. Both (66) and (67) are the solitary wave solutions of Klein-Gordon equation with quadratic nonlinear term

$$u_{tt} - c^2 \Delta_m u + au + bu^2 = 0, \Delta_m = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}.$$

In (1+1)-dimensional case, the results (66) and (67) of Klein-Gordon equation are also found to be agreement with that (u_1 in (17) and u_2 in (18)) obtained in Ref. [16].

8 Solitary wave solution of G-Boussinesq equation

We seek travelling wave solution of generalized Boussinesq equation (6) in the form

$$u(x,t) = u(\xi), \xi = x - Vt. \tag{68}$$

Substituting (68) into Eq. (6), yields an ODE for $u = u(\xi)$

$$(1 - V^2)u'' + a\left(u^{r+1}\right)'' + bu^{(4)} = 0.$$

Integrating the ODE with respect to ξ twice and taking the constants of integration to zero, yields

$$u'' = \frac{V^2 - 1}{b} u - \frac{a}{b} u^{r+1}. \tag{69}$$

Using the solutions (29) and (30) of ODE (9) in section 2, the positive bounded solution of ODE (69), i.e., solitary wave solution of generalized Boussinesq equation (6) can be given by

$$u(x,t) = \left[\frac{(V^2-1)(r+2)}{2a} \operatorname{sech}^2 \frac{r}{2} \sqrt{\frac{V^2-1}{b}} \left(x - Vt + \xi_0\right) \right]^{\frac{1}{r}}, \tag{70}$$

provided that $a > 0, b > 0, V^2 > 1$, or $a < 0, b < 0, V^2 < 1$,

and

$$u(x,t) = \left[\frac{(1-V^2)(r+2)}{2a} \operatorname{csch}^2 \frac{r}{2} \sqrt{\frac{V^2-1}{b}} \left(x - Vt + \xi_0\right) \right]^{\frac{1}{r}}, \tag{71}$$

provided that $a < 0, b > 0, V^2 > 1$, or $a > 0, b < 0, V^2 < 1$, respectively.

In particular, if $r = 1$, (70) and (71) become

$$u(x,t) = \frac{3(V^2-1)}{2a} \operatorname{sech}^2 \frac{1}{2} \sqrt{\frac{V^2-1}{b}} \left(x - Vt + \xi_0\right), (a > 0, b > 0, V^2 > 1, \text{ or } a < 0, b < 0, V^2 < 1) \tag{72}$$

and

$$u(x,t) = \frac{3(1-V^2)}{2a} \operatorname{csch}^2 \frac{1}{2} \sqrt{\frac{1-V^2}{b}} \left(x - Vt + \xi_0\right), (a < 0, b > 0, V^2 > 1, \text{ or } a > 0, b < 0, V^2 < 1), \tag{73}$$

respectively. Both (72) and (73) are solitary wave solutions of the Boussinesq equation

$$u_{xx} - u_{tt} + a\left(u^2\right)_{xx} + bu_{xxxx} = 0.$$

9 Solitary wave solution of GNLS equation

In order to seek travelling wave solution of GNLS equation (7), we suppose that

$$u(x,t) = F(\xi) e^{i(Vx + \omega t)}, \xi = x - Vt, \tag{74}$$

V and ω are to be determined later. Substituting (74) into Eq. (7), cancelling the common exponential factor $e^{i(Vx+\omega t)}$, yields an ODE for $F = F(\xi)$ as follows

$$F'' = (2\omega + V^2)F - 2aF^{r+1}. \tag{75}$$

Using the solutions (29) and (30) of ODE (9) in section 2, we come to the conclusion that if $2\omega + V^2 > 0, a > 0$, then Eq. (75) admits a solution

$$F(\xi) = \left[\frac{(2\omega + V^2)(r+2)}{4a}\operatorname{sech}^2\frac{r}{2}\sqrt{2\omega + V^2}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}}; \tag{76}$$

and if $2\omega + V^2 > 0, a < 0$, then Eq. (75) admits a solution

$$F(\xi) = \left[-\frac{(2\omega + V^2)(r+2)}{4a}\operatorname{csch}^2\frac{r}{2}\sqrt{2\omega + V^2}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}}. \tag{77}$$

Substituting (76) and (77) into (74) we have solitary wave solutions of GNLS equation (7) as follows

$$u(x,t) = \left[\frac{(2\omega + V^2)(r+2)}{4a}\operatorname{sech}^2\frac{r}{2}\sqrt{2\omega + V^2}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}} e^{i(Vx+\omega t)}, \tag{78}$$

where $2\omega + V^2 > 0, a > 0$; and

$$u(x,t) = \left[-\frac{(2\omega + V^2)(r+2)}{4a}\operatorname{csch}^2\frac{r}{2}\sqrt{2\omega + V^2}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}} e^{i(Vx+\omega t)}, \tag{79}$$

where $2\omega + V^2 > 0, a < 0$.

In particular, if $r = 2$, (78) and (79) become

$$u(x,t) = \sqrt{\frac{2\omega + V^2}{a}}\operatorname{sech}\sqrt{2\omega + V^2}\,(x - Vt + \xi_0)\, e^{i(Vx+\omega t)}(2\omega + V^2 > 0, a > 0) \tag{80}$$

and

$$u(x,t) = \sqrt{\frac{2\omega + V^2}{-a}}\left|\operatorname{csch}\sqrt{2\omega + V^2}\,(x - Vt + \xi_0)\right| e^{i(Vx+\omega t)}(2\omega + V^2 > 0, a < 0) \tag{81}$$

respectively. Both (80) and (81) are solitary wave solutions of NLS equation

$$iu_t + \frac{1}{2}u_{xx} + a\,|u|^2\,u = 0.$$

10 Solitary wave solution of GZE

In order to look for solitary wave solution of GZE $(8)_1 - (8)_2$. we suppose that

$$E(x,t) = u(\xi)e^{i\eta}, \xi = x - Vt, \eta = \frac{V}{2}x + \omega t, \qquad (82)_1$$

$$H(x,t) = H(\xi). \qquad (82)_2$$

Substituting (82) into GZE (8), after integrated it, yields

$$u'' = \left(\frac{V^2}{4} + \omega\right)u + \left(\frac{\alpha\beta}{V^2 - 1} + b\right)u^{r+1}, \qquad (83)_1$$

$$H = \frac{\beta}{V^2 - 1}u^r. \qquad (83)_2$$

Using the solutions (29) and (30) of ODE (9) in section 2, we can get the positive solutions of ODE (83). In view of $(82)_1$ and $(82)_2$, the solitary wave solutions of GZE can be given by

Case 1: $\frac{V^2}{4} + \omega > 0, \frac{\alpha\beta}{V^2-1} + b < 0$,

$$E(x,t) = \left[-\frac{\left(\frac{V^2}{4} + \omega\right)(r+2)}{2\left(\frac{\alpha\beta}{V^2-1} + b\right)}\operatorname{sech}^2\frac{r}{2}\sqrt{\frac{V^2}{4} + \omega}\,(x - Vt + \xi_0)\right]^{\frac{1}{r}} e^{i\left(\frac{V}{2}x + \omega t\right)}, \qquad (84)_1$$

$$H(x,t)=\frac{\beta}{V^2-1}\left[-\frac{\left(\frac{V^2}{4}+\omega\right)(r+2)}{2\left(\frac{\alpha\beta}{V^2-1}+b\right)}\operatorname{sech}^2\frac{r}{2}\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0)\right]. \quad (84)_2$$

Case 2: $\frac{V^2}{4}+\omega>0$, $\frac{\alpha\beta}{V^2-1}+b>0$,

$$E(x,t)=\left[\frac{\left(\frac{V^2}{4}+\omega\right)(r+2)}{2\left(\frac{\alpha\beta}{V^2-1}+b\right)}\operatorname{csch}^2\frac{r}{2}\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0)\right]^{\frac{1}{r}}e^{i\left(\frac{V}{2}x+\omega t\right)}, \quad (85)_1$$

$$H(x,t)=\frac{\beta}{V^2-1}\left[\frac{\left(\frac{V^2}{4}+\omega\right)(r+2)}{2\left(\frac{\alpha\beta}{V^2-1}+b\right)}\operatorname{csch}^2\frac{r}{2}\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0)\right]. \quad (85)_2$$

In particular, if $r=2$, (84) and (85) become

$$E(x,t)=\sqrt{\frac{-2\left(\frac{V^2}{4}+\omega\right)}{\frac{\alpha\beta}{V^2-1}+b}}\operatorname{sech}\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0)\,e^{i\left(\frac{V}{2}x+\omega t\right)}, \quad (86)_1$$

$$H(x,t)=\frac{\beta}{V^2-1}\left[-\frac{2\left(\frac{V^2}{4}+\omega\right)}{\frac{\alpha\beta}{V^2-1}+b}\operatorname{sech}^2\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0)\right], \quad (86)_2$$

where $\frac{V^2}{4}+\omega>0$, $\frac{\alpha\beta}{V^2-1}+b<0$, and

$$E(x,t)=\sqrt{\frac{2\left(\frac{V^2}{4}+\omega\right)}{\frac{\alpha\beta}{V^2-1}+b}}\left|\operatorname{csch}\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0)\right|e^{i\left(\frac{V}{2}x+\omega t\right)}, \quad (87)_1$$

$$H(x,t)=\frac{\beta}{V^2-1}\frac{2\left(\frac{V^2}{4}+\omega\right)}{\frac{\alpha\beta}{V^2-1}+b}\operatorname{csch}^2\sqrt{\frac{V^2}{4}+\omega}\,(x-Vt+\xi_0), \quad (87)_2$$

where $\frac{V^2}{4}+\omega>0$, $\frac{\alpha\beta}{V^2-1}+b>0$, respectively.

Both (86) and (87) are solitary wave solutions of the Zakharov equations as follows

$$iE_t+E_{xx}=\alpha EH+b\,|E|^2\,E,\qquad H_{tt}-H_{xx}=\beta\left(|E|^2\right)_{xx}.$$

11 Conclusions

In this paper, the ODE (9) were introduced and solved by transformation of dependent variable and the $\left(\frac{G'}{G}\right)$-expansion method. The solutions (29) and (30) of ODE (9) have played an important role for finding solitary wave solutions of a class of nonlinear evolution equations. The travelling wave reduction ODEs of a number of PDEs with a positive real number power term considered in this paper, after integrated it, can be reduced to second order ODEs which belongs in the same type as ODE (9). Therefore, the solitary wave solution of nonlinear PDEs considered in this paper can be obtained by using the solutions (29) and (30) of ODE (9). We have illustrated the method in detail with eight examples. If we are only interested in finding solitary wave solutions of nonlinear evolution equations with a positive real number power term, the method used in the paper is really a convenient, simple, and effective method. ODE(9) and its solutions (29) and (30) can be applied for finding solitary wave solutions of other nonlinear evolution equations with a power law nonlinearity.

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