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Cascading Failure in Star-like Network of N Clustered Networks with Interdependent And Interconnected Links

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Abstract: In real life, many systems show significant clustering, studying the properties of clustered network is more necessary.Previous studies are concentrated on single clustered network or coupled clustered networks with dependency links.Nevertheless real world systems are more complicated. In this work,we develop a framework for studying the cascade of failures in a star-like network of n clustered networks with interdependent and interconnected links.When the networks are fully coupled, we obtain the analytical solution of giant component P_∞.We carry out numerical simulation ,and find that the simulation results agree well with the theory.

Keywords: Cascading Failure; Clustered Networks; Interdependent and Interconnected Links; Star-Like Networks;

1 Introduction

In recent years,advances in the field of complex networks have been made[1-14].Most of the research have concentrated on a single network that do not connect with or depend on other networks.However,most of the real-world systems are complicated and are not isolated,they are often coupled with interconnected links[3-4,15-16], or interdependent links[17-22], or both. Leicht and Douza [16] investigated the cascade of failures in interacting networks by using a generating function ,which show the interconnected links influence the robustness of the system. Four years ago, robustness of two interdependent networks have been studied,observed that the system becomes less robust due to the dependency coupling. Gao et al. [23] developed a framework to study percolation of the "network of networks"(NON),which suggests that the percolation theory of a single network is a limiting case of a more general case of percolation of interdependent networks.

In addition, many networks in real life show significant clustering. Newman [24] showed random-graph model of a clustered network by using the method of generating function. Adam et al. [25] developed an analytical framework for studing the effect of clustering qualitatively on the cascade condition,which demonstrated that the cascade size decreasing with the increasing of clustering .Gleeson et al. [26] studied the influence of clustering on the bond percolation threshold compared with the case for non-clustered networks.Lately, Buldyrev et al. [21] investigated the cascade of failures in two interacting unclustered networks with dependency links,which showed that the interdependence links have influence on the robustness of the system.More recently, Huang et al. [27] investigated the robustness of two interdependent clustered networks with dependency links,and they found that clustering decreases the robustness of the system which is opposite to single clustered network.. Shao et al. [28]developed a framework to investigate the robustness of partially interdependent network formed of clustered networks,which showed that clustering coefficient became smaller as the coupling strength is reduced which effects the robustness of network.Hu et al. [29]investigated the cascade of failures in two random unclustered networks, which with both interdependent and interconnected links. They found that the change of interconnecting links leading to the change of the transition from second order phase transition to first order through hybrid phase transition .

For interacting networks, previous researches are concentrated on random clustered networks with dependency links. Nevertheless, in real life clustered networks with both dependency and connectivity links are more common. In this

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work, we study analytically and numerically the cascading failure of a star-like NON with clustering under no-feedback condition, which is interdependent and interconnected coupled.We first develop a framework for studying star-like network of n clustered networks with interdependent and interconnected links,then analyze the process of the cascade failures in a special case.Finally,we obtain the analytical expression of P_∞.

2 The model

In this paper, we consider a system of star-like network of n clustered networks with interdependent and interconnected links,based on the percolation approach.We suppose that each network i $(i = 1, 2, ..., n)$ consists of N_i nodes linked together by connectivity links. Two networks i and l are partially coupled by dependency links, so that a fraction q_{li} of nodes in network i depends on nodes in network l $(l = 1, 2, ..., n; l \neq i)$,i.e., they cannot function if the nodes in network l on which they depend do not function. And a fraction q_{li} of nodes in l depends on nodes in network i. If $q_{il} = q_{li}$,the partially dependent pair becomes fully dependent.We further assume that each node a in network i depends on only one node b in network l (uniqueness condition), and if node a in network i depends on node b in network l, and node b in network l depends on node c in network i, node a must coincide with node c (no-feedback condition).On the other hand, connectivity links are established by randomly connecting the nodes within each network and between networks i and l according to degree distributions $P^i_{k_{i1},...,s_i,t_i,...,k_{in}}$,which denotes the probability of a node in network i to have s_i single edges and t_i triangles to other nodes in network i and k_{il} links towards nodes in network l $(l = 1, 2, ..., n; l \neq i)$.The conventional degree of each node k_i is equal to $s_i + 2t_i$.When the density of triangles of each network is approximative to zero, it corresponding to the situation of random network.

In this study,we define $n + 2$ dimensional generating functions of the degree distributions describing all the connectivity links, $G^i_0(a_i, b_i, z_1, \ldots, z_n) = \sum\limits_{k_{i1},...,s_i,t_i,...,k_{in}} P^i_{k_{i1},...,s_i,t_i,...,k_{in}} z_1^{k_{i1}} \cdots a_i^{s_i} b_i^{t_i} \cdots z_n^{k_{in}}$. The generating function of the joint degree distribution $P^i_{s_i,t_i}$ and clustering coefficient of network i are given by
$G^{ii}_0(a_i, b_i) = \sum_{s_i,t_i=0}^{\infty} P^i_{s_i,t_i} a_i^{s_i} b_i^{t_i}$,
$c = \frac{3\times(\ number\ \ of\ \ triangles\ \ in\ \ network\ )}{number\ \ of\ \ connected\ \ triples} = \frac{N_i \sum_{s_i,t_i} t_i P^i_{s_i.t_i}}{N_i \sum_k \binom{k}{2} P^i_k}$.

The unknowns x_i satisfy the system of n equations.

$$x_i = p_i \prod_{l=1}^{K} (q_{li} y_{li} g_l - q_{li} + 1), \tag{1}$$

where the product is taken over the K networks interlinked with network i by the partial dependency links and

$$y_{li} = \frac{x_l}{q_{il} y_{il} g_i - q_{il} + 1}, \tag{2}$$

has the meaning of the fraction of nodes in network l survived after the damage from all the networks connected to network l except from network i is taken into account.The damage from network i must be excluded due to the no-feedback condition. In the absence of the no-feedback condition Eq. (1) becomes much simpler since $y_{li} = x_l$.

3 A special case

As an example,n clustered networks have doubly Poisson degree distributions $P^i_{s,t} = e^{-\mu_i} \frac{\mu_i^s}{s!} e^{-\gamma_i} \frac{\gamma_i^t}{t!}$,and the degree distribution between network i and l is Poisson distribution. The parameters μ_i and γ_i are the average numbers of single edges and triangles of each vertex of networks i, $\widetilde{k}_{il}$ is the average interlinks degree of network i..Then the clustering coefficients can be expressed as:

$$c_i = \frac{2\gamma_i}{2\gamma_i + (\mu_i + 2\gamma_i)^2}, \tag{3}$$

We first remove a fraction of $1 - p$ nodes in each networks.These damages spread to all other networks, and then back to the root network, back and force. In this case, $y_{l1} = p\,(l = 2, 3, \ldots, n)$,where the index 1 represents the central network and l denotes the other networks.Under the following simplifying conditions that $q_{1l} = q_{l1} = q, \mu_1 = \mu_l = \mu, \gamma_1 = \gamma_l =$

$\gamma, k = \mu + 2\gamma, \widetilde{k}_{1l} = \widetilde{k}_{l1} = \widetilde{k}, l = 2, 3, \ldots, n$, we have $x_2 = x_3 = \ldots = x_n$, so we obtain

$$x_1 = p(qp - qpf_2 - q + 1)^{n-1}, \qquad x_2 = p - pq + p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2}. \tag{4}$$

And

$$\begin{aligned} f_1 = \exp\Big\{ & \Big[\mu p(qp - qpf_2 - q + 1)^{n-1} + 2p(qp - qpf_2 - q + 1)^{n-1}\left(1 - p(qp - qpf_2 - q + 1)^{n-1}\right)\gamma\Big](f_1 - 1) \\ & + \gamma p^2 (qp - qpf_2 - q + 1)^{2n-2}\left(f_1^2 - 1\right) - (n-1)\,\tilde{k}\,(1 - f_2)\Big[p - pq + p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2}\Big]\Big\} \\ f_2 = \exp\Big\{ & \Big[\mu\left(p - pq + p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2}\right) \\ & + 2\left(p - pq + p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2}\right)\left(1 - \left(p - pq + p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2}\right)\right)\gamma\Big] \\ & \cdot (f_2 - 1) + \gamma\left(p - pq + p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2}\right)^2\left(f_2^2 - 1\right) - \tilde{k}\,(1 - f_1)\,p(qp - qpf_2 - q + 1)^{n-1}\Big\} \end{aligned} \tag{5}$$

where $u_i = v_i = f_{il} = 1 - g_i\,(x_1, x_2, \ldots, x_n)$.
From the definition of $P_{\infty,i}$,we obtain

$$\begin{aligned} P_{\infty,1} &= x_1 g_1 = p(qp - qpf_2 - q + 1)^{n-1}\,(1 - f_1)\,, \\ P_{\infty,2} &= x_2 g_2 = \Big[p^2 q\,(1 - f_1)\,(qp - qpf_2 - q + 1)^{n-2} - pq + p\Big]\,(1 - f_2) \end{aligned} \tag{6}$$

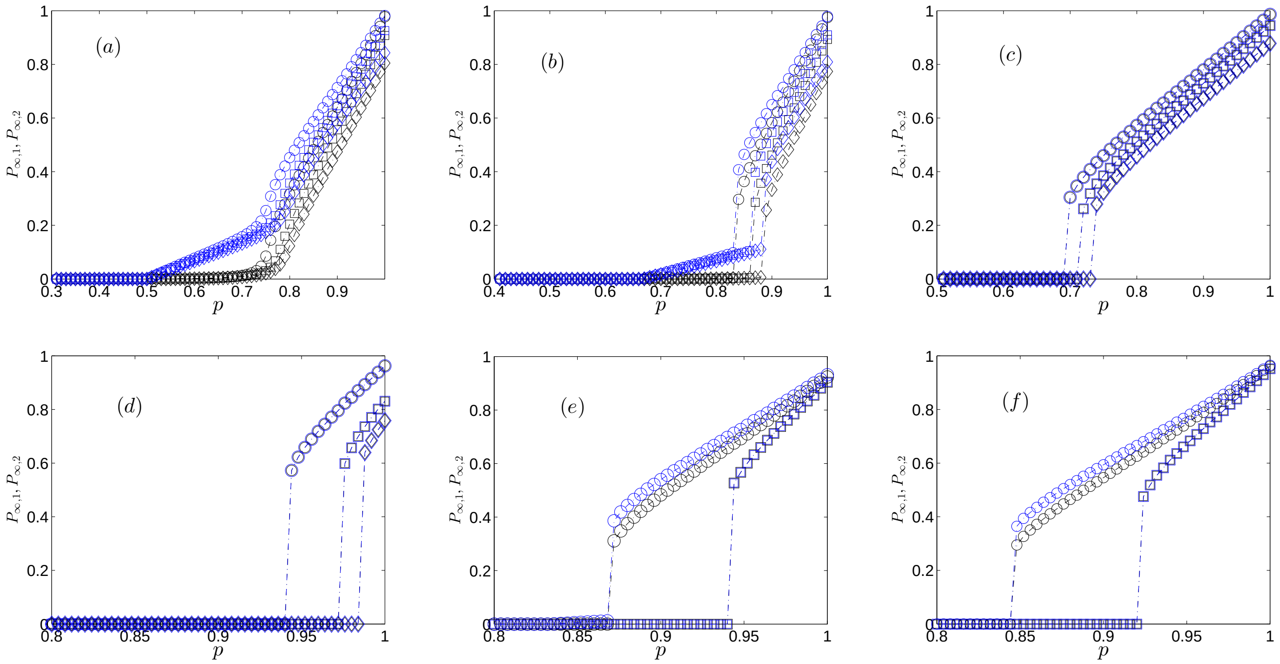

Figure 1: For a star-like network of n clustered networks,the giant component of the root network $P_{\infty,1}$(black),and other networks $P_{\infty,2}$ (blue) as a function of the remaining nodes p in every networks.Find that the simulation results(symbols) agree well with the theory(lines).

Figure 1 shows the fraction of the giant component $P_{\infty,1}$ and $P_{\infty,2}$ as a function of P from Eq(6), compares the numerical results and simulation results.Fig.1(a) shows the situation when $n = 5, k = 5, \widetilde{k} = 0.1, q = 0.6, c = 0, 0.1, 0.15$,(circle, c=0; square, c=0.1; demand, c=0.15). Fig.1(b) shows the situation when $n = 5, k = 5, \widetilde{k} = 0.1, q = 0.7, c = 0, 0.1, 0.15$,(circle, c=0; square, c=0.1; demand, c=0.15). Fig.1(c) shows the situation when $n = 2, k = 5, \widetilde{k} = 0.1, q = 1, c = 0, 0.1, 0.15$, (circle, c=0; square, c=0.1; demand, c=0.15). Fig.1(d) shows the situation when $n = 5, k = 5, \widetilde{k} = 0.1, q = 1, c = 0, 0.1, 0.12$, (circle, c=0; square, c=0.1; demand, c=0.12). Fig.1(e) shows the situation when $n = 5, k = 6, \widetilde{k} = 0.1, q = 0.8, 1, c = 0.1$, (circle, q=0.8; square, q=1). Fig.1(f) shows the situation when

$n = 5, k = 5, \widetilde{k} = 1, q = 0.8, 1, c = 0.1$, (circle, q=0.8; square, q=1).Find that the simulation results(symbols) agree well with the theory(lines).

Figure 1(a) ,(b) and (d) tell that as q increases, with other parameters fixed, the phase transitions undergoes from second-order to first-order, through hybrid phase transition, which means, when p decreases, the size of the giant component jumps at a critical point $p_c^{h,\mathrm{I}}$ from a large value to a small one and then continuously decreases to zero at $p_c^{h,\mathrm{II}}$.Furthermore, Fig.1(a) shows that, the values of second-order threshold p_c^{II} are the same for different values of c.It means that, for weak coupling strength q, the influence of clustering c on the robustness of network is little. However, Fig.1(d) demonstrates that the first order phase transition,p_c^{I} increases as c increases, it means that for strong coupling strength q, the system becomes more vulnerable as c increases. For the coupling strength q in Fig.1(b), as clustering coefficient c increases, the hybrid order phase transition points p_c^{I} increases and p_c^{II} keeps constant.From Fig.1(a),(b) and (d) note that when $p < 1$,that is the networks are partial dependent ,the central network is less robust than other networks,when $p = 1$,the giant component of the central network and other networks are the same .Fig.1(c) and (d) show that when the values of other parameters are fixed,with the increasing of n,the networks becomes more vulnerable,and the effect of clustering c in reducing the robustness becomes lager as n increases.Fig.1(d) and (e) demonstrate that when the values of other parameters are fixed,with the increasing of k,the networks becomes more robust.From Fig.1(d) and (f) ,we observe that when the values of other parameters are fixed,with the increasing of $\widetilde{k}$,the networks becomes more robust.From Fig.1(a) ,(b),(c),(d), we will see when the clustering efficient c increases,the system becomes vulnerable,and different n and q may have some influence on the effect of c.

Especially when $q = 1$,the star-like NON are fully interdependent.Eqs. (4)-(6) yield simple forms

$$x_1 = p^n(1-f_2)^{n-1}, \qquad x_2 = p^n(1-f_1)(1-f_2)^{n-2}. \tag{7}$$

$$\begin{aligned} f_1 &= e^{-\left[k+(n-1)\tilde{k}\right]p^n(1-f_1)(1-f_2)^{n-1}+\gamma\left[p^n(1-f_1)(1-f_2)^{n-1}\right]^2}, \\ f_2 &= e^{-\left(k+\tilde{k}\right)p^n(1-f_1)(1-f_2)^{n-1}+\gamma\left[p^n(1-f_1)(1-f_2)^{n-1}\right]^2}. \end{aligned} \tag{8}$$

$$P_\infty = P_{\infty,1} = P_{\infty,2} = p^n(1-f_1)(1-f_2)^{n-1} = p^n\left(1-e^{-\left[k+(n-1)\tilde{k}\right]P_\infty+\gamma P_\infty^2}\right)\left(1-e^{-\left(k+\tilde{k}\right)P_\infty+\gamma P_\infty^2}\right)^{n-1}. \tag{9}$$

From equation (8),we get

$$\begin{aligned} \ln f_1 &= -\left[k+(n-1)\tilde{k}\right]p^n(1-f_1)(1-f_2)^{n-1}+\gamma\left[p^n(1-f_1)(1-f_2)^{n-1}\right]^2, \\ \ln f_2 &= -\left(k+\tilde{k}\right)p^n(1-f_1)(1-f_2)^{n-1}+\gamma\left[p^n(1-f_1)(1-f_2)^{n-1}\right]^2. \end{aligned} \tag{10}$$

Then we have

$$\begin{aligned} \ln f_1 &= -\left[k+(n-1)\tilde{k}\right]p^n(1-f_1)(1-f_2)^{n-1}+\ln f_2+\left(k+\tilde{k}\right)p^n(1-f_1)(1-f_2)^{n-1}, \\ &= -(n-2)\tilde{k}p^n(1-f_1)(1-f_2)^{n-1}+\ln f_2, \\ f_1 &= 1+\tfrac{\ln f_1-\ln f_2}{(n-2)\tilde{k}p^n(1-f_2)^{n-1}}. \end{aligned} \tag{11}$$

A nontrivial solution first emerges in the critical case when both sides of this equation have equal derivatives,

$$1 = \tfrac{df_1}{df_1} = \tfrac{1}{(n-2)\tilde{k}p^n(1-f_2)^{n-1}f_1}. \quad f_1 = \tfrac{1}{(n-2)\tilde{k}p^n(1-f_2)^{n-1}}. \quad p_c = \left(\tfrac{1}{(n-2)\tilde{k}f_1(1-f_2)^{n-1}}\right)^{\frac{1}{n}}. \tag{12}$$

4 Conclusions

In summary, we have introduced the generating function of the joint degree distribution for studying the cascading failure in a star-like network of n clustered networks with interdependent and interconnected links.We obtain the analytical solution of P_∞ as the coupling strength $q = 1$.We carry out numerical simulation ,and find that the simulation results agree well with the theory.

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