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On an Efficient Technique to Solve Nonlinear Fractional-order Partial Differential Equations

Qazi Mahmood Ul Hassan, Syed Tauseef Mohyud-Din *
Department of Mathematics, Faculty of Sciences, HITEC University, Taxila Cantt Pakistan
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Abstract: This paper witnesses the combination of an efficient transformation and Exp-function method to construct generalized solitary wave solutions of the nonlinear Klein-Gordon equations of fractional-order. Computational work and subsequent numerical results re-confirm the efficiency of proposed algorithm. It is observed that suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.

Keywords: Klein–Gordon equation; fractional calculus; exp-function method; modified Riemann-Liouville derivative.

1 Introduction

The subject of factional calculus [1, 2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential equations (NFDEs) have gained much interest due to exact description of nonlinear phenomena of many real-time problems. The fractional calculus is also considered as a novel topic [3, 4]; has gained considerable popularity and importance during the recent past. It has been the subject of specialized conferences, workshops and treatises or so, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Some of the areas of present-day applications of fractional models [5-8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics and signal processing, bio-sciences, economics, geology, astrophysics, probability and statistics, chemical physics, and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations, see [1–8] and the references therein. Recently, He and Wu [9] developed a very efficient technique which is called exp-function method for solving various nonlinear physical problems. The through study of literature reveals that Exp-function method has been applied on a wide range of differential equations and is highly reliable. The exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. The subsequent works have shown the complete reliability and efficiency of this algorithm. He et. al. [10-11] used this scheme to find periodic solutions of evolution equations; Mohyud-Din [12-15] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [16] tried this novel approach for Fisher's equation; Wu et. al. [17, 18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [19] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolutions; Zhu [21, 22] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [24] for an explicit and numerical solutions of the fractional KdV equation; Ebaid [25] for the improvement on the Exp-function method when balancing the highest order linear and nonlinear terms. The basic motivation of this paper is the development of an efficient combination comprising an efficient transformation**,** exp-function method using Jumarie's derivative approach [27-30] and its subsequent application to construct generalized solitary wave solutions of the nonlinear Klein-Gordon equations [26] of fractional-order. It is to be highlighted that Ebaid [25] proved that $c = d$ and $p = q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation. The Klein-Gordon

* **Corresponding author**. *E-mail address*: syedtauseefs@hotmail.com

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equations [26] appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics and applied and physical sciences.

Theorem 1 (25) *Suppose that $u^{(r)}$ and u^{γ} are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $U(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}$, leads to $c=d$ and$p=q$, $\forall r,s,\Omega,\lambda\geq 1$.*

Theorem 2 *Suppose that $u^{(r)}$ and $u^{(s)}u^{k}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and$p=q$,$\forall r,s,k\geq 1$..*

Theorem 3 (25) *Suppose that $u^{(r)}$ and$(u^{(s)})^{\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where $r,s$and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and$p=q$,$\forall r,s\geq 1,\forall\Omega\geq 2$.*

Theorem 4 (25) *Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}u^{\lambda}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s,Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and$p=q$,$\forall r,s,\Omega,\lambda\geq 1$.*

2 Jumarie's fractional derivative

Jumarie's fractional derivative is a moldified Riemann-Liouville derivative defined as [27-30]

$$D_t^{\alpha} f(x)=\left\{\begin{array}{l}\frac{1}{\Gamma(-\alpha)}\int_0^x (x-t)^{-\alpha-1}(f(t)-f(0))\,dt,\alpha\leq 0,\\ \frac{1}{\Gamma(-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}(f(t)-f(0))\,dt, 0\leq\alpha\leq 1\\ \left[f^{\alpha-n}\ (x)^{n}\right]^{n}, n\leq\alpha\leq n+1, n\geq 1\end{array}\right\} \tag{1}$$

where f : R $\to$ R, x $\to$ f(x) denotes a continuous (but not necessarily differentiable) function.

Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative were summarized in Refs. [27-30].

$$D_x^{\alpha} c=0,\alpha\geq 0, c=constant \tag{2}$$

$$D_x^{\alpha}\left[cf\ (x)\right]=cD_x^{\alpha} f(x)\,\alpha\geq 0, c=constant \tag{3}$$

$$D_x^{\alpha} x^{\beta}=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha},\beta\geq\alpha\geq 0. \tag{4}$$

$$D_x^{\alpha}\left[f(x)\ g(x)\right]=\left[D_x^{\alpha} f(x)\,g(x)+f(x)\left[D_x^{\alpha}\ g(x)\right].\right. \tag{5}$$

$$D_x^{\alpha} f(x(t))=f_x^{'}(x)\,.x^{\alpha}(t)\,. \tag{6}$$

3 Exp-function method

We consider the general nonlinear FPDE of the type

$$P(u,\,u_t,\,u_x,u_{xx}\,u_{xxx},\ D_t^{\alpha}u,D_x^{\alpha}u,D_{xx}^{\alpha}u,\,\ldots)=0,\quad 0\,<\,\alpha\,\leq\,1, \tag{7}$$

where $D_t^{\alpha}u,D_x^{\alpha}u,D_{xx}^{\alpha}u$are the modified Riemann-Liouville derivative of *u* with respect to $t,\ x,xx$ respectively.

Using a transformation [31]

$$\eta=kx+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}+\eta_0, \tag{8}$$

k,ω,η_0 are all constants with $k,\omega,\neq 0$.

We can rewrite equation (1) in the following nonlinear ODE;

$$Q(u,\,u^{'},u^{''},u^{'''},u^{iv})=0, \tag{9}$$

where the prime denotes derivative with respect to η.

According to Exp-function method, we assume that the wave solution can be expressed in the following form

$$u(\eta) = \frac{\sum_{n-c}^{d} a_n \exp[n\eta]}{\sum_{m-p}^{q} b_m \exp[m\eta]}. \tag{10}$$

where p, q, c and d are positive integers which are known to be further determined ,a_n and b_m are unknown constants. We can rewrite Eq.(4) in the following equivalent form

$$u(\eta) = \frac{a_c \exp(c\eta) + ... + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + ... + b_{-q} \exp(-q\eta)}. \tag{11}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and$p$ by using [25],

$$p = c, q = d \tag{12}$$

4 Numerical application

In this section, we apply exp-function method for fractional order nonlinear Klein-Gordon.

Example 1 *Consider the fractional order nonlinear Klein-Gordon equation*

$$D_t^{\alpha} u + (u_x)^2 + u - u^2 = t.e^x, 0 < \alpha \leq 1 \tag{13}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = e^{-x}.$$

Using (8) *equation* (13) *can be converted to an ordinary differential equation*

$$\omega^2 u'' + k^2 u'^2 + u - u^2 = te^{-x}. \tag{14}$$

where the prime denotes the derivative with respect to η. The solution of the equation (13) *can be expressed in the form, equation* (11)*. To determine the value of c and p, by using [25],*

$$p = c, q = d \tag{15}$$

Case 4.1.I. *We can freely choose the values of c andd, but we will illustrate that the final solution does not strongly depend upon the choice of values of candd. For simplicity, we set $p = c = 1$and $q = d = 1$ equation* (11) *reduces to*

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \tag{16}$$

Substituting equation (16) *into equation* (14)*, we have*

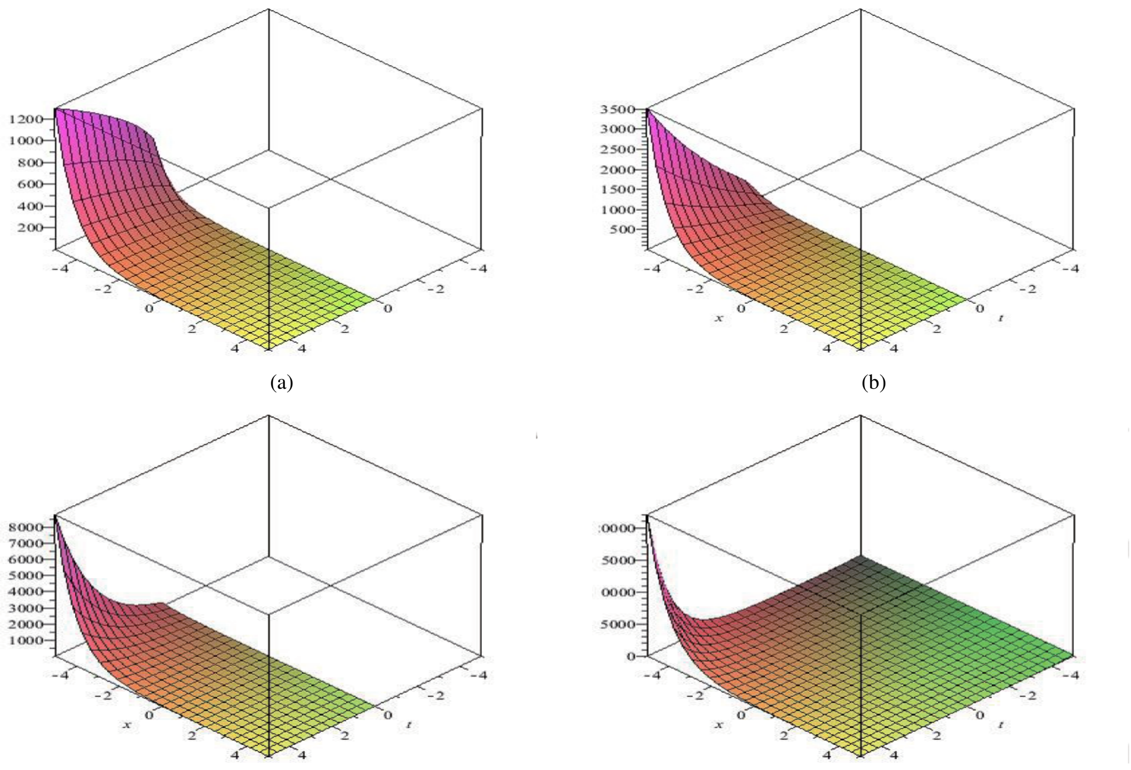
$$\frac{1}{A}\left[\begin{array}{l} c_4 \exp(4\eta) = c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \\ +c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) \end{array}\right] = 0 \tag{17}$$

where $A = (b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^4$,$c_i$ are constants obtained by Maple 16. Equating the coefficients of $\exp(n\eta)$ *to be zero, we obtain*

$$\{c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\} \tag{18}$$

Solution of (12) *will yield*

$$\left\{k = -1, a_{-1} = -\tfrac{1}{16}\frac{b_0^2\left(-1 + \omega^2 + 4\left(t\left(\frac{1}{e^x}\right)\right)\right)}{a_1}, a_0 = \tfrac{1}{2}b_0\omega^2 + \tfrac{1}{2}b_0, a_1 = a_1, b_{-1} = 0, b_0 = b_0, b_1 = 0\right\}. \tag{19}$$



(a) (b)

(c) (d)

Figure 1: Soliton solutions of equation (13) for $a_1 = b_0 = \omega = 1$ and $k = -1$.

We, therefore, obtained the following generalized solitary solution $u\,(x,t)$*of equation* (13)*.*

$$\frac{1}{b_0}\left\{ -\frac{1}{16}\frac{b_0^2\left(-1+\omega^2+4\left(t\left(\frac{1}{e^x}\right)\right)\right)e^{-\frac{-x+\omega t^\alpha}{\Gamma(1+\alpha)}}}{a_1} + \frac{1}{2}b_0\omega^2 + \frac{1}{2}b_0 + a_1 e^{\frac{-x+\omega t^\alpha}{\Gamma(1+\alpha)}} \right\} \tag{20}$$

Case 4.1.II. *If* $p = c = 2$ *and* $q = d = 1$ *then trial solution, equation* (13) *reduces to*

$$u\,(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0. + b_{-1}\exp[-\eta]}. \tag{21}$$

Proceeding as before, we obtain

$$\left\{ \begin{array}{l} a_2 = 0, b_2 = 0, b_1 = 0, b_{-1} = 0, a_0 = \frac{1}{2}\omega^2 b_0 + \frac{1}{2}b_0, a_{-1} = a_{-1}, \\ a_1 = -\frac{1}{16}\frac{b_0^2\left(-e^x+e^x\omega^4+4t\right)}{a_{-1}e^x}, k = -1, b_0 = b_0 \end{array} \right\} \tag{22}$$

Hence we get the generalized solitary wave solution of equation (13) *for* $\alpha = 1$*as follow*

$$u\,(x,t) = \frac{-\frac{1}{16}\frac{b_0^2\left(-e^x+e^x\omega^4+4t\right)e^{(kx+\omega t)}}{a_{-1}e^x} + \frac{1}{2}\omega^2 b_0 + \frac{1}{2}b_0 + a_{-1}e^{(-kx-\omega t)}}{b_0}. \tag{23}$$

In both cases, for different choices of c, p, d *and* q*, we get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.*

5 Conclusion

In this paper, we applied exp-function method to construct generalized solitary solutions of the nonlinear fractional order Klien-Gordon equations. It is observed that the Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems.

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