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Numerical Solution of Second Order Singularly Perturbed Differential–Difference Equation with Negative Shift

K. Phaneendra[1] *, GBSL Soujanya[2], Y.N. Reddy[3]
[1] Department of Mathematics, University College of Science, Saifabad, Osmania University, Hyderabad-500007, INDIA
[2] Department of Mathematics, Kakatiya University, Warangal-506009, INDIA
[3] Department of Mathematics, National Institute of Technology, Warangal-506004, INDIA
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Abstract: In this paper, we present a numerical finite difference method to solve the boundary-value problem for singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term. In this method, we first approximate the shifted term by Taylor series and apply a fourth order finite difference scheme, provided shifts are of $o(\varepsilon)$. The existence and uniqueness of the discrete problem along with stability estimates are discussed. We have also discussed the convergence of the method. The effect of small shifts on the boundary layer solution of the problem has been given by considering several numerical experiments.
Keywords: Boundary layer; Taylor series; Tridiagonal system; Differential-difference equation; Negative shift

1 Introduction

Delay differential equation play an important role in the mathematical modeling of various practical phenomena in the biosciences and control theory. The problems in which the highest order derivative term is multiplied by a small positive parameter are known to be perturbed problems and the parameter is known as the perturbation parameter. Depending on the solution behavior of the problem in the limiting case when perturbation parameter goes to zero, such type of problems are classified into two classes, namely, (i) regularly perturbed and (ii) singularly perturbed. If the solution of the original problem tends to the solution of the reduced problem (i.e., the problem which is obtained by putting $\varepsilon = 0$ in the original problem) as the perturbation parameter tends to zero, the problem is known as regularly perturbed, otherwise it is known as singularly perturbed. Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense information and then react to it. If we restrict the class of delay differential equations to a class in which the highest derivative is multiplied by a small positive parameter, then it is said to be a singularly perturbed delay differential equation. In the literature the researchers used negative shift for delay. The boundary value problems for singularly perturbed differential equations with delay or negative shift are ubiquitous in the modeling of various physical and biological phenomena.

The literature on delay differential equations is mainly centered on first order initial value problems [1], [2]. The boundary value problems of delay differential equations are ubiquitous in the variational problems in control theory [3]. The authors in [6], [7] gave an asymptotic approach in study of class of boundary-value problems for linear second-order differential-difference equations in which the highest order derivative is multiplied by small parameter. In [6], they presented mathematical model of the determination of expected time for generation of action potentials in nerve cell by random synaptic inputs in the dendrites.

In [7], authors presented the problems that have solutions that exhibit rapid oscillations. Restrictions on the sizes of the shifts in terms of the small parameter are found such that, generally, the shifted terms cannot be replaced with truncated Taylor series. In particular, it is shown that, even when the shifts are small relative to the width of an oscillation, they can affect the solution to leading order. The conclusion is that oscillatory solutions are more sensitive to small delays

* **Corresponding author**. *E-mail address*: kollojuphaneendra@yahoo.co.in

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than are layer solutions. It is shown that a suitably modified version of the standard WKB method can be used to obtain leading-order oscillatory solutions of these differential-difference equations.

In [4], authors presented a numerical approach to solve singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term. In this method they first approximate the shifted term by Taylor series and apply a difference scheme, provided shifts are $o(\varepsilon)$. In [5], authors presented a numerical method to solve singularly perturbed differential-difference equation which contains only negative shift not in the differentiated terms. In this method they present a numerical method composed of a standard upwind finite difference scheme on a special type of mesh shifts are either $o(\varepsilon)$ or $O(\varepsilon)$. Authors in [9] derived a fifth order method to solve singularly perturbed differential –difference equations with negative shift.

The objective of this paper is to present a numerical finite difference approach to solve the boundary-value problem for singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term. In this method, we first approximate the shifted term by Taylor series and apply a fourth order finite difference scheme, provided shifts are of o(ε). The existence and uniqueness of the discrete problem along with stability estimates are discussed. we have also discussed the convergence of the method. The effect of small shifts on the boundary layer solution of the problem has been given by considering several numerical experiments.

2 Numerical scheme

We consider the boundary value problem for a singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x) \tag{1}$$

on (0,1), under the boundary conditions

$$y(x) = \phi(x) \text{ on } -\delta \le x \le 0, \quad y(1) = \gamma, \tag{2}$$

where ϵ is a small parameter, $0 < \varepsilon << 1$ and δ is also a small shifting parameter, $0 < \delta << 1$, $a(x), b(x), f(x), \delta(\epsilon)$ and $\phi(x)$ are smooth functions and γ is a constant. Now there are two cases according to the sign of $a(x)$. If $a(x) \ge M > 0$ throughout the interval [0, 1], where M is a positive constant, then boundary layer will be in the neighborhood of 0, i.e., on the left side of the interval [0, 1]. If $a(x) \le M < 0$ throughout the interval [0, 1], then boundary layer will be in the neighborhood of 1, i.e., on the right side of the interval [0, 1].

Since the solution $y(x)$ of boundary value problem (1) and (2) is sufficiently differentiable, so we expand the retarded term $y'(x-\delta)$ by Talor series, we obtain

$$y'(x-\delta) \approx y'(x) - \delta y''(x) \tag{3}$$

using (3) in (1),we obtain

$$(\epsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{4}$$

Equation (4) is a second order singular perturbation problem.

We solve (4) subject to the boundary conditions (2) by using fourth order finite difference method described as follows:

By Taylor series expansion, we obtain the difference formulae for $y_i',\ y_i''$assuming that y has continuous fourth order derivatives in the interval [0, 1] as:

$$y_i'' \cong \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) \tag{5}$$

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y^{(3)}(\eta) \tag{6}$$

where $\xi, \eta \in [x_{i-1},\ x_{i+1}]$. Substituting (5) and (6) in (4) at $x = x_i$, we get the difference operator L_h, defined by

$$L_h y_i = e_i y_{i-1} - f_i y_i + g_i y_{i+1} = f_i + \tau_i[y] \tag{7}$$

for $1 \le i \le n-1$, where

$$e_i = \frac{(\varepsilon - a_i\delta)}{h^2} - \frac{a_i}{2h}, f_i = \frac{2(\varepsilon - a_i\delta)}{h^2} - b_i, g_i = \frac{(\varepsilon - a_i\delta)}{h^2} + \frac{a_i}{2h},$$

and $\tau_i[y] = -\frac{h^2\varepsilon}{12}y^{(4)}(\xi) + \frac{h^2 a_i}{6}y'''(\eta)$ where $\xi,\ \eta \in [x_{i-1},\ x_{i+1}]$, here $\tau_i[y]$ are local truncation errors of the difference approximation.

We now rewrite the central difference formulae for y_i' and y_i'' in new form as given below

$$y_i^{"} \cong \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \frac{h^2}{12}y_i^{(4)} + R_1, \quad y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6}y_i''' + R_2 \tag{8}$$

where

$$R_1 = -\frac{2h^4 y^{(5)}(\xi)}{6!}, \ \ R_2 = -\frac{h^4 y^{(4)}(\eta)}{5!}$$

for $\xi, \eta \in [x_{i-1},\ x_{i+1}]$. Substituting $y_i',\ y_i''$ from (9) and (10) in (4) at $x = x_i$, it becomes

$$L_i y_i - \frac{h^2}{12}\left[(\varepsilon - a_i\delta)\, y_i^{(4)} + 2a_i y_i'''\right] + \tilde{R} = f_i \tag{9}$$

where $\tilde{R} = (\varepsilon - a_i\delta)\, R_1 + a_i R_2$

Now, we find $y_i''', y_i^{(4)}$ from the differential equation (4) and substituting these in (11), we get

$$\tilde{L}_h y_i = E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i + \tilde{\tau}_i[y] \tag{10}$$

where

$$E_i = e_i + \frac{hC_i}{24} - \frac{D_i}{12} + \frac{ha_iC_i'}{12} - \frac{a_iD_i'}{6},\ G_i = g_i - \frac{hC_i}{24} - \frac{D_i}{12} - \frac{ha_iC_i'}{12} - \frac{a_iD_i'}{6}$$

$$F_i = f_i + \frac{h^2 B_i}{12} - \frac{D_i}{6} + \frac{h^2 a_i B_i'}{6} - \frac{a_i D_i'}{3}, H_i = f_i + \frac{h^2 A_i}{12} + \frac{h^2 A_i' a_i}{6}.$$

Here

$$A_i = f_i'' + \frac{2a_i' f_i' \delta}{(\varepsilon - a_i\delta)} + \frac{a_i'' f_i \delta}{(\varepsilon - a_i\delta)} + \frac{2(a_i')^2 f_i \delta^2}{(\varepsilon - a_i\delta)^2} - \frac{a_i a_i' \delta f_i}{(\varepsilon - a_i\delta)^2} - \frac{a_i f_i'}{(\varepsilon - a_i\delta)}$$

$$B_i = \frac{-2a_i' b_i' \delta}{(\varepsilon - a_i\delta)} - \frac{a_i'' b_i \delta}{(\varepsilon - a_i\delta)} - \frac{2(a_i')^2 b_i \delta^2}{(\varepsilon - a_i\delta)^2} + \frac{a_i a_i' \delta b_i}{(\varepsilon - a_i\delta)^2} + \frac{a_i b_i'}{(\varepsilon - a_i\delta)}$$

$$C_i = -(a_i'' + 2b_i') + \frac{2a_i'(a_i' + b_i)\delta}{(\varepsilon - a_i\delta)} + \frac{2(a_i')^2 a_i \delta^2}{(\varepsilon - a_i\delta)^2} + \frac{a_i^2 a_i' \delta}{(\varepsilon - a_i\delta)^2} + \frac{(a_i b_i + a_i a_i')}{(\varepsilon - a_i\delta)} - \frac{a_i a_i'' \delta}{(\varepsilon - a_i\delta)}$$

$$D_i = -(2a_i' + b_i) - \frac{2a_i a_i' \delta}{(\varepsilon - a_i\delta)} + \frac{a_i^2}{(\varepsilon - a_i\delta)},$$

$$A_i' = \frac{f_i'}{\varepsilon - a_i\delta} + \frac{a_i' \delta f_i}{(\varepsilon - a_i\delta)^2}, B_i' = \frac{-b_i'}{\varepsilon - a_i\delta} - \frac{a_i' b}{(\varepsilon - a_i\delta)^2}, C_i' = \frac{-(a_i' + b_i)}{\varepsilon - a_i\delta} - \frac{a_i a_i' \delta}{(\varepsilon - a_i\delta)^2}, D_i' = \frac{-a_i}{\varepsilon - a_i\delta}.$$

Equation (12) is a three term recurrence relation and gives a tridiagonal system. We solve this system by using Discrete invariant imbedding algorithm.

3 Discrete invariant imbedding algorithm

Let us set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \tag{11}$$

for $i = N-1, N-2, \ldots., 2, 1$. Where $W_i = W(x_i)$and $T_i = T(x_i)$ which are to be determined. From (13), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{12}$$

substituting (14) in (13), we have

$$E_i\left(W_{i-1}y_i+T_{i-1}\right)-F_iy_i+G_iy_{i+1}=H_i, y_i=\left(\frac{G_i}{F_i-E_iW_{i-1}}\right)y_{i+1}+\left(\frac{E_iT_{i-1}-H_i}{F_i-E_iW_{i-1}}\right).$$

By comparing (13) and (15), we get the recurrence relations

$$W_i=\left(\frac{G_i}{F_i-E_iW_{i-1}}\right), T_i=\left(\frac{E_iT_{i-1}-H_i}{F_i-E_iW_{i-1}}\right). \tag{13}$$

To solve these recurrence relations for $i=2,3,\ldots\ldots,N-1$, we need the initial conditions for W_0 and T_0. If we choose $W_0=0$, then we get $T_0=\phi_0$. With these initial values, we compute W_i and T_i for i = 2, 3,. . . ., N-1 from (16) and (17) in forward process, and then obtain y_i in the backward process from (13).

Under the conditions

$$E_i>0, G_i>0, F_i\geq E_i+G_i \quad and \quad E_i\leq G_i \tag{14}$$

The discrete invariant imbedding algorithm is stable [8].

One can easily show that if the assumptions $a(x)>0$, $b(x)<0$ and $(\varepsilon-\delta a(x))>0$ hold, then the above conditions (18) hold and thus the invariant imbedding algorithm is stable.

4 Stability and convergence analysis

Theorem 1 *Under the assumptions $(\varepsilon-\delta a(x))>0$, $a(x)\geq M>0$ and b(x) < 0, $\forall x\in[0,\ 1]$, the solution to the system of the difference equations (12), together with the given boundary conditions exists, is unique and satisfies*

$$\|y\|_{h,\infty}\leq 2M^{-1}\|f\|_{h,\infty}+(|\phi_0|+|\gamma|)$$

where $\|\cdot\|_{h,\infty}$is the discrete $l_\infty-norm$, given by $\|x\|_{h,\infty}=\max\limits_{0\leq i\leq N}\{\ |x_i|\}$.

Proof. Let $L_h(.)$ denote the difference operator on left hand side of (12) and w_i be any mesh function satisfying

$$L_h(w_i)=f_i$$

By rearranging the difference scheme (12) and using non-negativity of the coefficients E_i, F_i and G_i, we obtain

$$F_i\ |w_i|\ \leq\ |H_i|+E_i\ |w_{i-1}|+G_i\ |w_{i+1}|$$

$$\left[(\ \varepsilon\text{-}a_i\delta)-\tfrac{h^2D_i}{12}-\tfrac{h^2a_iD_i'}{6}\right]\tfrac{(|w_{i+1}|-2|w_i|+|w_{i-1}|)}{h^2}+\left[a_i-\tfrac{h^2C_i}{12}-\tfrac{h^2a_iC_i'}{6}\right]\left(\tfrac{|w_{i+1}|-|w_{i-1}|}{2h}\right)$$
$$+b_i\,|w_i|-\tfrac{h^2B_i}{12}\,|w_i|+|H_i|\ \geq 0$$

Now using the assumption $(\varepsilon-\delta a_i)\geq M$, the definition of l_∞-norm and manipulating, we obtain

$$\left((\ \varepsilon\text{-}\delta M)-\tfrac{h^2\|D\|_{h,\infty}}{12}-\tfrac{h^2\|a\|_{h,\infty}\|D'\|_{h,\infty}}{6}\right)\tfrac{(|w_{i+1}|-2|w_i|+|w_{i-1}|)}{h^2}+$$
$$\left(\|a\|_{h,\infty}-\tfrac{h^2\|C\|_{h,\infty}}{12}-\tfrac{h^2\|a\|_{h,\infty}\|C'\|_{h,\infty}}{6}\right)\left(\tfrac{|w_{i+1}|-|w_{i-1}|}{2h}\right)+b_i\,|w_i|-\tfrac{h^2\|B\|_{h,\infty}}{12}\,|w_i|+|H_i|\ \geq 0 \tag{15}$$

To prove the uniqueness and existence, let $\{u_i\}$, $\{v_i\}$ be two sets of solution of the difference equation (12) satisfying boundary conditions. Then

$$w_i=u_i-v_i$$

satisfies

$$L_h(w_i)=f_i$$

where

$$f_i = 0 \text{ and } w_0 = w_N = 0.$$

Summing (19) over over i = 1, 2, ..., N-1, we obtain

$$\begin{gathered}-\left[(\varepsilon\text{-}\delta M)-\frac{h^2\|D\|_{h,\infty}}{12}-\frac{h^2\|a\|_{h,\infty}\left\|D'\right\|_{h,\infty}}{6}\right]\frac{|w_1|}{h^2}-\left[(\varepsilon\text{-}\delta M)-\frac{h^2\|D\|_{h,\infty}}{12}-\frac{h^2\|a\|_{h,\infty}\left\|D'\right\|_{h,\infty}}{6}\right]\frac{|w_{N-1}|}{h^2}\\-\left[\|a\|_{h,\infty}-\frac{h^2\|C\|_{h,\infty}}{12}-\frac{h^2\|a\|_{h,\infty}\left\|C'\right\|_{h,\infty}}{6}\right]\frac{|w_1|}{2h}+\sum_{i=1}^{N-1}\left(b_i-\frac{h^2\|B\|_{h,\infty}}{12}\right)|w_i|\ \geq 0\end{gathered} \tag{16}$$

Since

$$(\varepsilon-\delta M)>0,\ \|a\|_{h,\infty}\geq 0,\ \|D\|_{h,\infty}\geq 0,\ \|D'\|_{h,\infty}\geq 0, \|C\|_{h,\infty}\geq 0, \|D'\|_{h,\infty}\geq 0,$$

$$\|B\|_{h,\infty}\geq 0,\ b_i<0 \text{ and } |w_i|\geq 0\ \forall i,\ i=1,2,......,N-1,$$

therefore for inequality (20) to hold, we must have $w_i = 0\ \forall i,\ i = 1, 2, ...N-1.$

This implies the uniqueness of the solution of the tridiagonal system of difference equations (12). For linear equations, the existence is implied by uniqueness. Now to establish the estimate, let $w_i = y_i - l_i$,

Where y_i satisfies difference equations (10), the boundary conditions and

$$l_i = (1-ih)\,\phi_0 + (ih)\,\gamma,$$

then $w_0 = w_N = 0$,

and $w_i,\ i = 1, 2, ...N-1$

$$L_h(w_i) = f_i$$

Now let

$$|w_n| = \|w\|_{h,\infty} \geq |w_i|\,,\ i = 0, 1, ..., N.$$

Then summing (19) from $i = n$ to $N-1$ and using the assumption on $a(x)$, which gives

$$\begin{gathered}-\left[(\varepsilon-\delta M)-\frac{h^2\|D\|_{h,\infty}}{12}-\frac{h^2M\left\|D'\right\|_{h,\infty}}{6}\right]\frac{(|w_{n+1}|-2|w_n|+|w_{n-1}|)}{h^2}-\left[(\varepsilon-\delta M)-\frac{h^2\|D\|_{h,\infty}}{12}-\frac{h^2M\left\|D'\right\|_{h,\infty}}{6}\right]\frac{|w_{N-1}|}{h^2}\\-\left[M-\frac{h^2\|C\|_{h,\infty}}{12}-\frac{h^2M\left\|C'\right\|_{h,\infty}}{6}\right]\frac{|w_n|}{2h}+\sum_{i=1}^{N-1}\left(b_i-\frac{h^2\|B\|_{h,\infty}}{12}\right)|w_i|+\sum_{i=n}^{N-1}|H_i|\geq 0\end{gathered} \tag{17}$$

Inequality (21), together with the condition on $b(x)$ implies that

$$\frac{M}{2}\,|w_n| \leq h\sum_{i=n}^{N-1}|H_i| \leq h\sum_{i=0}^{N}|H_i| \leq \|H\|_{h,\infty}\,,$$

i.e., we have

$$|w_n| \leq 2M^{-1}\,\|H\|_{h,\infty} \tag{18}$$

Also, we have

$$\|y\|_{h,\infty} = \max_{0\leq i\leq N}\{\ |y_i|\ \}\ \leq\ \|w\|_{h,\infty} + \|l\|_{h,\infty}\ \leq\ |w_n| + \|l\|_{h,\infty}\,. \tag{19}$$

Now to complete the estimate, we have to find out the bound on l_i

$$\|l\|_{h,\infty} = \max_{0\leq i\leq N}\{\ |l_i|\ \}\ \leq \max_{0\leq i\leq N}\{\ |(1-ih)|\ \ |\phi_0| + |ih|\ \ |\gamma|\}\ \leq \max_{0\leq i\leq N}\{\ (1-ih)\ \ |\phi_0| + (ih)\ \ |\gamma|\}\,,$$

i.e., we have

$$\|\ l\ \|_{h,\infty} \leq |\phi_0| + |\gamma|\,. \tag{20}$$

From (22) – (24), we obtain the estimate

$$\|y\|_{h,\infty} \leq 2M^{-1}\,\|f\|_{h,\infty} + (|\phi_0| + |\gamma|)\,.$$

This theorem implies that the solution to the system of the difference equations (12) are uniformly bounded, independent of mesh size h and the perturbation parameter ε. Thus the scheme is stable for all step sizes. ■

Corollary 1 *Under the conditions for theorem 1, the error* $e_i = y(x_i) - y_i$ *between the solution* $y(x)$ *of the continues problem and the solution* y_i *of the discretized problem, with boundary conditions, satisfies the estimate*

$$\|e\|_{h,\infty} \leq 2M^{-1} \|\tau\|_{h,\infty},$$

where

$$|\tau_i| \leq \max_{x_{i-1}\leq x\leq x_{i+1}} \left\{ \frac{h^4 a(x)}{5!} \left|y^{(5)}(x)\right| \right\} + \max_{x_{i-1}\leq x\leq x_{i+1}} \left\{ \frac{2h^4(\varepsilon - \delta a(x))}{6!} \left|y^{(6)}(x)\right| \right\}$$

Proof. Truncation error τ_i is given by

$$\tau_i = (\varepsilon - a(x)\delta) \left\{ \left(\frac{y_{i+1} - y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y_i^{(4)} \right) - y_i'' \right\} + a(x) \left\{ \left(\frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y_i^{(3)} \right) - y_i' \right\}$$

$$|\tau_i| \leq \max_{x_{i-1}\leq x\leq x_{i+1}} \left\{ \frac{h^4 a(x)}{5!} \left|y^{(5)}(x)\right| \right\} + \max_{x_{i-1}\leq x\leq x_{i+1}} \left\{ \frac{2h^4(\varepsilon - \delta a(x))}{6!} \left|y^{(6)}(x)\right| \right\}$$

One can easily show that the error e_i, satisfies

$$L_h\left(e(x_i)\right) = L_h\left(y(x_i)\right) - L_h\left(y_i\right) = \tau_i,\ i = 1, 2, ..., N-1$$

and $e_0 = e_N = 0$. ■

Theorem 2 *Then Theorem 1 implies that*

$$\|e\|_{h,\infty} \leq 2M^{-1} \|\tau\|_{h,\infty}. \quad (21)$$

The estimate (25) establishes the convergence of the difference scheme for the fixed values of the parameter ε. *Under the assumptions* $(\varepsilon - \delta a(x)) > 0$,$a(x) \leq M < 0$ *and* $b(x) < 0$, $\forall x \in [0,\ 1]$, *the solution to the system of the difference equations (12), together with the given boundary conditions exists, is unique and satisfies*

$$\|y\|_{h,\infty} \leq 2M^{-1} \|f\|_{h,\infty} + (|\phi_0| + |\gamma|).$$

Proof. The proof of estimate can be done on similar lines as we did in theorem 1. ■

5 Numerical results and discussion

To demonstrate the efficiency of the method, we consider two numerical experiments with left-end boundary layer and two numerical experiments with right-end boundary layer. We compare the results with the exact solution of the problems. To show the layer behavior, we plot the graphs of the exact and computed solution of the problem for different values of ε and for different values of δ of $o(\varepsilon)$, which are represented by solid and dotted lines respectively.

Example 1 *Consider an example of the BVP (1), (2) with constant coefficients, with* $a(x) = 1, b(x) = -1, f(x) = 0, \phi(x) = 1$ *and* $\gamma = 1$. *The singularly perturbed delay differential equation is*

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0\ ;\ x \in [0,1]$$

with $y(0) = 1$ *and* $y(1) = 1$.

The exact solution is given by

$$y(x) = \frac{((1 - e^{m_2})e^{m_1 x} + (e^{m_1} - 1)e^{m_2 x})}{(e^{m_1} - e^{m_2})}$$

where $m_1 = {(-1 - \sqrt{1 + 4(\varepsilon - \delta)}\ )}\big/{2(\varepsilon - \delta)}$ *and* $m_2 = {(-1 + \sqrt{1 + 4(\varepsilon - \delta)}\ )}\big/{2(\varepsilon - \delta)}$.

The maximum errors are presented in Tables 1(a), 1(b) for $\varepsilon = 0.1$, *0.01 and different choices of* δ *and grid size. We plot the graphs of the solution of the problem for* $\varepsilon = 0.1$, *0.01 and for different values of* δ *of* $o(\varepsilon)$, *shown in figures 1 and 2 respectively.*

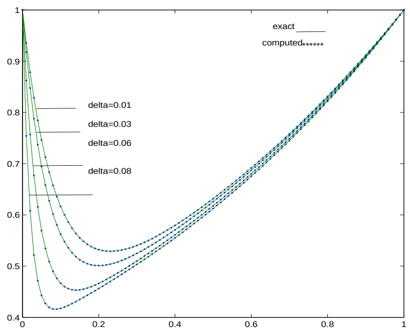

Figure 1: Graph of the solution of the Example (1) for $\varepsilon = 0.1$ and for different of δ of o(ε).

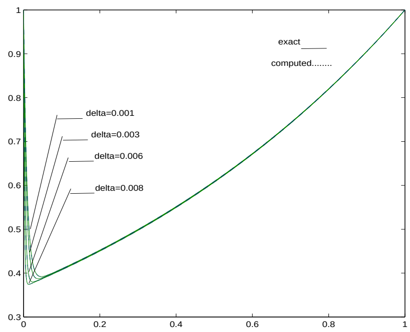

Figure 2: The maximum errors of example (1) for ε=0.01 and for different δ and grid size.

Example 2 *Now we consider an example of the BVP with variable coefficients. Consider equation (1), (2) with* $a(x) = e^{-0.5x}, b(x) = -1, f(x) = 0, \phi(x) = 1$ *and* $\gamma = 1$. *The variable coefficient singularly perturbed delay differential equation is*

$$\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) - y(x) = 0 \text{ with } y(0) = 1,\ y(1) = 1$$

for which exact solution is not known.

We plot the graphs of the solution of the problem for $\varepsilon = 0.1$, *0.01 and for different values of* δ *of* $o(\varepsilon)$, *shown in figures 3 and 4 respectively.*

Remark 1 *We have considered numerical results for several test examples to show the effect of small shifts on boundary layer solution of the problem. From the numerical experiments presented here, we observe as* δ *increases, the thickness of the boundary layer decreases and maximum error decreases as the grid size* h *decreases, which shows the convergence to the computed solution.*

Layer on the right side: *If* $a(x) \leq M < 0$ *throughout the interval [0, 1], then the boundary layer will be in the neighborhood of 1, i.e., on the right side of the interval [0, 1]. To demonstrate the efficiency of the method, we consider two numerical experiments.*

Example 3 *Consider (1), (2) with* $a(x) = -1, b(x) = -1, f(x) = 0, \phi(x) = 1$ *and* $\gamma = -1$. *The singularly perturbed delay differential equation is*

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0\ ;\ x \in [0,1]$$

with $y(0) = 1$ *and* $y(1) = -1$. *The exact solution is given by*

$$y(x) = \frac{(1+e^{m_2})e^{m_1 x} - (e^{m_1}+1)e^{m_2 x})}{(e^{m_2} - e^{m_1})}$$

Where $m_1 = {(1-\sqrt{1+4(\varepsilon+\delta)}\ )}\big/{2(\varepsilon+\delta)}$ *and* $m_2 = {(1+\sqrt{1+4(\varepsilon+\delta)}\ )}\big/{2(\varepsilon+\delta)}$.
The maximum errors are presented in Tables 3(a), 3(b) for $\varepsilon = 0.01, 0.001$ *different choices of* δ *and grid size. Also we plot the graph of the solution of the problem for* $\varepsilon = 0.01$ *and for different values of* δ *of* $o(\varepsilon)$, *shown in figure 5.*

Example 4 *Now we consider an example of the BVP with variable coefficients. Consider equation (1), (2) with* $a(x) = -e^x, b(x) = -x, f(x) = 0, \phi(x) = 1$ *and* $\gamma = 1$. *The variable coefficient singularly perturbed delay differential equation is*

$$\varepsilon y''(x) - e^x y'(x-\delta) - xy(x) = 0, \text{ with } y(0) = 1,\ y(1) = 1$$

for which exact solution is not known. We plot the graphs of the solution of the problem for $\varepsilon = 0.01$, *0.001and for different values of* δ *of* $o(\varepsilon)$, *shown in figures 6 and 7 respectively.*

Table 1: The maximum errors of example (1) for $\varepsilon = 0.1$ and for different δ and grid size

$\delta\backslash$ N	10^2	10^3	10^4
0.01	$6.2706e-008$	$6.3283e-012$	$3.2276e-012$
0.03	$1.6378e-007$	$1.6441e-011$	$2.5074e-012$
0.06	$1.4301e-006$	$1.4254e-010$	$1.5683e-011$
0.08	$2.1860e-005$	$2.1528e-009$	$5.8608e-011$

Table 2: The maximum errors of example (1) for ε=0.01 and for different δ and grid size

$\delta\backslash$ N	10^2	10^3	10^4
0.01	$5.4191e-004$	$5.0759e-008$	$3.2564e-011$
0.03	$1.4385e-003$	$1.3787e-007$	$2.9030e-011$
0.06	$9.9988e-003$	$1.2835e-006$	$1.1959e-010$
0.08	$6.1971e-002$	$2.0628e-005$	$2.0281e-009$

Remark 2 *We have considered several numerical experiments to show the effect of small shifts on boundary layer solution. From the numerical experiments presented here, we observe that as* δ *increases, the thickness of right boundary layer increases. As the grid size* h *decreases, the maximum error decreases, which shows the convergence to the computed solution.*

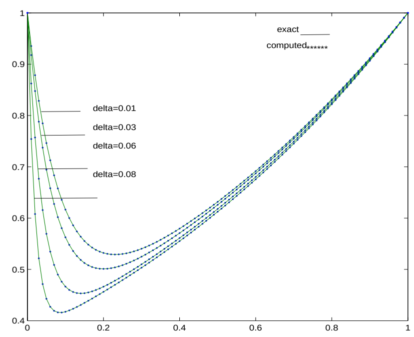

Figure 3: Graph of the solution of the Example (1) for ε = 0.01 and for different of δ of o(ε)

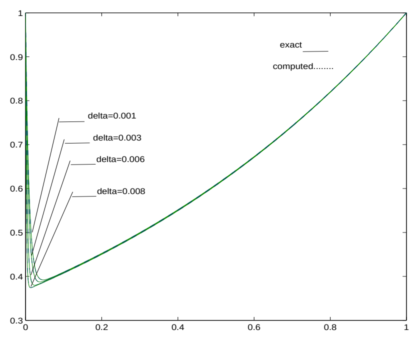

Figure 4: Graph of the solution of the Example (2) for ε = 0.1 and for different of δ of o(ε)

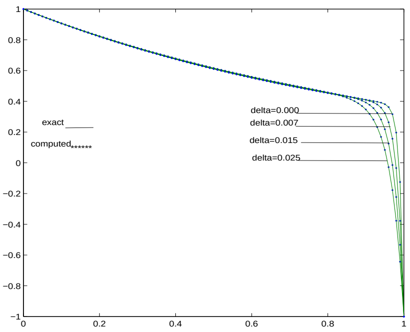

Figure 5: Graph of the solution of the Example (2) for ε = 0.01 and for different of δ of o(ε)

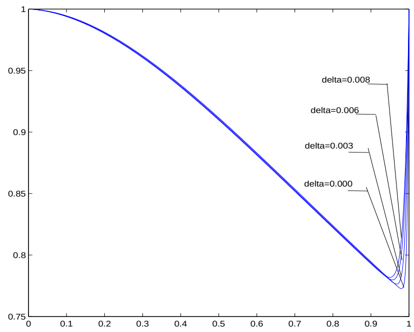

Figure 6: Graph of the solution of the Example (3) for ε = 0.1 and for different of δ of o(ε)

Table 3: he maximum errors of example (3) for ε=0.1 and for different δ and grid size

$\delta\backslash$ N	10^2	10^3	10^4
0.000	$7.7315e-004$	$7.2899e-008$	$2.8151e-011$
0.007	$9.0069e-005$	$8.9708e-009$	$3.7588e-011$
0.015	$1.9607e-005$	$1.9777e-009$	$3.5737e-011$
0.025	$5.3312e-006$	$5.3411e-010$	$1.4266e-011$

Table 4: The maximum errors of example (3) for ε=0.01 and for different δ and grid size

$\delta\backslash$ N	10^2	10^3	10^4
0.0007	$1.8573e-001$	$8.4819e-005$	$8.4629e-009$
0.0015	$8.0711e-002$	$1.7927e-005$	$1.8318e-009$
0.0025	$3.2547e-002$	$4.7036e-006$	$4.9565e-010$

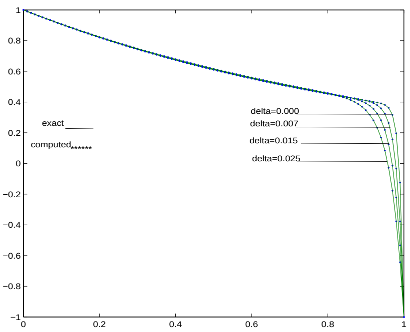

Figure 7: Graph of the solution of the Example (4) for $\varepsilon = 0.1$ and for different of δ of $o(\varepsilon)$

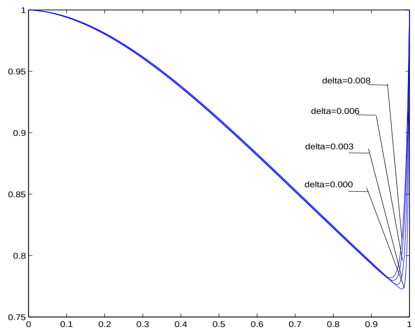

Figure 8: Graph of the solution of the Example (4) for $\varepsilon = 0.01$ and for different of δ of $o(\varepsilon)$

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